Introduction to Matrices for Engineers

1. What is a Matrix?

A matrix is a two dimensional array of numbers or expressions arranged in a set of rows and columns. The matrix **A** has *m* rows and *n* columns and the size of the array is written as m× n, where:

$$\underset{\text{number of rows}\nearrow}{m} \times \underset{\nwarrow\text{number of columns}}{n}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{mn} \end{bmatrix} \begin{matrix} \nwarrow \\ \longleftarrow \text{ rows} \\ \swarrow \end{matrix}$$

$$\nwarrow \quad \nwarrow \quad \nearrow$$
$$\text{columns}$$

A := uppercase denotes a matrix
a := lower case denotes an entry of a matrix.

For examples:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Where A is 2×3 matrix , B is 2×2 matrix .

1.1 Why use Matrices?

We use matrices in mathematics and engineering because often we need to deal with several variables at once e.g. the coordinates of a point in the plane are written (x, y) or in space as (x, y, z) and these are often written as column matrices in the form:

$$\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

It turns out that many operations that are needed to be performed on coordinates of points are linear operations and so can be organized in terms of rectangular arrays of numbers, matrices. Then we find that matrices themselves can under certain conditions be added, subtracted and multiplied so that arises a whole new set of algebraic rules for their manipulation. In general, an $(n \times m)$–matrix A looks like:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1,m-1} & a_{1,m} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2,m-1} & a_{2,m} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3,m-1} & a_{3,m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \dots & a_{n-1,m-1} & a_{n-1,m} \\ a_{n,1} & a_{n,2} & a_{n,3} & \dots & a_{n,m-1} & a_{n,m} \end{pmatrix}$$

Where the element a_{ij}, located in the i^{th} row and the j^{th} column, is a *scalar* quantity; a numerical constant, or a single valued expression.

1.2 Some special Matrices :

Matrices with the same number of rows as columns are called *square matrices* and are of particular importance. So if $m = n$, that is the same number of rows as columns, the matrix is *square*, otherwise it is a *rectangular matrix*. The *diagonal elements* of a square matrix are those elements where the row and column index are the same. For example, the diagonal elements of the n×n matrix **A** are a_{11}, a_{22},… and a_{nn}. The other elements are *non-diagonal* elements. The diagonal elements form the *diagonal* of the matrix:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

and in this case we have:

a) A matrix A is said to be diagonal matrix if $a_{ij} = 0$, $i \neq j$.
b) A diagonal matrix A may be denoted by $diag(d_1, d_2, \dots, d_n)$ where

$$a_{ii} = d_i \quad a_{ij} = 0 \qquad j \neq i.$$

All non-diagonal elements in a matrix are zero for example :

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Two special diagonal matrices are the *identity matrix* and *null matrix*. The identity matrix of dimension *n*, denoted $\mathbf{I}_n$, is the $n\times n$ matrix with 1's on the diagonal and 0's elsewhere as below:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

The *null* matrix **0**, which has the value of zero for all of its elements:

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Note:

If A is a square matrix then I A=A I=A . The basic idea is that if you multiply a matrix by the identity matrix, you get the original matrix. So, in some ways, the identity matrix is for matrices what the number 1 is for scalars.

c) A triangular matrix is a special kind of square matrix. A square matrix is called lower triangular if all the entries above the main diagonal are zero i.e. $a_{ij} = 0$, $i < j$. A matrix of the form below:

$$L = \begin{bmatrix} \ell_{1,1} & & & & 0 \\ \ell_{2,1} & \ell_{2,2} & & & \\ \ell_{3,1} & \ell_{3,2} & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ \ell_{n,1} & \ell_{n,2} & \ldots & \ell_{n,n-1} & \ell_{n,n} \end{bmatrix}$$

is called *a lower triangular matrix or left triangular matrix* .

for example :

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 8 & 0 \\ 4 & 9 & 7 \end{bmatrix}$$

d) A square matrix is called upper triangular if all the entries *below* the main diagonal are zero i.e. $a_{ij} = 0$, $i > j$. A matrix of the form below:

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \ldots & u_{1,n} \\ & u_{2,2} & u_{2,3} & \ldots & u_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & u_{n-1,n} \\ 0 & & & & u_{n,n} \end{bmatrix}$$

is called an *upper triangular matrix* or *right triangular matrix*. The variable *L* (standing for lower or left) is commonly used to represent a lower triangular matrix, while the variable *U* (standing for upper) is commonly used for upper triangular matrix.

For example:

$$\begin{bmatrix} 1 & 4 & 1 \\ 0 & 6 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

e) A square matrix A is called symmetric when the entries of a symmetric matrix are symmetric with respect to the main diagonal. So if the entries are written as $A = (a_{ij})$, then $a_{ij} = a_{ji}$, for all indices *i* and *j*. The following 3 × 3 matrix is symmetric:

$$\begin{bmatrix} 1 & 7 & 3 \\ 7 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$$

1.3 Vectors as Matrices:

Matrices may have any positive number of rows and columns, including one. We have already encountered matrices with one row or one column: vectors! A vector of dimension n can be viewed either as a 1×n matrix or as an n×1 matrix. A 1×n matrix is known as a row vector, and an n×1 matrix is known as a column vector. Row vectors are written horizontally, and column vectors are written vertically. Vector is often used to define the coordinates of a point in a multi-dimensional space. For example a vector having a single row is defined to be a *row vector* :

$$\mathbf{x} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \end{bmatrix}$$

while a vector having a single column is defined to be a *column vector* :

$$\mathbf{y} = \begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{m1} \end{bmatrix}$$

1.4 Transposition

Consider a matrix **A** with dimensions $m \times n$. The *transpose* of **A** (denoted $\mathbf{A}^{\mathbf{T}}$) is the $n \times m$ matrix where the columns are formed from the rows of **A**. In other words, $\mathbf{A}^{\mathbf{T}}{}_{ij} = \mathbf{A}_{ji}$. This “flips” the matrix diagonally.

Example :

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

For vectors, transposition turns row vectors into column vectors and vice versa:

$$\begin{bmatrix} x & y & z \end{bmatrix}^T = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T = \begin{bmatrix} x & y & z \end{bmatrix}$$

Transposition notation is often used to write column vectors inline in a paragraph, like this: $[1, 2, 3]^T$.

For matrices **A**, **B** and scalar *c* we have the following properties of transpose:

1. $(A^T)^T = A$
   The operation of taking the transpose is an involution (self-inverse).

2. $(A+B)^T = A^T + B^T$
   The transpose respects addition.

3. $(AB)^T = B^T A^T$
   Note that the order of the factors reverses. From this one can deduce that a square matrix **A** is invertible if and only if $\mathbf{A}^T$ is invertible.

4. $(c\,A)^T = c\,A^T$
   The transpose of a scalar is the same scalar. This states that the transpose is a linear map from the space of $m \times n$ matrices to the space of all $n \times m$ matrices.

Example:

- $\begin{bmatrix} 1 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

Example :

If $A = \begin{bmatrix} 4 & 2 & 6 \\ 1 & 8 & 7 \end{bmatrix}$ find A^T then evaluate AA^T ?

Solution

$$A^T = \begin{bmatrix} 4 & 1 \\ 2 & 8 \\ 6 & 7 \end{bmatrix} \quad , \quad AA^T = \begin{bmatrix} 4 & 2 & 6 \\ 1 & 8 & 7 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 8 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 56 & 62 \\ 62 & 114 \end{bmatrix}$$

Notes:

1) If a square matrix A and its transpose AT are identical, then A is said to be a symmetric matrix.

Example if $A = \begin{bmatrix} 5 & -4 & 2 \\ -4 & 6 & 9 \\ 2 & 9 & 13 \end{bmatrix}$ then $A^T = \begin{bmatrix} 5 & -4 & 2 \\ -4 & 6 & 9 \\ 2 & 9 & 13 \end{bmatrix}$

So, A is a symmetric matrix " symmetrical about its main diagonal".

2) If a square matrix A is such that $A^T = -A$ then A is said to be skew symmetric.

Example :

Let $A = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}$, $A^T = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix} = -A$

Hence A is skew symmetric .

H.W

1) If $A = \begin{bmatrix} 3 & 1 \\ 2 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 3 & 8 \end{bmatrix}$

a) Find A^T, B^T and $(AB)^T$.

b) Deduce that $(AB)^T = B^T A^T$.

2) If $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ -1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -7 & 0 \\ 0 & 2 & 5 \\ 3 & 4 & 5 \end{bmatrix}$ find $(AB)^T$, $A^T B^T$

3) If $A = \begin{bmatrix} 1 & 13 \\ 15 & 7 \end{bmatrix}$ find A^T , $(A^T)^T$, show that $(A^T)^T = A$.

2. Elementary Matrix Arithmetic

2.1 Matrix Addition:

The operation of addition of two matrices is only defined when both matrices have the same dimensions. If **A** and **B** are both ($m \times n$), then the sum :

$$\mathrm{C} = \mathrm{A} + \mathrm{B}$$

is also ($m \times n$) and is defined to have each element the sum of the corresponding elements of **A** and **B**, thus :

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

Matrix addition is both associative, that is:

$$\mathrm{A} + (\mathrm{B} + \mathrm{C}) = (\mathrm{A} + \mathrm{B}) + \mathrm{C}$$

and commutative:

$$\mathrm{A} + \mathrm{B} = \mathrm{B} + \mathrm{A}$$

The subtraction of two matrices is similarly defined; if **A** and **B** have the same dimensions, then the difference is :

$$\mathrm{C} = \mathrm{A} - \mathrm{B}$$

Examples: consider two metrices A and B find A+B ? , A-B?

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 & 0 \\ 7 & 5 \\ 2 & 1 \end{bmatrix}$$

A+B=

$$\begin{bmatrix} 1 & 3 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 7 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 3+0 \\ 1+7 & 0+5 \\ 1+2 & 2+1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 8 & 5 \\ 3 & 3 \end{bmatrix}$$

A-B=

$$\begin{bmatrix} 1 & 3 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 7 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1-0 & 3-0 \\ 1-7 & 0-5 \\ 1-2 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -6 & -5 \\ -1 & 1 \end{bmatrix}$$

2.2 **Matrix multiplication**:

2.2.1 **Multiplication of a Matrix by a Scalar Quantity**

If **A** is a matrix and *k* is a scalar quantity, the product **B** = k**A** is defined to be the matrix of the same dimensions as **A** whose elements are simply all scaled by the constant *k*,

$$b_{ij} = k \times a_{ij}$$

Example:

If $A=\begin{bmatrix} -1 & 2 & 0 \\ 4 & 1 & -2 \end{bmatrix}$ find 3A?

Solution:

$$3A = 3\begin{bmatrix} -1 & 2 & 0 \\ 4 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 0 \\ 12 & 3 & -6 \end{bmatrix}$$

2.2.2 **Multiplication of Matrices:**

Two matrices may be multiplied together only if they meet conditions on their dimensions that allow them to *conform*. Let **A** have dimensions $m \times n$, and **B** be $n \times p$, that is **A** has the same number as columns as the number of rows in **B**, then the product:

$$\mathrm{C} = \mathrm{AB}$$

is defined to be an $m \times p$ matrix with elements :

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

The element in position *ij* is the sum of the products of elements in the i^{th} row of the first matrix (**A**) and the corresponding elements in the j^{th} column of the second matrix (**B**). Notice that the product **AB** is not defined unless the above condition is satisfied, that is the number of columns of the first matrix must equal the number of rows in the second, that is:

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

For example, below we show how to compute *c*24:

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \end{bmatrix}$$

$$c_{24} = a_{21}b_{14} + a_{22}b_{24}$$

Matrix multiplication is associative, that is:

$$\mathbf{A}\,(\mathbf{BC}) = (\mathbf{AB})\,\mathbf{C}$$

but is not commutative in general that is :

$$\mathbf{AB} \neq \mathbf{BA}$$

In fact unless the two matrices are square, reversing the order in the product will cause the matrices to be non-conformal. The order of the terms in the product is therefore very important. In general:

Distributive	$A(B+C) = AB + AC$
Distributive	$(A+B)C = AC + BC$
Associative	$(AB)C = A(BC)$
Non–Commutative	$AB \neq BA$ (usually)
Moving Constants	$A(\lambda B) = \lambda(AB)$

Examples:

$$\text{Let } A=\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix},\quad B=\begin{pmatrix} -1 & 4 \\ 1 & -2 \end{pmatrix},\quad C=\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Consider the following:

$$AB=\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} -1 & 4 \\ 1 & -2 \end{pmatrix}=\begin{pmatrix} 1.-1+0.1 & 1.4+0.-2 \\ 3.-1+2.1 & 3.4+2.-2 \end{pmatrix}=\begin{pmatrix} -1 & 4 \\ -1 & 8 \end{pmatrix}$$

$$BA=\begin{pmatrix} -1 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}=\begin{pmatrix} -1.1+4.3 & -1.0+4.2 \\ 1.1+-2.3 & 1.0+-2.2 \end{pmatrix}=\begin{pmatrix} 11 & 8 \\ -5 & -4 \end{pmatrix}$$

Now, AB ≠ BA, and we see that two matrices are not the same if they are multiplied the other way around. Also consider:

$$AC=\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}=\begin{pmatrix} 2 \\ 12 \end{pmatrix}$$

$$BC=\begin{pmatrix} -1 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}=\begin{pmatrix} 10 \\ -4 \end{pmatrix}$$

$$AB+BC=\begin{pmatrix} 2 \\ 12 \end{pmatrix}+\begin{pmatrix} 10 \\ -4 \end{pmatrix}=\begin{pmatrix} 12 \\ 8 \end{pmatrix}$$

$$A+B=\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}+\begin{pmatrix} -1 & 4 \\ 1 & -2 \end{pmatrix}=\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$$

$$(A+B)C=\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}=\begin{pmatrix} 12 \\ 8 \end{pmatrix}$$

Example:

$$\text{Let } A = \begin{pmatrix} 1 & 7 \\ 3 & 2 \end{pmatrix}, \quad I = I_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$AI = \begin{pmatrix} 1 & 7 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1.1+7.0 & 1.0+7.1 \\ 3.1+2.0 & 3.0+2.1 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 3 & 2 \end{pmatrix}$$

$$IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 7 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1.1+0.3 & 1.7+0.2 \\ 0.1+1.3 & 0.7+1.2 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 3 & 2 \end{pmatrix}$$

H.W 1:

Given $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ -1 & 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 & 9 \\ 1 & 0 & 0 \\ 3 & -2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 1 & 5 \\ 2 & 3 & 1 \\ 0 & 1 & 5 \end{bmatrix}$

find BC, A(BC), AB and (AB) C

H.W2:

if $A = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} -7 & 1 \\ 0 & 4 \end{bmatrix}$, $D = [3, 2, 1]$

and $E = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & -1 \end{bmatrix}$. find, if possible,

(a) A+B, C−A and D−E

(b) AB, BA, CA, AC, DA, DB, BD, EB, BE and AE

(c) 7C, −3D and kE, where k is a scalar.

3. The matrix determinant

One of the most important properties of square matrices is the determinant. This is a number obtained from the entries. When written using the elements of the matrix, the determinant is enclosed between vertical bars. Some examples about matrices and how calculates their determinants :

3.1 Determinant of a (2 × 2)–Matrix:

Let A= $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the determinant of A, denoted det(A) or |A| is given by :

$$\det(A) = ad - bc$$

Example :

1) Given the following matrix A, specify the determinant det(A)?

$$A = \begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}$$

2) Calculate the determinant of the following 2×2 matrices :

$$\det\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} = 2.5 - 3.1 = 10 - 3 = 7$$

$$\det\begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix} = (-1)(-6) - 2.3 = 6 - 6 = 0$$

3) Calculate the determinant of the following 2×2 matrices:

$$a. \begin{pmatrix} 1 & 3 \\ 5 & -2 \end{pmatrix} \qquad b. \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$$

$$c. \begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix} \qquad d. \begin{pmatrix} 4 & -3 \\ 1 & 2 \end{pmatrix}$$

Before we go on to larger matrices, we need to define few concepts…

3.1.1 Minors

Let A be the (n × m) matrix :

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1,m-1} & a_{1,m} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2,m-1} & a_{2,m} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3,m-1} & a_{3,m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \dots & a_{n-1,m-1} & a_{n-1,m} \\ a_{n,1} & a_{n,2} & a_{n,3} & \dots & a_{n,m-1} & a_{n,m} \end{pmatrix}$$

Then, if (m=n) the minor m_{ij} , for each i, j, is the determinant of the (n − 1 × n − 1)–matrix obtained by deleting the i^{th} row and the j^{th} column. For example, in the below notation, the minor m_{11} is the determinant of the matrix obtained by eliminating the first row and the first column :

$$m_{11} = \det \begin{pmatrix} a_{22} & a_{23} & \dots & a_{2,m-1} & a_{2,m} \\ a_{32} & a_{33} & \dots & a_{3,m-1} & a_{3,m} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-1,2} & a_{n-1,3} & \dots & a_{n-1,m-1} & a_{n-1,m} \\ a_{n,2} & a_{n,3} & \dots & a_{n,m-1} & a_{n,m} \end{pmatrix}$$

The same for m_{21} :

$$m_{21} = \det \begin{pmatrix} a_{12} & a_{13} & \dots & a_{1,m-1} & a_{1,m} \\ a_{32} & a_{33} & \dots & a_{3,m-1} & a_{3,m} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-1,2} & a_{n-1,3} & \dots & a_{n-1,m-1} & a_{n-1,m} \\ a_{n,2} & a_{n,3} & \dots & a_{n,m-1} & a_{n,m} \end{pmatrix}$$

Example :

Compute all the minors of matrix A ?

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 4 & 3 \\ -5 & 0 & -2 \end{pmatrix}$$

Solution :

$$m_{11} = \begin{vmatrix} 4 & 3 \\ 0 & -2 \end{vmatrix} = -8 \qquad m_{12} = \begin{vmatrix} 0 & 3 \\ -5 & -2 \end{vmatrix} = 15 \qquad m_{13} = \begin{vmatrix} 0 & 4 \\ -5 & 0 \end{vmatrix} = 20$$

$$m_{21} = \begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix} = -2 \qquad m_{22} = \begin{vmatrix} 2 & -1 \\ -5 & -2 \end{vmatrix} = -9 \qquad m_{23} = \begin{vmatrix} 2 & 1 \\ -5 & 0 \end{vmatrix} = 5$$

$$m_{31} = \begin{vmatrix} 1 & -1 \\ 4 & 3 \end{vmatrix} = 7 \qquad m_{32} = \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} = 6 \qquad m_{33} = \begin{vmatrix} 2 & 1 \\ 0 & 4 \end{vmatrix} = 8$$

3.1.2 Minors and Cofactors

The numbers called 'cofactors' are almost the same as minors, except some have a minus sign in accordance with the following pattern:

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

The cofactor and the minor alwayes have the same numerical value, with the possible exception of their sign. In general, the cofactor of a matrix is defined by the relation :

$$C_{ij} = (-1)^{i+j}\,\mathrm{m}_{ij}$$

To illustrate these definitions, consider the following 3 by 3 matrix,

$$\begin{bmatrix} 1 & 4 & 7 \\ 3 & 0 & 5 \\ -1 & 9 & 11 \end{bmatrix}$$

To compute the minor $m_{2,3}$ and the cofactor $C_{2,3}$, we find the determinant of the above matrix with row 2 and column 3 removed, $m_{2,3}$ =

$$= \det \begin{bmatrix} 1 & 4 & \square \\ \square & \square & \square \\ -1 & 9 & \square \end{bmatrix} = \det \begin{bmatrix} 1 & 4 \\ -1 & 9 \end{bmatrix} = (9 - (-4)) = 13$$

So the cofactor of the (2,3) entry is :

$$C_{2,3} = (-1)^{2+3}(m_{2,3}) = -13$$

The cofactors from the previous example for matrix A are:

$$\begin{array}{lll} c_{11} = m_{11} = -8 & c_{12} = -m_{12} = -15 & c_{13} = m_{13} = 20 \\ c_{21} = -m_{21} = 2 & c_{22} = m_{22} = -9 & c_{23} = -m_{23} = -5 \\ c_{31} = m_{31} = 7 & c_{32} = -m_{32} = -6 & c_{33} = m_{33} = 8 \end{array}$$

Cofactor expansion of the determinant:

The cofactors feature prominently for the expansion of determinants, which is a method of computing larger determinants in terms of smaller ones. Given the n×n matrix (a_{ij}) , the determinant of *A* (denoted det(*A*)) can be written as the sum of the cofactors of any row or column of the matrix multiplied by the entries that generated them. In other words, the cofactor expansion along the j^{th} column gives:

$$\det(\mathbf{A}) = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j} + \ldots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$$

The cofactor expansion along the i^{th} row gives:

$$\det(\mathbf{A}) = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} + \ldots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

3.2 Determinant of a (3 × 3)–Matrix

In order to calculate the determinant of a (3×3)–matrix, choose any row or column. Then, multiply each entry by its corresponding cofactor, and add the three products. This gives the determinant. In another word the determinant of a high order matrix is found by expanding in terms of the elements of any selected row or column and the cofactors of those elements. For 3×3 matrix if the expansion is done by selecting the first row, the determinant is defined as a sum of order (n − 1) determinants as :

$$det\ A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Similarly, the expansion in terms of the elements of the second column of the determinant is :

$$det\ A = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$$

Example :

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 4 & 3 \\ -5 & 0 & -2 \end{pmatrix}$$

Compute the determinant for matrix A?
Solution :

We compute the determinant using the top row:

$$\det A = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} = 2.(-8) + 1.(-15) + (-1).20 = -16 - 15 - 20 = -51$$

Suppose, we use the second column instead:

$$\det A = a_{12}c_{12} + a_{22}c_{22} + a_{32}c_{32} = 1.(-15) + 4.(-9) + 0.(-6) = -15 - 36 - 0 = -51$$

It doesn’t matter which row or column is used, but the top row is normal. Note that it is not necessary to work out all the minors (or cofactors), just three.

Example :

Find the determinant of matrix A?

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 2 & 0 & -2 \end{pmatrix}$$

Solution :

Choose any row or column; let us choose the first row. Multiply each element in first row with its cofactor.

$$C_{11} = (-1)^{1+1} M_{11} = 1\begin{vmatrix} 0 & 2 \\ 0 & -2 \end{vmatrix} = 1(0.(-2) - 2.0) = 0$$

$$C_{12} = (-1)^{1+2} M_{12} = (-1)\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = 1(1 \times (-2) - 2 \times 2) = 6$$

$$C_{13} = (-1)^{1+3} M_{13} = 1\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} = 1(1 \times (0) - 2 \times 0) = 0$$

Finally, calculate det (A) …

$$det\ A = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$$

$$det\ A = 2 \times 0 + 6 \times 1 + 3 \times 0 = 6$$

H.W
Find the determinant of matrix B?

$$B = \begin{pmatrix} 1 & 0 & 4 \\ -2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

3.3 Determinant of an (n × n)–Matrix

The procedure for larger matrices is exactly the same as for a (3 × 3)–matrix: choose a row or column, multiply the entry by the corresponding cofactor, and add them up. But of course each minor is itself the determinant of an (n−1×n−1)–matrix, so for example, in a (4×4) determinant, it is first necessary to do four (3 × 3) determinants.

Example :

Calculate the determinant of matrix A?

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ -1 & 0 & 3 & 1 \\ 3 & 1 & 2 & 0 \end{pmatrix}$$

Solution

We will proceed to a cofactor expansion along the fourth column which means that

$$det\ A = a_{14}C_{14} + a_{24}C_{24} + a_{34}C_{34} + a_{44}C_{44}$$

$$C_{24} = (-1)^{2+4}M_{24} = 1\begin{vmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$

$$C_{34} = (-1)^{3+4}M_{34} = -1\begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

$$C_{24} = 18\ , \quad C_{34} = -2$$

$$det\ A = 0 \times C_{14} + 1 \times 18 + 1 \times (-2) + 0 \times C_{44} = 16$$

H.W

Show that the determinant of A in the previous example is 16 by a cofactor expansion along the :

1) The first row

2) The third column

4 Inverse of a matrix

Let A be an n×n matrix, and let I be the n×n identity matrix. Sometimes, there exists a matrix A^{-1} (called the inverse of A) with the property:

$$A\,A^{-1} = I = A^{-1}A$$

In this section, we demonstrate a method for finding inverses:

4.1 Inverse of a 2 × 2 Matrix

Let A be a 2×2 matrix then the inverse of A, A^{-1} is given by:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

To check , we multiply:

$$A^{-1}A = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} da-bc & db-bd \\ -ca+ac & -cb+ad \end{pmatrix}$$
$$= \frac{1}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In a similar fashion we could show that $A\,A^{-1} = I$. Of course, the inverse could also be written in another form:

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Note, that if det A = 0, then we have a division by zero, which we can't do. In this situation there is no inverse of A.

4.2 Inverse of 3 × 3 (and higher) Matrices

Recall the definition of a minor from Section 3.1.1: given an (n × n)–matrix A, the minor m_{ij} is the determinant of the (n−1×n−1)–matrix obtained by omitting the i^{th} row and the j^{th} column.

Example

Calculate the inverse of a (3 × 3)–matrix A

$$A = \begin{pmatrix} 1 & 0 & 4 \\ -2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

Find the minors

$$m_{11} = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1 \qquad m_{12} = \begin{vmatrix} -2 & 0 \\ 3 & 1 \end{vmatrix} = -2 \qquad m_{13} = \begin{vmatrix} -2 & 1 \\ 3 & 2 \end{vmatrix} = -7$$

$$m_{21} = \begin{vmatrix} 0 & 4 \\ 2 & 1 \end{vmatrix} = -8 \qquad m_{22} = \begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix} = -11 \qquad m_{23} = \begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} = 2$$

$$m_{31} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4 \qquad m_{32} = \begin{vmatrix} 1 & 4 \\ -2 & 0 \end{vmatrix} = 8 \qquad m_{33} = \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} = 1$$

Recall also the pattern of + and − signs from which we obtain the cofactors:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Now, we put the minors into a matrix and change their signs according to the pattern to get the matrix of cofactors:

$$\begin{pmatrix} 1 & 2 & -7 \\ 8 & -11 & -2 \\ -4 & -8 & 1 \end{pmatrix}$$

The next stage is take the transpose:

$$\begin{pmatrix} 1 & 8 & -4 \\ 2 & -11 & -8 \\ -7 & -2 & 1 \end{pmatrix}$$

and finally we must divide by the determinant, which is det A = -27 to calculate the inverse of matrix A:

$$A^{-1} = \frac{1}{-27} \begin{pmatrix} 1 & 8 & -4 \\ 2 & -11 & -8 \\ -7 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{27} & -\frac{8}{27} & \frac{4}{27} \\ -\frac{2}{27} & \frac{11}{27} & \frac{8}{27} \\ \frac{7}{27} & \frac{2}{27} & -\frac{1}{27} \end{pmatrix}$$

To check the result, the following condition must satisfy;

$$A\,A^{-1} = I = A^{-1}A$$

$$A^{-1}A = -\frac{1}{27}\begin{pmatrix} 1 & 8 & -4 \\ 2 & -11 & -8 \\ -7 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 4 \\ -2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1-16-12 & 0+8-8 & 4+0-4 \\ 2+22-24 & 0-11-16 & 8+0-8 \\ -7+4+3 & 0-2+2 & -28+0+1 \end{pmatrix}$$

$$= -\frac{1}{27}\begin{pmatrix} -27 & 0 & 0 \\ 0 & -27 & 0 \\ 0 & 0 & -27 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The same procedure works for (n × n)–matrices.

(I) Work out the minors.
(II) Put in the − signs to form the cofactors.
(III) Take the transpose.
(IV) Divide by the determinant.

Note:

Furthermore, an n × n matrix has an inverse if and only if the determinant is not zero. So, it’s a good idea to calculate the determinant first, just to check whether the rest of the procedure is necessary.

5 Linear Systems

5.1 Finding inverse matrix

In this section, we discuss one very important application of finding inverses of matrices. Often, when solving problems in mathematics, we need to solve simultaneous equations, e.g. something like:

$$\begin{array}{cccc} 2x & + & y & = & 3 \\ 5x & + & 3y & = & 7 \end{array}$$

From which we would obtain x = 2 and y = −1. The process we have used up until now is a little messy: we combine the equations to try and eliminate one of the unknown variables. There is a more systematic way using matrices. We can write the equations in a slightly different way:

$$\begin{pmatrix} 2x+y \\ 5x+3y \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

Now we can check that the first matrix is equal to the product:

$$\begin{pmatrix} 2x+y \\ 5x+3y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The next stage is to use the inverse of the (2 × 2)–matrix, so let's calculate that now.

$$\text{Let } A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}, \text{ then } A^{-1} = \frac{1}{2.3-1.5} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}.$$

Now, we take the matrix equation above, and multiply by A^{-1}

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

Then, doing the multiplication:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

and so x = 2 and y = −1, as required.

For larger systems:

The same thing works with 3 equations in x, y and z. Suppose we have :

$$\begin{array}{ccccccc} x & + & 2y & + & 2z & = & -1 \\ & & 3y & - & 2z & = & 2 \\ 2x & - & y & + & 8z & = & 7 \end{array}$$

Then, the matrix form is :

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & 3 & -2 \\ 2 & -1 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 7 \end{pmatrix}$$

Now, we denote the 3 × 3 matrix by A, and calculate the inverse of A. The minors are as follows:

$$m_{11} = \begin{vmatrix} 3 & -2 \\ -1 & 8 \end{vmatrix} = 22 \qquad m_{12} = \begin{vmatrix} 0 & -2 \\ 2 & 8 \end{vmatrix} = 4 \qquad m_{13} = \begin{vmatrix} 0 & 3 \\ 2 & -1 \end{vmatrix} = -6$$

$$m_{21} = \begin{vmatrix} 2 & 2 \\ -1 & 8 \end{vmatrix} = 18 \qquad m_{22} = \begin{vmatrix} 1 & 2 \\ 2 & 8 \end{vmatrix} = 4 \qquad m_{23} = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5$$

$$m_{31} = \begin{vmatrix} 2 & 2 \\ 3 & -2 \end{vmatrix} = -10 \qquad m_{32} = \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = -2 \qquad m_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 3$$

The following matrix of cofactors:

$$\begin{pmatrix} 22 & -4 & -6 \\ -18 & 4 & 5 \\ -10 & 2 & 3 \end{pmatrix}$$

Calculate the determinant by taking any row or column, let's choose the top row:

$$\det A = 1.22 + 2. - 4 + 2. - 6 = 22 - 8 - 12 = 2$$

The determinant is 2, so we will be able to find the inverse. From the matrix of cofactors, we take the transpose, and then divide by the determinant to get A^{-1}:

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 22 & -18 & -10 \\ -4 & 4 & 2 \\ -6 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 11 & -9 & -5 \\ -2 & 2 & 1 \\ -3 & \frac{5}{2} & \frac{3}{2} \end{pmatrix}$$

Return to solving the simultaneous equations, where we had:

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & 3 & -2 \\ 2 & -1 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 7 \end{pmatrix}$$

Multiplying both sides on the left by A^{-1}, we have:

$$\begin{pmatrix} 11 & -9 & -5 \\ -2 & 2 & 1 \\ -3 & \frac{5}{2} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ 0 & 3 & -2 \\ 2 & -1 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 & -9 & -5 \\ -2 & 2 & 1 \\ -3 & \frac{5}{2} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11.-1+-9.2+-5.7 \\ -2.-1+2.2+1.7 \\ -3.-1+\frac{5}{2}.2+\frac{3}{2}.7 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -64 \\ 13 \\ \frac{37}{2} \end{pmatrix}$$

So, we get the values of x,y,z .

5.2 Cramer's Method

Cramer's method is a convenient method for manually solving low-order non-homogeneous sets of linear equations. If the equations are written in matrix form **:**

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

Then the i^{th} element of the vector **x** may be found directly from a ratio of determinants

$$x_i = \frac{\det \mathbf{A}_{(i)}}{\det \mathbf{A}}$$

where A(i) is the matrix formed by replacing the i^{th} column of A with the column vector b. For example, solve the following equations:

$$\begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & 2x_3 & = & 2 \\ x_1 & + & 10x_2 & - & 3x_3 & = & 5 \\ -x_1 & + & x_2 & + & x_3 & = & -3 \end{array} .$$

Then

$$\det \mathbf{A} = \begin{vmatrix} 2 & -1 & 2 \\ 1 & 10 & -3 \\ -1 & 1 & 1 \end{vmatrix} = 46$$

$$\begin{aligned} x_1 &= \frac{1}{46} \begin{vmatrix} 2 & -1 & 2 \\ 5 & 10 & -3 \\ -3 & 1 & 1 \end{vmatrix} \\ &= 2 \end{aligned}$$

$$\begin{aligned} x_2 &= \frac{1}{46} \begin{vmatrix} 2 & 2 & 2 \\ 1 & 5 & -3 \\ -1 & -3 & 1 \end{vmatrix} \\ &= 0 \end{aligned}$$

$$\begin{aligned} x_3 &= \frac{1}{46} \begin{vmatrix} 2 & -1 & 2 \\ 1 & 10 & 5 \\ -1 & 1 & -3 \end{vmatrix} \\ &= -1 \end{aligned}$$