Section 3-10 : Implicit Differentiation

To this point we've done quite a few derivatives, but they have all been derivatives of functions of the form $y = f(x)$. Unfortunately, not all the functions that we're going to look at will fall into this form.

Let's take a look at an example of a function like this.

Example 1 Find y' for $xy = 1$.

Solution

There are actually two solution methods for this problem.

Solution 1 :

This is the simple way of doing the problem. Just solve for *y* to get the function in the form that we're used to dealing with and then differentiate.

$$y = \frac{1}{x} \qquad \Rightarrow \qquad y' = -\frac{1}{x^2}$$

So, that's easy enough to do. However, there are some functions for which this can't be done. That's where the second solution technique comes into play.

Solution 2 :

In this case we're going to leave the function in the form that we were given and work with it in that form. However, let's recall from the first part of this solution that if we could solve for *y* then we will get *y* as a function of *x*. In other words, if we could solve for *y* (as we could in this case but won't always be able to do) we get $y = y(x)$. Let's rewrite the equation to note this.

$$xy = x\,y(x) = 1$$

Be careful here and note that when we write $y(x)$ we don't mean *y* times *x*. What we are noting here is that *y* is some (probably unknown) function of *x*. This is important to recall when doing this solution technique.

The next step in this solution is to differentiate both sides with respect to *x* as follows,

$$\frac{d}{dx}\left(x\,y(x)\right) = \frac{d}{dx}(1)$$

The right side is easy. It's just the derivative of a constant. The left side is also easy, but we've got to recognize that we've actually got a product here, the *x* and the $y(x)$. So, to do the derivative of the left side we'll need to do the product rule. Doing this gives,

$$(1)\,y(x) + x\frac{d}{dx}\left(y(x)\right) = 0$$

Now, recall that we have the following notational way of writing the derivative.

$$\frac{d}{dx}\left(y\left(x\right)\right)=\frac{dy}{dx}=y'$$

Using this we get the following,

$$y+xy'=0$$

Note that we dropped the $\left(x\right)$ on the *y* as it was only there to remind us that the *y* was a function of *x* and now that we've taken the derivative it's no longer really needed. We just wanted it in the equation to recognize the product rule when we took the derivative.

So, let's now recall just what were we after. We were after the derivative, y', and notice that there is now a y' in the equation. So, to get the derivative all that we need to do is solve the equation for y'.

$$y'=-\frac{y}{x}$$

There it is. Using the second solution technique this is our answer. This is not what we got from the first solution however. Or at least it doesn't look like the same derivative that we got from the first solution. Recall however, that we really do know what *y* is in terms of *x* and if we plug that in we will get,

$$y'=-\frac{1/x}{x}=-\frac{1}{x^2}$$

which is what we got from the first solution. Regardless of the solution technique used we should get the same derivative.

The process that we used in the second solution to the previous example is called **implicit differentiation** and that is the subject of this section. In the previous example we were able to just solve for *y* and avoid implicit differentiation. However, in the remainder of the examples in this section we either won't be able to solve for *y* or, as we'll see in one of the examples below, the answer will not be in a form that we can deal with.

In the second solution above we replaced the *y* with $y\left(x\right)$ and then did the derivative. Recall that we did this to remind us that *y* is in fact a function of *x*. We'll be doing this quite a bit in these problems, although we rarely actually write $y\left(x\right)$. So, before we actually work anymore implicit differentiation problems let's do a quick set of "simple" derivatives that will hopefully help us with doing derivatives of functions that also contain a $y\left(x\right)$.

Example 2 Differentiate each of the following.

(a) $\left(5x^3-7x+1\right)^5$, $\left[f(x)\right]^5$, $\left[y(x)\right]^5$

(b) $\sin(3-6x)$, $\sin\left(y(x)\right)$

(c) $\mathbf{e}^{x^2-9x}$, $\mathbf{e}^{y(x)}$

Solution

These are written a little differently from what we're used to seeing here. This is because we want to match up these problems with what we'll be doing in this section. Also, each of these parts has several functions to differentiate starting with a specific function followed by a general function. This again, is to help us with some specific parts of the implicit differentiation process that we'll be doing.

(a) $\left(5x^3-7x+1\right)^5$, $\left[f(x)\right]^5$, $\left[y(x)\right]^5$

With the first function here we're being asked to do the following,

$$\frac{d}{dx}\left[\left(5x^3-7x+1\right)^5\right]=5\left(5x^3-7x+1\right)^4\left(15x^2-7\right)$$

and this is just the chain rule. We differentiated the outside function (the exponent of 5) and then multiplied that by the derivative of the inside function (the stuff inside the parenthesis).

For the second function we're going to do basically the same thing. We're going to need to use the chain rule. The outside function is still the exponent of 5 while the inside function this time is simply $f(x)$. We don't have a specific function here, but that doesn't mean that we can't at least write down the chain rule for this function. Here is the derivative for this function,

$$\frac{d}{dx}\left[f(x)\right]^5=5\left[f(x)\right]^4 f'(x)$$

We don't actually know what $f(x)$ is so when we do the derivative of the inside function all we can do is write down notation for the derivative, *i.e.* $f'(x)$.

With the final function here we simply replaced the *f* in the second function with a *y* since most of our work in this section will involve *y*'s instead of *f*'s. Outside of that this function is identical to the second. So, the derivative is,

$$\frac{d}{dx}\left[y(x)\right]^5=5\left[y(x)\right]^4 y'(x)$$

(b) $\sin(3-6x)$, $\sin\left(y(x)\right)$

The first function to differentiate here is just a quick chain rule problem again so here is it's derivative,

$$\frac{d}{dx}\left[\sin\left(3-6x\right)\right]=-6\cos\left(3-6x\right)$$

For the second function we didn't bother this time with using $f\left(x\right)$ and just jumped straight to $y\left(x\right)$ for the general version. This is still just a general version of what we did for the first function. The outside function is still the sine and the inside is given by $y\left(x\right)$ and while we don't have a formula for $y\left(x\right)$ and so we can't actually take its derivative we do have a notation for its derivative. Here is the derivative for this function,

$$\frac{d}{dx}\left[\sin\left(y\left(x\right)\right)\right]=y'\left(x\right)\cos\left(y\left(x\right)\right)$$

(c) $\mathbf{e}^{x^2-9x}$, $\mathbf{e}^{y(x)}$

In this part we'll just give the answers for each and leave out the explanation that we had in the first two parts.

$$\frac{d}{dx}\left(\mathbf{e}^{x^2-9x}\right)=\left(2x-9\right)\mathbf{e}^{x^2-9x} \qquad\qquad \frac{d}{dx}\left(\mathbf{e}^{y(x)}\right)=y'\left(x\right)\mathbf{e}^{y(x)}$$

So, in this set of examples we were just doing some chain rule problems where the inside function was $y\left(x\right)$ instead of a specific function. This kind of derivative shows up all the time in doing implicit differentiation so we need to make sure that we can do them. Also note that we only did this for three kinds of functions but there are many more kinds of functions that we could have used here.

So, it's now time to do our first problem where implicit differentiation is required, unlike the first example where we could actually avoid implicit differentiation by solving for *y*.

Example 3 Find y' for the following function.

$$x^2+y^2=9$$

Solution

Now, this is just a circle and we can solve for *y* which would give,

$$y=\pm\sqrt{9-x^2}$$

Prior to starting this problem, we stated that we had to do implicit differentiation here because we couldn't just solve for *y* and yet that's what we just did. So, why can't we use "normal" differentiation here? The problem is the "$\pm$". With this in the "solution" for *y* we see that *y* is in fact two different functions. Which should we use? Should we use both? We only want a single function for the derivative and at best we have two functions here.

So, in this example we really are going to need to do implicit differentiation so we can avoid this. In this example we'll do the same thing we did in the first example and remind ourselves that *y* is really a function of *x* and write *y* as $y(x)$. Once we've done this all we need to do is differentiate each term with respect to *x*.

$$\frac{d}{dx}\left(x^2+\left[y(x)\right]^2\right)=\frac{d}{dx}(9)$$

As with the first example the right side is easy. The left side is also pretty easy since all we need to do is take the derivative of each term and note that the second term will be similar the part (a) of the second example. All we need to do for the second term is use the chain rule.

After taking the derivative we have,

$$2x+2\left[y(x)\right]^1 y'(x)=0$$

At this point we can drop the (x) part as it was only in the problem to help with the differentiation process. The final step is to simply solve the resulting equation for y'.

$$2x+2yy'=0$$

$$y'=-\frac{x}{y}$$

Unlike the first example we can't just plug in for *y* since we wouldn't know which of the two functions to use. Most answers from implicit differentiation will involve both *x* and *y* so don't get excited about that when it happens.

As always, we can't forget our interpretations of derivatives.

Example 4 Find the equation of the tangent line to

$$x^2+y^2=9$$

at the point $\left(2, \sqrt{5}\right)$.

Solution

First note that unlike all the other tangent line problems we've done in previous sections we need to be given both the *x* and the *y* values of the point. Notice as well that this point does lie on the graph of the circle (you can check by plugging the points into the equation) and so it's okay to talk about the tangent line at this point.

Recall that to write down the tangent line all we need is the slope of the tangent line and this is nothing more than the derivative evaluated at the given point. We've got the derivative from the previous example so all we need to do is plug in the given point.

$$m=\left.y'\right|_{x=2,\,y=\sqrt{5}}=-\frac{2}{\sqrt{5}}$$

The tangent line is then.

$$y = \sqrt{5} - \frac{2}{\sqrt{5}}(x-2)$$

Now, let's work some more examples. In the remaining examples we will no longer write $y(x)$ for *y*. This is just something that we were doing to remind ourselves that *y* is really a function of *x* to help with the derivatives. Seeing the $y(x)$ reminded us that we needed to do the chain rule on that portion of the problem. From this point on we'll leave the *y*'s written as *y*'s and in our head we'll need to remember that they really are $y(x)$ and that we'll need to do the chain rule.

There is an easy way to remember how to do the chain rule in these problems. The chain rule really tells us to differentiate the function as we usually would, except we need to add on a derivative of the inside function. In implicit differentiation this means that every time we are differentiating a term with *y* in it the inside function is the *y* and we will need to add a y' onto the term since that will be the derivative of the inside function.

Let's see a couple of examples.

Example 5 Find y' for each of the following.

(a) $x^3 y^5 + 3x = 8y^3 + 1$

(b) $x^2 \tan(y) + y^{10} \sec(x) = 2x$

(c) $\mathbf{e}^{2x+3y} = x^2 - \ln(xy^3)$

Solution

(a) $x^3 y^5 + 3x = 8y^3 + 1$

First differentiate both sides with respect to *x* and remember that each *y* is really $y(x)$ we just aren't going to write it that way anymore. This means that the first term on the left will be a product rule.

We differentiated these kinds of functions involving *y*'s to a power with the chain rule in the Example 2 above. Also, recall the discussion prior to the start of this problem. When doing this kind of chain rule problem all that we need to do is differentiate the *y*'s as normal and then add on a y', which is nothing more than the derivative of the "inside function".

Here is the differentiation of each side for this function.

$$3x^2 y^5 + 5x^3 y^4 y' + 3 = 24y^2 y'$$

Now all that we need to do is solve for the derivative, y'. This is just basic solving algebra that you are capable of doing. The main problem is that it's liable to be messier than what you're used to doing. All we need to do is get all the terms with y' in them on one side and all the terms without y'

in them on the other. Then factor y' out of all the terms containing it and divide both sides by the "coefficient" of the y'. Here is the solving work for this one,

$$\begin{aligned} 3x^2y^5+3 &= 24y^2y'-5x^3y^4y' \\ 3x^2y^5+3 &= \left(24y^2-5x^3y^4\right)y' \\ y' &= \frac{3x^2y^5+3}{24y^2-5x^3y^4} \end{aligned}$$

The algebra in these problems can be quite messy so be careful with that.

(b) $x^2\tan\left(y\right)+y^{10}\sec\left(x\right)=2x$

We've got two product rules to deal with this time. Here is the derivative of this function.

$$2x\tan\left(y\right)+x^2\sec^2\left(y\right)y'+10y^9y'\sec\left(x\right)+y^{10}\sec\left(x\right)\tan\left(x\right)=2$$

Notice the derivative tacked onto the secant! Again, this is just a chain rule problem similar to the second part of Example 2 above.

Now, solve for the derivative.

$$\begin{aligned} \left(x^2\sec^2\left(y\right)+10y^9\sec\left(x\right)\right)y' &= 2-y^{10}\sec\left(x\right)\tan\left(x\right)-2x\tan\left(y\right) \\ y' &= \frac{2-y^{10}\sec\left(x\right)\tan\left(x\right)-2x\tan\left(y\right)}{x^2\sec^2\left(y\right)+10y^9\sec\left(x\right)} \end{aligned}$$

(c) $\mathbf{e}^{2x+3y}=x^2-\ln\left(xy^3\right)$

We're going to need to be careful with this problem. We've got a couple chain rules that we're going to need to deal with here that are a little different from those that we've dealt with prior to this problem.

In both the exponential and the logarithm we've got a "standard" chain rule in that there is something other than just an *x* or *y* inside the exponential and logarithm. So, this means we'll do the chain rule as usual here and then when we do the derivative of the inside function for each term we'll have to deal with differentiating *y*'s.

Here is the derivative of this equation.

$$\mathbf{e}^{2x+3y}\left(2+3y'\right)=2x-\frac{y^3+3xy^2y'}{xy^3}$$

In both of the chain rules note that the y' didn't get tacked on until we actually differentiated the *y*'s in that term.

Now we need to solve for the derivative and this is liable to be somewhat messy. In order to get the y' on one side we'll need to multiply the exponential through the parenthesis and break up the quotient.

$$\begin{aligned} 2\mathbf{e}^{2x+3y} + 3y'\mathbf{e}^{2x+3y} &= 2x - \frac{y^3}{xy^3} - \frac{3xy^2y'}{xy^3} \\ 2\mathbf{e}^{2x+3y} + 3y'\mathbf{e}^{2x+3y} &= 2x - \frac{1}{x} - \frac{3y'}{y} \\ \left(3\mathbf{e}^{2x+3y} + 3y^{-1}\right)y' &= 2x - x^{-1} - 2\mathbf{e}^{2x+3y} \\ y' &= \frac{2x - x^{-1} - 2\mathbf{e}^{2x+3y}}{3\mathbf{e}^{2x+3y} + 3y^{-1}} \end{aligned}$$

Note that to make the derivative at least look a little nicer we converted all the fractions to negative exponents.

Okay, we've seen one application of implicit differentiation in the tangent line example above. However, there is another application that we will be seeing in every problem in the next section.

In some cases we will have two (or more) functions all of which are functions of a third variable. So, we might have $x(t)$ and $y(t)$, for example and in these cases, we will be differentiating with respect to *t*. This is just implicit differentiation like we did in the previous examples, but there is a difference however.

In the previous examples we have functions involving *x*'s and *y*'s and thinking of *y* as $y(x)$. In these problems we differentiated with respect to *x* and so when faced with *x*'s in the function we differentiated as normal and when faced with *y*'s we differentiated as normal except we then added a y' onto that term because we were really doing a chain rule.

In the new example we want to look at we're assuming that $x = x(t)$ and that $y = y(t)$ and differentiating with respect to *t*. This means that every time we are faced with an *x* or a *y* we'll be doing the chain rule. This in turn means that when we differentiate an *x* we will need to add on an x' and whenever we differentiate a *y* we will add on a y'.

These new types of problems are really the same kind of problem we've been doing in this section. They are just expanded out a little to include more than one function that will require a chain rule.

Let's take a look at an example of this kind of problem.

Example 6 Assume that $x = x(t)$ and $y = y(t)$ and differentiate the following equation with respect to *t*.

$$x^3 y^6 + \mathbf{e}^{1-x} - \cos(5y) = y^2$$

Solution

So, just differentiate as normal and add on an appropriate derivative at each step. Note as well that the first term will be a product rule since both *x* and *y* are functions of *t*.

$$3x^2 x' y^6 + 6x^3 y^5 y' - x' \mathbf{e}^{1-x} + 5y' \sin(5y) = 2yy'$$

There really isn't all that much to this problem. Since there are two derivatives in the problem we won't be bothering to solve for one of them. When we do this kind of problem in the next section the problem will imply which one we need to solve for.

At this point there doesn't seem be any real reason for doing this kind of problem, but as we'll see in the next section every problem that we'll be doing there will involve this kind of implicit differentiation.