Special Functions

special functions are particular mathematical functions which have more or less established names and notations due to their importance in mathimatical analysis, functional analysis, physics, or other applications. There is no general formal difinition but the list of mathematical functions contains functions which are accepted as special.

1- Error Function : The error function, also called the Gauss error function or probability integral, is a special or non-elementary function of sigmoid shap which occors in probability, statistics, material science and partial differential equations ; it is defined as

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad \text{— [1]}$$

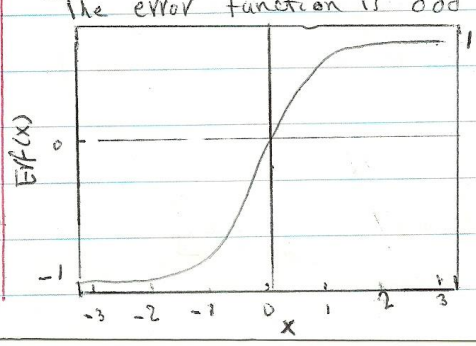
, and the Complementary error function, denoted erfc, is defined as

$$erfc(x) = 1 - erf(x)$$
$$= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

The Complex error function, denoted W(x), and also known as Faddeeva function is defined is

[1] $$W(x) = e^{-x^2} erfc(-jx)$$

The error function is odd i.e $erf(-z) = -erf(z)$



Taylor Series: The error function is an entire function, it has no singularities (except that at infinity), and its Taylor expansion always converges.

The defining Integral cannot be evaluated in closed form in terms of elementary functions. but by expanding the integrand e^{-z^2}, into its Taylor series and integrating term by term, one obtain the error function's Taylor series as

$$erf(z) = \frac{2}{\sqrt{\pi}}\left(z - \frac{z^3}{3} + \frac{z^5}{10} - \frac{z^7}{42} + \cdots\right.$$

$$= \frac{2}{\sqrt{\pi}}\left(\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!\,(2n+1)}\right)$$

Results:

$erf(0) = 0 \qquad erf(\infty) = 1$

The tabulated values of error function and Complementary error are given in Tables 1 and 2

In Petroleum Engineering the error function appears in many field such as Miscible displacement field and Reservoir stimulation problems.

1- Miscible Displacement problems; for more detailed see Brigham W.D. et.al " Experimental of Mixing Displacement in porous media" SPEJ Vol. 1 Nov. 1 P.P. 1 – 8

2- Calculations Related to Hydraulic Fracturing

The following equation is used to calculate the area of one face of fracture

$$A(t) = \frac{q_i\, w}{4\pi C^2}\left[e^{x^2} erf(x) + \frac{2x}{\sqrt{\pi}} - 1\right]$$

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Table - 1

Error Function Table

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

	Hundredths digit of x									
x	0	1	2	3	4	5	6	7	8	9
0.0	0.00000	0.01128	0.02256	0.03384	0.04511	0.05637	0.06762	0.07886	0.09008	0.10128
0.1	0.11246	0.12362	0.13476	0.14587	0.15695	0.16800	0.17901	0.18999	0.20094	0.21184
0.2	0.22270	0.23352	0.24430	0.25502	0.26570	0.27633	0.28690	0.29742	0.30788	0.31828
0.3	0.32863	0.33891	0.34913	0.35928	0.36936	0.37938	0.38933	0.39921	0.40901	0.41874
0.4	0.42839	0.43797	0.44747	0.45689	0.46623	0.47548	0.48466	0.49375	0.50275	0.51167
0.5	0.52050	0.52924	0.53790	0.54646	0.55494	0.56332	0.57162	0.57982	0.58792	0.59594
0.6	0.60386	0.61168	0.61941	0.62705	0.63459	0.64203	0.64938	0.65663	0.66378	0.67084
0.7	0.67780	0.68467	0.69143	0.69810	0.70468	0.71116	0.71754	0.72382	0.73001	0.73610
0.8	0.74210	0.74800	0.75381	0.75952	0.76514	0.77067	0.77610	0.78144	0.78669	0.79184
0.9	0.79691	0.80188	0.80677	0.81156	0.81627	0.82089	0.82542	0.82987	0.83423	0.83851
1.0	0.84270	0.84681	0.85084	0.85478	0.85865	0.86244	0.86614	0.86977	0.87333	0.87680
1.1	0.88021	0.88353	0.88679	0.88997	0.89308	0.89612	0.89910	0.90200	0.90484	0.90761
1.2	0.91031	0.91296	0.91553	0.91805	0.92051	0.92290	0.92524	0.92751	0.92973	0.93190
1.3	0.93401	0.93606	0.93807	0.94002	0.94191	0.94376	0.94556	0.94731	0.94902	0.95067
1.4	0.95229	0.95385	0.95538	0.95686	0.95830	0.95970	0.96105	0.96237	0.96365	0.96490
1.5	0.96611	0.96728	0.96841	0.96952	0.97059	0.97162	0.97263	0.97360	0.97455	0.97546
1.6	0.97635	0.97721	0.97804	0.97884	0.97962	0.98038	0.98110	0.98181	0.98249	0.98315
1.7	0.98379	0.98441	0.98500	0.98558	0.98613	0.98667	0.98719	0.98769	0.98817	0.98864
1.8	0.98909	0.98952	0.98994	0.99035	0.99074	0.99111	0.99147	0.99182	0.99216	0.99248
1.9	0.99279	0.99309	0.99338	0.99366	0.99392	0.99418	0.99443	0.99466	0.99489	0.99511
2.0	0.99532	0.99552	0.99572	0.99591	0.99609	0.99626	0.99642	0.99658	0.99673	0.99688
2.1	0.99702	0.99715	0.99728	0.99741	0.99753	0.99764	0.99775	0.99785	0.99795	0.99805
2.2	0.99814	0.99822	0.99831	0.99839	0.99846	0.99854	0.99861	0.99867	0.99874	0.99880
2.3	0.99886	0.99891	0.99897	0.99902	0.99906	0.99911	0.99915	0.99920	0.99924	0.99928
2.4	0.99931	0.99935	0.99938	0.99941	0.99944	0.99947	0.99950	0.99952	0.99955	0.99957
2.5	0.99959	0.99961	0.99963	0.99965	0.99967	0.99969	0.99971	0.99972	0.99974	0.99975
2.6	0.99976	0.99978	0.99979	0.99980	0.99981	0.99982	0.99983	0.99984	0.99985	0.99986
2.7	0.99987	0.99987	0.99988	0.99989	0.99989	0.99990	0.99991	0.99991	0.99992	0.99992
2.8	0.99992	0.99993	0.99993	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995	0.99996
2.9	0.99996	0.99996	0.99996	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998
3.0	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999	0.99999
3.1	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999
3.2	0.99999	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

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Table Two

Complementary Error Function Table

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt$$

	Hundredths digit of x									
x	0	1	2	3	4	5	6	7	8	9
0.0	1.00000	0.98872	0.97744	0.96616	0.95489	0.94363	0.93238	0.92114	0.90992	0.89872
0.1	0.88754	0.87638	0.86524	0.85413	0.84305	0.83200	0.82099	0.81001	0.79906	0.78816
0.2	0.77730	0.76648	0.75570	0.74498	0.73430	0.72367	0.71310	0.70258	0.69212	0.68172
0.3	0.67137	0.66109	0.65087	0.64072	0.63064	0.62062	0.61067	0.60079	0.59099	0.58126
0.4	0.57161	0.56203	0.55253	0.54311	0.53377	0.52452	0.51534	0.50625	0.49725	0.48833
0.5	0.47950	0.47076	0.46210	0.45354	0.44506	0.43668	0.42838	0.42018	0.41208	0.40406
0.6	0.39614	0.38832	0.38059	0.37295	0.36541	0.35797	0.35062	0.34337	0.33622	0.32916
0.7	0.32220	0.31533	0.30857	0.30190	0.29532	0.28884	0.28246	0.27618	0.26999	0.26390
0.8	0.25790	0.25200	0.24619	0.24048	0.23486	0.22933	0.22390	0.21856	0.21331	0.20816
0.9	0.20309	0.19812	0.19323	0.18844	0.18373	0.17911	0.17458	0.17013	0.16577	0.16149
1.0	0.15730	0.15319	0.14916	0.14522	0.14135	0.13756	0.13386	0.13023	0.12667	0.12320
1.1	0.11979	0.11647	0.11321	0.11003	0.10692	0.10388	0.10090	0.09800	0.09516	0.09239
1.2	0.08969	0.08704	0.08447	0.08195	0.07949	0.07710	0.07476	0.07249	0.07027	0.06810
1.3	0.06599	0.06394	0.06193	0.05998	0.05809	0.05624	0.05444	0.05269	0.05098	0.04933
1.4	0.04771	0.04615	0.04462	0.04314	0.04170	0.04030	0.03895	0.03763	0.03635	0.03510
1.5	0.03389	0.03272	0.03159	0.03048	0.02941	0.02838	0.02737	0.02640	0.02545	0.02454
1.6	0.02365	0.02279	0.02196	0.02116	0.02038	0.01962	0.01890	0.01819	0.01751	0.01685
1.7	0.01621	0.01559	0.01500	0.01442	0.01387	0.01333	0.01281	0.01231	0.01183	0.01136
1.8	0.01091	0.01048	0.01006	0.00965	0.00926	0.00889	0.00853	0.00818	0.00784	0.00752
1.9	0.00721	0.00691	0.00662	0.00634	0.00608	0.00582	0.00557	0.00534	0.00511	0.00489
2.0	0.00468	0.00448	0.00428	0.00409	0.00391	0.00374	0.00358	0.00342	0.00327	0.00312
2.1	0.00298	0.00285	0.00272	0.00259	0.00247	0.00236	0.00225	0.00215	0.00205	0.00195
2.2	0.00186	0.00178	0.00169	0.00161	0.00154	0.00146	0.00139	0.00133	0.00126	0.00120
2.3	0.00114	0.00109	0.00103	0.00098	0.00094	0.00089	0.00085	0.00080	0.00076	0.00072
2.4	0.00069	0.00065	0.00062	0.00059	0.00056	0.00053	0.00050	0.00048	0.00045	0.00043
2.5	0.00041	0.00039	0.00037	0.00035	0.00033	0.00031	0.00029	0.00028	0.00026	0.00025
2.6	0.00024	0.00022	0.00021	0.00020	0.00019	0.00018	0.00017	0.00016	0.00015	0.00014
2.7	0.00013	0.00013	0.00012	0.00011	0.00011	0.00010	0.00009	0.00009	0.00008	0.00008
2.8	0.00008	0.00007	0.00007	0.00006	0.00006	0.00006	0.00005	0.00005	0.00005	0.00004
2.9	0.00004	0.00004	0.00004	0.00003	0.00003	0.00003	0.00003	0.00003	0.00003	0.00002
3.0	0.00002	0.00002	0.00002	0.00002	0.00002	0.00002	0.00002	0.00001	0.00001	0.00001
3.1	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001
3.2	0.00001	0.00001	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

where:

$A(t)$: Area of one face of fracture, ft^2 (at time t)

t: time (min)

w: width of the fracture in ft.

q_i: injection rate ft^3/min.

C: fracturing fluid coefficient $ft/\sqrt{min}$ [$C\downarrow$, $A(t)\uparrow$, low fluid loss]

$$X = \frac{2C\sqrt{\pi t}}{W}$$

$$C = 0.0469\left(\frac{K\Delta P\phi}{\mu}\right)^{\frac{1}{2}} \qquad ft/\sqrt{min}$$

K: Effective formation permeability of fracturing fluid, Darcy

ϕ: formation porosity, fraction

μ: Fracturing fluid viscosity @ bottom hole temperature (cP)

ΔP: Differential pressure across the face of the fracture in psi

$$\Delta P = G_f D - P_{ws}$$

G_f: fracture gradient psi/ft

D: Depth, ft

P_{ws}: Bottom hole pressure, psi

Approximation the value of Error function with Elementary Functions:

$$erf(x) \cong \frac{x}{|x|}\sqrt{1 - \exp\left\{-x^2\left(\frac{\frac{4}{\pi} + ax^2}{1 + ax^2}\right)\right\}}$$

where

$$a = \frac{8(\pi - 3)}{3\pi(4 - \pi)} \simeq 0.140012$$

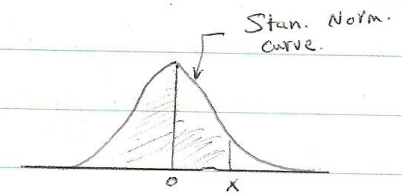
Related Functions: The error function is essentially identical to the standard normal cumulative distribution function Φ, as they differ only by scaling and translation.

$$\Phi(x) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right] = \frac{1}{2}\operatorname{erfc}\left(-\frac{x}{\sqrt{2}}\right)$$

i.e $\operatorname{erf}(x) = 2\Phi(x\sqrt{2}) - 1$ and $\operatorname{erfc}(x) = 2\Phi(-x\sqrt{2})$

$\Phi(x)$ is the area under the standard normal curve

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\,dt$$



[[The ordinates of the standard normal curve

$$y = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{\pi}}$$

[[for more detailes see " Spiegel M.R, Mathematical Handbook Table 46 and Table 47 pp. 256 - 257"

Ex:-

Show that $\operatorname{erf}(-x) = -\operatorname{erf} x$

Solution:

by difinition $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-t^2}\,dt$ --- [1]

let $t = -u$ $dt = -du$ sub in [1]

$$\operatorname{erf}(x) = -\frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-u^2}\,du$$

$$= -\operatorname{erf}(-x)$$

Exponential Integral [E_i - Function]

For real, nonzero values of x, the exponential Integral $E_i(x)$ can be defined as

$$E_i(x) = \int_{-\infty}^{x} \frac{e^t}{t}\, dt$$

The function is given as a special function because $\int \frac{e^t}{t}$ is not an elementary function

Properties: 1- Integrating the tylor series for $\frac{e^{-t}}{t}$ and extracting the logarithmic singularity, the following series representation can be deriven

$$E_i(x) = \gamma + \ln x + \sum_{n=1}^{\infty} \frac{x^n}{n\, n!} \qquad x > 0$$

where

$\gamma = 0.57721\ 56649$ Euler's Constant

$$= \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right)$$

$$E_i(-x) = E_1 = \int_{x}^{\infty} \frac{e^{-t}}{t}\, dt$$

here $E_1(x) = -E_i(-x)$

2- For $x < 0.02$

$$E_1 = E_i(-x) = \ln(1.78x) \quad \text{[logarithmic Approximation]}$$

[2] Applications

- Time-dependent heat transfer
- Radial Diffusivity Equation for transient or unsteady state flow with line sources and sinks

[line source $r_e >>> r_w$ i.e $r_w \to 0$

Fig 1 represent the values of $E_i(-x)$, Table 3 also the values of $-E_i(-x)$

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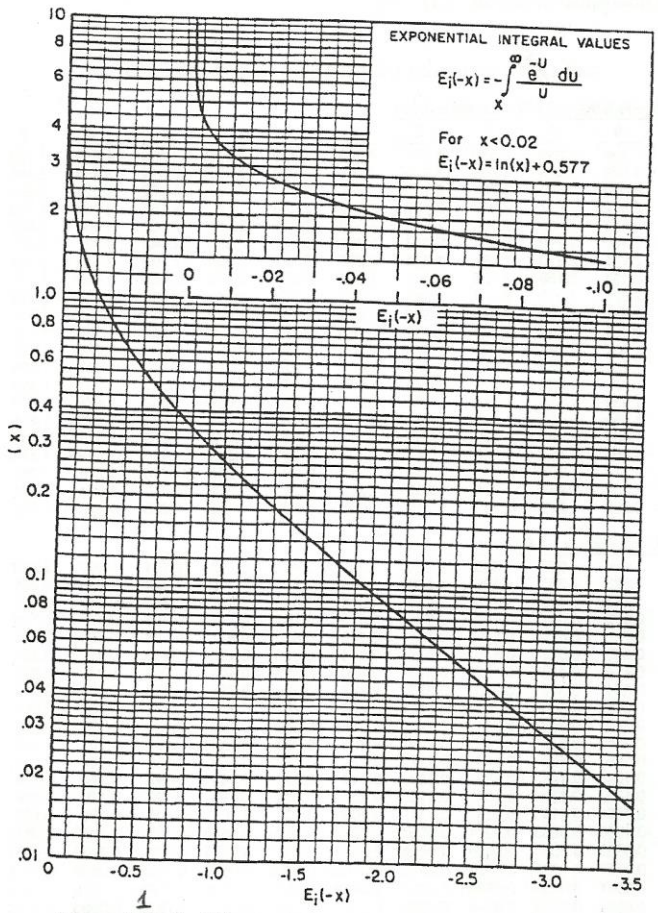

Figure 1. The E_i-function. (*After Craft, Hawkins, and Terry, 1991.*)

from drilling or completion operations. Many other wells are stimulated by acidization or hydraulic fracturing. Eq. 1.7 fails to model such wells properly; its derivation holds the explicit assumption of uniform permeability throughout the drainage area of the well up to the wellbore. Hawkins[4] pointed out that if the damaged or stimulated zone is considered equivalent to an altered zone of uniform permeability (k_s) and outer radius (r_s), the *additional* pressure drop across this zone (Δp_s) can be modeled by the steady-state radial flow equation (see Fig. 1.1). Thus,

$$\Delta p_s = 141.2 \frac{qB\mu}{k_s h} \ln(r_s/r_w) - 141.2 \frac{qB\mu}{kh} \ln(r_s/r_w)$$

$$= 141.2 \frac{qB\mu}{kh} \left(\frac{k}{k_s} - 1\right) \ln(r_s/r_w). \quad \ldots (1.9)$$

TABLE 3* – VALUES OF THE EXPONENTIAL INTEGRAL, $-Ei(-x)$

$-Ei(-x)$, $0.000 < 0.209$, Interval – 0.001

x	0	1	2	3	4	5	6	7	8	9
0.00	$+\infty$	6.332	5.639	5.235	4.948	4.726	4.545	4.392	4.259	4.142
0.01	4.038	3.944	3.858	3.779	3.705	3.637	3.574	3.514	3.458	3.405
0.02	3.355	3.307	3.261	3.218	3.176	3.137	3.098	3.062	3.026	2.992
0.03	2.959	2.927	2.897	2.867	2.838	2.810	2.783	2.756	2.731	2.706
0.04	2.681	2.658	2.634	2.612	2.590	2.568	2.547	2.527	2.507	2.487
0.05	2.468	2.449	2.431	2.413	2.395	2.377	2.360	2.344	2.327	2.311
0.06	2.295	2.279	2.264	2.249	2.235	2.220	2.206	2.192	2.178	2.164
0.07	2.151	2.138	2.125	2.112	2.099	2.087	2.074	2.062	2.050	2.039
0.08	2.027	2.015	2.004	1.993	1.982	1.971	1.960	1.950	1.939	1.929
0.09	1.919	1.909	1.899	1.889	1.879	1.869	1.860	1.850	1.841	1.832
0.10	1.823	1.814	1.805	1.796	1.788	1.779	1.770	1.762	1.754	1.745
0.11	1.737	1.729	1.721	1.713	1.705	1.697	1.689	1.682	1.674	1.667
0.12	1.660	1.652	1.645	1.638	1.631	1.623	1.616	1.609	1.603	1.596
0.13	1.589	1.582	1.576	1.569	1.562	1.556	1.549	1.543	1.537	1.530
0.14	1.524	1.518	1.512	1.506	1.500	1.494	1.488	1.482	1.476	1.470
0.15	1.464	1.459	1.453	1.447	1.442	1.436	1.431	1.425	1.420	1.415
0.16	1.409	1.404	1.399	1.393	1.388	1.383	1.378	1.373	1.368	1.363
0.17	1.358	1.353	1.348	1.343	1.338	1.333	1.329	1.324	1.319	1.314
0.18	1.310	1.305	1.301	1.296	1.291	1.287	1.282	1.278	1.274	1.269
0.19	1.265	1.261	1.256	1.252	1.248	1.243	1.239	1.235	1.231	1.227
0.20	1.223	1.219	1.215	1.210	1.206	1.202	1.198	1.195	1.191	1.187

$-Ei(-x)$, $0.00 < x < 2.09$, Interval = 0.01

0.0	$+\infty$	4.038	3.335	2.959	2.681	2.468	2.295	2.151	2.027	1.919
0.1	1.823	1.737	1.660	1.589	1.524	1.464	1.409	1.358	1.309	1.265
0.2	1.223	1.183	1.145	1.110	1.076	1.044	1.014	0.985	0.957	0.931
0.3	0.906	0.882	0.858	0.836	0.815	0.794	0.774	0.755	0.737	0.719
0.4	0.702	0.686	0.670	0.655	0.640	0.625	0.611	0.598	0.585	0.572
0.5	0.560	0.548	0.536	0.525	0.514	0.503	0.493	0.483	0.473	0.464
0.6	0.454	0.445	0.437	0.428	0.420	0.412	0.404	0.396	0.388	0.381
0.7	0.374	0.367	0.360	0.353	0.347	0.340	0.334	0.328	0.322	0.316
0.8	0.311	0.305	0.300	0.295	0.289	0.284	0.279	0.274	0.269	0.265
0.9	0.260	0.256	0.251	0.247	0.243	0.239	0.235	0.231	0.227	0.223
1.0	0.219	0.216	0.212	0.209	0.205	0.202	0.198	0.195	0.192	0.189
1.1	0.186	0.183	0.180	0.177	0.174	0.172	0.169	0.166	0.164	0.161
1.2	0.158	0.156	0.153	0.151	0.149	0.146	0.144	0.142	0.140	0.138
1.3	0.135	0.133	0.131	0.129	0.127	0.125	0.124	0.122	0.120	0.118
1.4	0.116	0.114	0.113	0.111	0.109	0.108	0.106	0.105	0.103	0.102
1.5	0.1000	0.0985	0.0971	0.0957	0.0943	0.0929	0.0915	0.0902	0.0889	0.0876
1.6	0.0863	0.0851	0.0838	0.0826	0.0814	0.0802	0.0791	0.0780	0.0768	0.0757
1.7	0.0747	0.0736	0.0725	0.0715	0.0705	0.0695	0.0685	0.0675	0.0666	0.0656
1.8	0.0647	0.0638	0.0629	0.0620	0.0612	0.0603	0.0595	0.0586	0.0578	0.0570
1.9	0.0562	0.0554	0.0546	0.0539	0.0531	0.0524	0.0517	0.0510	0.0503	0.0496
2.0	0.0489	0.0482	0.0476	0.0469	0.0463	0.0456	0.0450	0.0444	0.0438	0.0432

$2.0 < x < 10.9$, Interval = 0.1

x	0	1	2	3	4	5	6	7	8	9
2	4.89×10^{-2}	4.26×10^{-2}	3.72×10^{-2}	3.25×10^{-2}	2.84×10^{-2}	2.49×10^{-2}	2.19×10^{-2}	1.92×10^{-2}	1.69×10^{-2}	1.48×10^{-2}
3	1.30×10^{-2}	1.15×10^{-2}	1.01×10^{-2}	8.94×10^{-3}	7.89×10^{-3}	6.87×10^{-3}	6.16×10^{-3}	5.45×10^{-3}	4.82×10^{-3}	4.27×10^{-2}
4	3.78×10^{-3}	3.35×10^{-3}	2.97×10^{-3}	2.64×10^{-3}	2.34×10^{-3}	2.07×10^{-3}	1.84×10^{-3}	1.64×10^{-3}	1.45×10^{-3}	1.29×10^{-3}
5	1.15×10^{-3}	1.02×10^{-3}	9.08×10^{-4}	8.09×10^{-4}	7.19×10^{-4}	6.41×10^{-4}	5.71×10^{-4}	5.09×10^{-4}	4.53×10^{-4}	4.04×10^{-4}
6	3.60×10^{-4}	3.21×10^{-4}	2.86×10^{-4}	2.55×10^{-4}	2.28×10^{-4}	2.03×10^{-4}	1.82×10^{-4}	1.62×10^{-4}	1.45×10^{-4}	1.29×10^{-4}
7	1.15×10^{-4}	1.03×10^{-4}	9.22×10^{-5}	8.24×10^{-5}	7.36×10^{-5}	6.58×10^{-5}	5.89×10^{-5}	5.26×10^{-5}	4.71×10^{-5}	4.21×10^{-5}
8	3.77×10^{-5}	3.37×10^{-5}	3.02×10^{-5}	2.70×10^{-5}	2.42×10^{-5}	2.16×10^{-5}	1.94×10^{-5}	1.73×10^{-5}	1.55×10^{-5}	1.39×10^{-5}
9	1.24×10^{-5}	1.11×10^{-5}	9.99×10^{-6}	8.95×10^{-6}	8.02×10^{-6}	7.18×10^{-6}	6.44×10^{-6}	5.77×10^{-6}	5.17×10^{-6}	4.64×10^{-6}
10	4.15×10^{-6}	3.73×10^{-6}	3.34×10^{-6}	3.00×10^{-6}	2.68×10^{-6}	2.41×10^{-6}	2.16×10^{-6}	1.94×10^{-6}	1.74×10^{-6}	1.56×10^{-6}

*Adapted from Nisle, R.G.: "How To Use The Exponential Integral," *Pet. Eng.* (Aug. 1956) B171-173.

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The following equation represent the transient behavoir

$$P(r,t) = P_i + \frac{70.6\, Q_o \mu_o B_o}{Kh} Ei\left[-\frac{948\, \phi \mu_o C_t r^2}{Kt}\right]$$

[[it gaves the value of pressure @ any time @ any radius]]

if $r = r_w$ then $P = P_{wf}$

$$P_{wf} = P_i + \frac{70.6\, Q_o \mu_o B_o}{Kh} Ei\left(-\frac{948\, \phi \mu_o C_t r_w^2}{Kt}\right)$$

P: Pressure psi , Q_o = flow rate STB/D , μ_o = oil viscosity c.P

B_o = OFVF RB/STB , ϕ: porosity, fraction , C_t : total compressibility

r = radius r_w = well radius ft , K: permeability md , t: time hr.

Remark: when the parameter X is less than 0.01 , the log-approximation can be used to evaluate the pressure

$$P(r,t) = P_i - \frac{162.6\, Q_o B_o \mu_o}{Kh}\left[\log \frac{Kt}{\phi \mu_o C_t r^2} - 3.23\right] \quad \text{--- } \boxed{A}$$

and

$$P_{wf} = P_i - \frac{162.6\, Q_o B_o \mu_o}{Kh}\left[\log \frac{Kt}{\phi \mu_o C_t r_w^2} - 3.23\right] \quad \text{--- } \boxed{B}$$

equations (A) and (B) Cannot be used until the flow time t exceeds the limit imposed by the following Constraint

$$t > 9.48 \times 10^4 \frac{\phi \mu C_t r^2}{K} \quad ; t, \text{hr}$$

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The Gamma Function

The gamma Function was first introduced by swiss mathematician Leonhard Euler (1707-1783). The gamma function belongs to the category of the special transcendental functions, it appears in various area as asymptotic series, definite integration, hypergeometric series, Riemann Zeta function.

Mathematically

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\, dt \qquad n > 0$$

The graph of Gamma Function is shown in ~~Fig 16~~ fig 16 - 1 p. 101, Spiegel: Mathematical Handbook

Table 20 p. 235 of the same book gives the value of gamma function $1 \le n \le 2$

Properties of The Gamma Function

- Theorem 1: $\Gamma(1) = 1$

proof: since

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\, dt \Rightarrow \Gamma(1) = \int_0^\infty t^{1-1} e^{-t}\, dt$$

$$\therefore \Gamma(1) = \left[e^{-t}\right]_\infty^0 \Rightarrow \Gamma(1) = 1$$

- Theorem 2: $\Gamma(n+1) = n\,\Gamma(n) \qquad n > 0$

proof:

$$\cancel{\Gamma(n+1) = \int e^{n+1}}$$

$$\Gamma(n+1) = \int_0^\infty t^{n+1-1} e^{-t}\, dt = \int t^n e^{-t}\, dt$$

$$= \left[(-e^{-t})t^n\right] - \int_0^\infty (-e^{-t})(n)\, t^{n-1}\, dt$$

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16 THE GAMMA FUNCTION

DEFINITION OF THE GAMMA FUNCTION $\Gamma(n)$ FOR $n > 0$

16.1
$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}\,dt \qquad n > 0$$

RECURSION FORMULA

16.2
$$\Gamma(n+1) = n\,\Gamma(n)$$

16.3
$$\Gamma(n+1) = n! \qquad \text{if } n = 0, 1, 2, \ldots \text{ where } 0! = 1$$

THE GAMMA FUNCTION FOR $n < 0$

For $n < 0$ the gamma function can be defined by using 16.2, i.e.

16.4
$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

GRAPH OF THE GAMMA FUNCTION

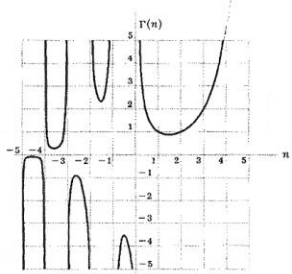

Fig. 16-1

SPECIAL VALUES FOR THE GAMMA FUNCTION

16.5
$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

16.6
$$\Gamma(m+\tfrac{1}{2}) = \frac{1\cdot 3\cdot 5\cdots(2m-1)}{2^m}\sqrt{\pi} \qquad m = 1, 2, 3, \ldots = \frac{(2m)!\sqrt{\pi}}{4^m m!}$$

16.7
$$\Gamma(-m+\tfrac{1}{2}) = \frac{(-1)^m 2^m\sqrt{\pi}}{1\cdot 3\cdot 5\cdots(2m-1)} \qquad m = 1, 2, 3, \ldots$$

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$$\Gamma(n+1) = 0 + n\int_0^\infty t^{n-1} e^{-t}\, dt$$

↑ Dif. of $\Gamma(n)$ ↑

$\therefore \Gamma(n+1) = n\,\Gamma(n) \Rightarrow \left[\!\left[\Gamma(n) = \dfrac{\Gamma(n+1)}{n} \;\; \forall\, n < 0 \right]\!\right]$

- Theorem 3

If n is a non-negative integer then $\Gamma(n+1) = n!$

proof:

since $\Gamma(n+1) = n\,\Gamma(n)$ *

$= n\,(n-1)\,\Gamma(n-1)$

$= n\,(n-1)(n-2)\,\Gamma(n-2)$ } Theorem 2

By further use of theorem 2

$\Gamma(n+1) = n\,(n-1)(n-2)(n-3) \;\text{----}\; 3\cdot 2\cdot 1\,\Gamma(1)$

$\Gamma(n+1) = n!\,\Gamma(1)$ since $\Gamma(1) = 1$

$\therefore \Gamma(n+1) = n!$ $\quad 0! = 1$

Then $n\,\Gamma(n) = n!$ $\Rightarrow$ ((n: Integer)), ~~$0! = 1$~~

Results $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

The formula $\Gamma(n+1) = n\,\Gamma(n)$ also used to evaluate the value of n is not an integer

Ex: Find the value of $\Gamma(\frac{7}{3})$, here $n = \frac{7}{3} = 1 + \frac{4}{3}$

$\therefore \Gamma(\frac{7}{3}) = \frac{4}{3}\,\Gamma(\frac{4}{3})$ also $\frac{4}{3} = 1 + \frac{1}{3}$

$\Gamma(\frac{7}{3}) = \frac{4}{3}\cdot\frac{1}{3}\,\Gamma(\frac{1}{3}) \Rightarrow \Gamma(\frac{7}{3}) = \frac{4}{9}\,\Gamma(\frac{1}{3}) \simeq 1.19$

---

* $\Gamma(n+1) = n\,\Gamma(n)$ letting $n = n-1$

$\Gamma(n) = (n-1)\,\Gamma(n-1)$ letting $n = n-1$

$\Gamma(n-1) = (n-2)\,\Gamma(n-2)$ and so on

- The Gamma Function For negative number

For $P \leq 0$, $\Gamma(P)$ ~~has~~ not so far defined, since

$$\Gamma(P+1) = P\,\Gamma(P) \quad \text{then}$$

$$\Gamma(P) = \frac{1}{P}\,\Gamma(P+1)$$

if $P = -0.5$

$$\Gamma(-0.5) = \frac{1}{(-0.5)}\,\Gamma(-0.5+1)$$

$$\Gamma(-0.5) = \frac{1}{-0.5}\,\Gamma\left(\frac{1}{2}\right)$$

$$\therefore \Gamma(-0.5) = -2\sqrt{\pi}$$

if P is an integer, let us start with $P = 0$

$$\Gamma(0) = \frac{1}{0}\,\Gamma(0+1) \Rightarrow \Gamma(0) = \infty$$

$$\Gamma(-1) = \frac{1}{-1}\,\Gamma(-1+1) \Rightarrow \Gamma(-1) = \infty$$

$$\Gamma(-2) = \frac{1}{-2}\,\Gamma(-2+1) = \frac{1}{-2}\,\Gamma(-1) \quad \therefore \Gamma(-2) = \infty$$

Result: $\Gamma(P)$ becomes infinite not only at zero, but at all the negative integer.

Relationship Among Gamma Equation

$$\Gamma(P)\,\Gamma(1-P) = \frac{\pi}{\sin P\pi}$$

Incomplete Gamma Functions: These Functions are defined by

[3] $$\Gamma(x,\alpha) = \int_{\alpha}^{\infty} e^{-t}\, t^{x-1}\, dt$$

$$\gamma(x,\alpha) = \int_{0}^{\alpha} e^{-t}\, t^{x-1}\, dt$$

and

$$\Gamma(x,\alpha) + \gamma(x,\alpha) = \Gamma(x)$$

Problems

1- Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Solution: By difinition

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\, dt \quad \text{--- } \boxed{1}$$

let $t = x^2$ $\quad dt = 2x\,dx$, sub. in $\boxed{1}$

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty x^{-1} e^{-x^2} x\, dx$$

$$= 2\int_0^\infty e^{-x^2} dx \quad \text{--- } \boxed{2}$$

Also

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz$$

and

$$erf(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2} dz$$

Since z is a dummy variable then

$$erf(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} dx \quad \text{--- } \boxed{3}$$

$$\sqrt{\pi}\, erf(\infty) = 2\int_0^\infty e^{-x^2} dx \quad \text{--- } \boxed{4}$$

make use of eq $\boxed{2}$

Then $\sqrt{\pi}\, erf(\infty) = \Gamma(\frac{1}{2})$ but $erf(\infty) = \underline{1}$

$$\therefore \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \quad \blacksquare$$

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2- Express the following Integrals in term of Gamma Function

$$1- \int_0^1 (x \ln x)^3 \, dx$$

Solution:

let $\ln x = -y \Rightarrow x = e^{-y} \Rightarrow dx = -e^{-y} dy$

when $x = 0$ $y = \infty$ when $x = 1$ $y = 0$

The above Integration transformed to

$$I = \int_{\infty}^{0} (e^{-y} y)^3 (-e^{-y} dy)$$

$$= -\int_0^{\infty} e^{-4y} y^3 \, dy$$

2nd substitution

let $z = 4y$ $dz = 4 dy$ or $dy = \frac{1}{4} dz$

Then

$$I = -\frac{1}{4} \int_0^{\infty} e^{-z} \frac{z^3}{4^3} dz$$

$$= -\frac{1}{4^4} \int_0^{\infty} e^{-z} z^3 dz \quad [\text{Gamma Function}]$$

here $n-1 = 3$ $\therefore$ $n = 4$

Therefore

$$I = -\frac{\Gamma(4)}{4^3} \quad \text{and} \quad \Gamma(n) = n! \ (n: \text{integer})$$

$$\therefore I = \frac{-3 \times 2}{4 \times 4 \times 4 \times 4} = -\frac{3}{128}$$

2- $\displaystyle\int_0^1 \ln\left(\frac{1}{x}\right)^{\frac{1}{3}} dx$

solution: since $\ln\left(\frac{1}{a}\right) = -\ln a$

$$I = \int_0^1 (-\ln x)^{\frac{1}{3}} dx$$

let $-\ln x = y \qquad x = e^{-y} \qquad dx = -e^{-y} dy$

$x = 0 \quad y = \infty$

$x = 1 \quad y = 0$

$$\therefore \quad I = -\int_\infty^0 (+y)^{\frac{1}{3}} e^{-y} dy$$

$$I = \int_0^\infty y^{\frac{1}{3}} e^{-y} dy \qquad [\text{Gamma function}]$$

$\therefore \quad P - 1 = \frac{1}{3} \qquad P = 1 + \frac{1}{3}$

or

$$I = \Gamma\left(1 + \frac{1}{3}\right)$$

$$= \frac{1}{3}\Gamma\left(\frac{1}{3}\right) \qquad \left[\Gamma(P+1) = P\Gamma(P)\right]$$

3- Show that $\Gamma\left(\frac{1}{2} + x\right)\Gamma\left(\frac{1}{2} - x\right) = \dfrac{\pi}{\cos \pi x}$

given that $\Gamma(x)\Gamma(1-x) = \dfrac{\pi}{\sin \pi x}$

Solution:

Since $\Gamma(x)\Gamma(1-x) = \dfrac{\pi}{\sin \pi x}$

let $x = x + \frac{1}{2}$ then

$$\Gamma\left(x + \frac{1}{2}\right)\Gamma\left(1 - x - \frac{1}{2}\right) = \frac{\pi}{\sin \pi\left(x + \frac{1}{2}\right)}$$

$$\Gamma\left(x + \frac{1}{2}\right)\left(\frac{1}{2} - x\right) = \frac{\pi}{\sin\left(\pi x + \frac{\pi}{2}\right)}$$

$$= \frac{\pi}{\cos \pi x} \qquad \left[\sin \pi x + \frac{\pi}{2} = \cos \pi x\right]$$

Beta Function

The beta function is defined as a definite Integral and some time It is known as Eularian integral, Mathematically the Beta function is defined as:-

$$B(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt \quad (x>0, y>0)$$

If we write the function as

$$B(p,q) = \int_0^1 t^{p-1} (1-t)^{q-1} \, dt \quad (p,q>0)$$

and

letting $t = 1-x$ then $dt = -dx$

$$B(p,q) = \int_0^1 x^{q-1} (1-x)^{p-1}$$

so that the function is symmetrical in its two parameters

Accordingly, the function can be written in terms of trigonometric function, make the following substitution

$t = \sin^2\theta$ sice $\sin^2\theta + \cos^2\theta = 1$

then $1-t = \cos^2\theta$ and $dt = 2\sin\theta\cos\theta$

$t=0$ $\theta=0$ $t=1$ $\theta = \frac{\pi}{2}$ $[\sin\frac{\pi}{2} = 1]$

$$\therefore B(p,q) = 2\int_0^{\frac{\pi}{2}} (\sin^2\theta)^{p-1} (\cos^2\theta)^{q-1} \sin\theta\cos\theta \, d\theta$$

4 $$B(q,p) = B(p,q) = 2\int_0^{\frac{\pi}{2}} \sin^{2p-1}\theta \cos^{2q-1}\theta \, d\theta$$

Some Important Results

$$\beta(p,q) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\, dt$$

In particular, if $0 < (p,q) < 1$, $p+q=1$ then

$$B(1-q,q) = \beta(q,1-q) = \int_0^\infty \frac{y^{-q}}{1+y}\, dy = \frac{\pi}{\sin \pi q}$$

Ex:

Evaluate the following integral

$$I = \int_0^{\frac{\pi}{2}} (\cot\theta)^{\frac{1}{2}}\, d\theta$$

Solution

$$I = \int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}}\theta \cos^{\frac{1}{2}}\theta\, d\theta \qquad [\text{Beta eq.}]$$

here

$2p-1 = -\frac{1}{2}$ $\therefore$ $p = \frac{1}{4}$

$2q-1 = \frac{1}{2}$ $\therefore$ $q = \frac{3}{4}$

$\therefore$ $I = \frac{1}{2} B(\frac{1}{4}, \frac{3}{4})$ here $p = \frac{1}{4}$, $q = \frac{3}{4}$

$p+q = 1$

Since $B(1-q,q) = \frac{\pi}{\sin \pi q} = B(q, 1-q)$

$\therefore$ $I = \frac{1}{2} \frac{\pi}{\sin \frac{\pi}{4}}$ also $I = \frac{1}{2} \frac{\pi}{\sin \frac{3\pi}{4}}$

$= \frac{\pi}{\sqrt{2}}$

Relationship of Beta Function To Gamma Function

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\Gamma(q)\,\Gamma(1-q) = B(q,1-q) = \frac{\pi}{\sin q\pi}$$

Examples

1- Evaluate

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^n}} \qquad n>0$$

Solution:

let $x^n = z$ or $x = z^{\frac{1}{n}}$ and $dx = \frac{1}{n} z^{\frac{1-n}{n}}$

Therefore

$$I = \frac{1}{n}\int_0^1 z^{\frac{1-n}{n}} (1-z)^{-\frac{1}{2}}\, dz \qquad [\text{ Beta function }]$$

$$p-1 = \frac{1-n}{n} \qquad p = \frac{1}{n} \qquad q-1 = -\frac{1}{2} \qquad q = \frac{1}{2}$$

$$I = \frac{1}{n} B\left(\frac{1}{n}, \frac{1}{2}\right)$$

$$= \frac{1}{n}\,\frac{\Gamma(\frac{1}{n})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{n})} = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(\frac{1}{n})}{\Gamma(\frac{1}{n}+\frac{1}{2})}$$

2- Evaluate

$$I = \int_0^\infty \frac{dt}{\sqrt{t}\,(1+t)}$$

Since $B(p,q) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}$ [prop. of B. function]

her

$$I = \int_0^\infty \frac{t^{-\frac{1}{2}}}{(1+t)}\, dt \quad , \quad p-1 = -\frac{1}{2} \quad p = \frac{1}{2} \text{ and } q = \frac{1}{2}$$

Also $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$$\therefore I = B\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})}$$

$$I = \pi$$

3- prove $\displaystyle\int_0^\infty \frac{dx}{1+x^n} = \frac{1}{n}\,\frac{\pi}{\sin\frac{\pi}{n}}$

solution:

let $x^n = z \qquad x = z^{\frac{1}{n}} \qquad dx = \frac{1}{n} z^{\frac{1-n}{n}} dz$

$$I = \frac{1}{n}\int_0^\infty \frac{1}{(1+z)} \cdot z^{\frac{1-n}{n}}\, dz$$

$p-1 = \dfrac{1-n}{n} \Rightarrow \boxed{p = \frac{1}{n}} \qquad p+q = 1 \quad \therefore q = \dfrac{n-1}{n}$

which is identical to $B(p,q) = \displaystyle\int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\, dt$

and the particular $p+q=1$

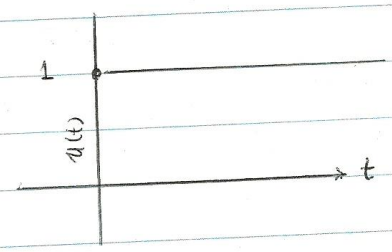
$$B(p,q) = \frac{\pi}{\sin \pi q}$$

$$\therefore I = \frac{1}{n}\cdot\frac{\pi}{\sin\frac{\pi}{n}} \qquad \boxed{n \neq 0}$$

Special Functions

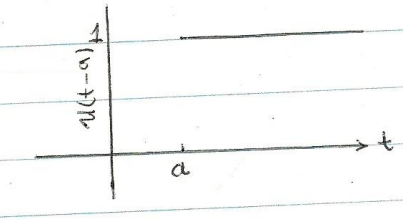
The unit step Function, also called The Heaviside step function is a discontinuous function whose value is zero for negative argument and one for positive argument, mathematically

$$\mathcal{U}(t) = \mathcal{U} = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$$



and the shifted unit step is

$$\mathcal{U}(t-a) = \begin{cases} 1 & t>a \\ 0 & t<a \end{cases}$$



The function is used in the mathematics of Control theory and signal processing to represent a signal that switches on a specified time, and stays switched on indefinitely.

Ex: Sketch the function $g(t)$:

$$g(t) = 3\,\mathcal{U}(t) - 4(t-1)\,\mathcal{U}(t-1) + 4(t-3)\,\mathcal{U}(t-3)$$

Solution:

a- $t<1$, $\mathcal{U}(t-1)$ and $\mathcal{U}(t-3) = 0$

so $\boxed{g(t) = 1}$

b- $1<t<3$, $\mathcal{U}(t-1) = 1$, $\mathcal{U}(t-3) = 0$, $g = 3 - 4(t-1)$ $\boxed{g(t) = 7-4t}$

c- $t>3$, $\mathcal{U}(t) = \mathcal{U}(t-1) = \mathcal{U}(t-3) = 1$

$g(t) = 3 - 4(t-1) + \mathcal{U}(t-3) \Rightarrow \boxed{g(t) = -5}$

5

t	g(t)	t	g(t)
0.5	3	2	-1
1	3	4	-5
1.5	+1.5	5	-5

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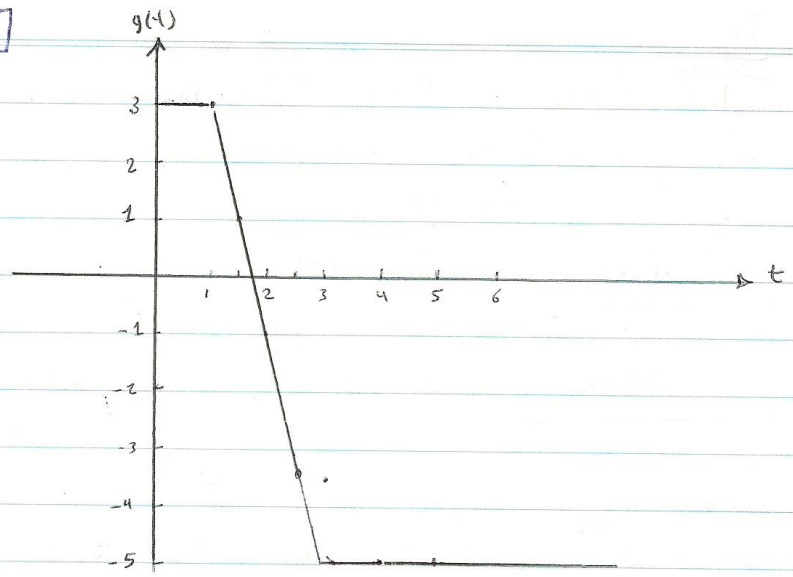
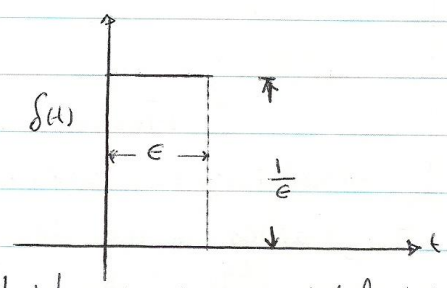


	$g(t) = 3$	$0 < t < 1$
$g(t) =$	$7 - 4t$	$1 < t < 3$
$g(t) =$	-5	$t > 3$

$\boxed{6}$ The unit Impulse Function (Dirac Delta Function

The difinition of UIF is

$$\delta(t) = \begin{cases} \frac{1}{\epsilon} & 0 \leq t \leq \epsilon \\ 0 & t > \epsilon \end{cases}$$



as $\epsilon \to 0$, the hight of rectanglar shaded region increases indefinitely and the width decreases in such a way that the area is always equal to 1

$$\int_0^\infty \delta(t)\, dt = 1.0$$

Special case

$$\delta(0) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

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$\boxed{7}$ The Sine and Cosine Integrals

The difinition of these function are

$$Si(x) = \int_0^x \frac{\sin t}{t}\, dt$$

$$Si(x) = -\int_x^\infty \frac{\sin t}{t}\, dt \qquad \left[\operatorname{sinc}(t) = \frac{\sin t}{t}\right]^*$$

$$Ci(x) = \int_\infty^x \frac{\cos t}{t}\, dt \qquad t>0$$

Table $\boxed{A\text{-}4}$ illustrates the values of sine and cosine integral.

$Si(0) = 0$, $Ci(0) = \infty$,

$Si(\infty) = \frac{\pi}{2}$ $(\simeq 1.5708)$

and

$Ci(0) = \infty$ $\qquad$ $Ci(\infty) = 0$

$\boxed{8}$ Fresnel Integral

Fresnel Integrals, $S(x)$ and $C(x)$ are two transcendental functions They arises in the description of diffraction phenomena, and are defined through the following integral representation

$$S(x) = \int_0^x \sin(t^2)\, dt \qquad C(x) = \int_0^x \cos(t^2)\, dt$$

$* \quad \lim_{t \to 0} \sin t / t = 1$.

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Table A1 ***(continued)***

x	$Y_0(x)$	$Y_1(x)$	x	$Y_0(x)$	$Y_1(x)$	x	$Y_0(x)$	$Y_1(x)$
0.0	$(-\infty)$	$(-\infty)$	2.5	0.498	0.146	5.0	−0.309	0.148
0.5	−0.445	−1.471	3.0	0.377	0.325	5.5	−0.339	−0.024
1.0	0.088	−0.781	3.5	0.189	0.410	6.0	−0.288	−0.175
1.5	0.382	−0.412	4.0	−0.017	0.398	6.5	−0.173	−0.274
2.0	0.510	−0.107	4.5	−0.195	0.301	7.0	−0.026	−0.303

Table A2 Gamma Function [see (24) in App. A3.1]

α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$
1.00	1.000 000	1.20	0.918 169	1.40	0.887 264	1.60	0.893 515	1.80	0.931 384
1.02	0.988 844	1.22	0.913 106	1.42	0.886 356	1.62	0.895 924	1.82	0.936 845
1.04	0.978 438	1.24	0.908 521	1.44	0.885 805	1.64	0.898 642	1.84	0.942 612
1.06	0.968 744	1.26	0.904 397	1.46	0.885 604	1.66	0.901 668	1.86	0.948 687
1.08	0.959 725	1.28	0.900 718	1.48	0.885 747	1.68	0.905 001	1.88	0.955 071
1.10	0.951 351	1.30	0.897 471	1.50	0.886 227	1.70	0.908 639	1.90	0.961 766
1.12	0.943 590	1.32	0.894 640	1.52	0.887 039	1.72	0.912 581	1.92	0.968 774
1.14	0.936 416	1.34	0.892 216	1.54	0.888 178	1.74	0.916 826	1.94	0.976 099
1.16	0.929 803	1.36	0.890 185	1.56	0.889 639	1.76	0.921 375	1.96	0.983 743
1.18	0.923 728	1.38	0.888 537	1.58	0.891 420	1.78	0.926 227	1.98	0.991 708
1.20	0.918 169	1.40	0.887 264	1.60	0.893 515	1.80	0.931 384	2.00	1.000 000

Table A3 Factorial Function and Its Logarithm with Base 10

n	$n!$	$\log(n!)$	n	$n!$	$\log(n!)$	n	$n!$	$\log(n!)$
1	1	0.000 000	6	720	2.857 332	11	39 916 800	7.601 156
2	2	0.301 030	7	5 040	3.702 431	12	479 001 600	8.680 337
3	6	0.778 151	8	40 320	4.605 521	13	6 227 020 800	9.794 280
4	24	1.380 211	9	362 880	5.559 763	14	87 178 291 200	10.940 408
5	120	2.079 181	10	3 628 800	6.559 763	15	1 307 674 368 000	12.116 500

Table A4 Error Function, Sine and Cosine Integrals [see (35), (40), (42) in App. A3.1]

x	erf x	$\mathrm{Si}(x)$	$\mathrm{ci}(x)$	x	erf x	$\mathrm{Si}(x)$	$\mathrm{ci}(x)$
0.0	0.0000	0.0000	∞	2.0	0.9953	1.6054	−0.4230
0.2	0.2227	0.1996	1.0422	2.2	0.9981	1.6876	−0.3751
0.4	0.4284	0.3965	0.3788	2.4	0.9993	1.7525	−0.3173
0.6	0.6039	0.5881	0.0223	2.6	0.9998	1.8004	−0.2533
0.8	0.7421	0.7721	−0.1983	2.8	0.9999	1.8321	−0.1865
1.0	0.8427	0.9461	−0.3374	3.0	1.0000	1.8487	−0.1196
1.2	0.9103	1.1080	−0.4205	3.2	1.0000	1.8514	−0.0553
1.4	0.9523	1.2562	−0.4620	3.4	1.0000	1.8419	0.0045
1.6	0.9763	1.3892	−0.4717	3.6	1.0000	1.8219	0.0580
1.8	0.9891	1.5058	−0.4568	3.8	1.0000	1.7934	0.1038
2.0	0.9953	1.6054	−0.4230	4.0	1.0000	1.7582	0.1410

The Fresnel Integrals admit the following series expansions that Converges for all x

$$S(x) = \int_0^x \sin(t^2)\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)!\,(4n+3)}$$

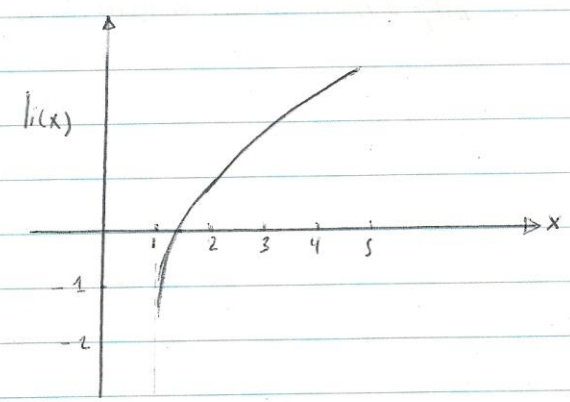
$$C(x) = \int_0^x \cos(t^2)\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n)!\,(4n+1)}$$

[9] Logarithmic Integral Function

The logarithmic integral has an integral representation defined for all positive real numbers $x \neq 1$, by the definite integral

$$li(x) = \int_0^x \frac{dt}{\ln t}$$

The function $\frac{1}{\ln t}$ has a singularity at $t = 1$



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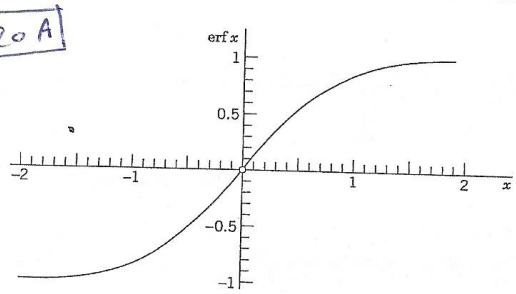

Fig. 553. Error function

erf $(\infty) = 1$, *complementary error function*

(37) $$\operatorname{erfc} x = 1 - \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$$

Fresnel integrals[1] (Fig. 554)

(38) $$C(x) = \int_0^x \cos(t^2)\, dt, \qquad S(x) = \int_0^x \sin(t^2)\, dt$$

$C(\infty) = \sqrt{\pi/8}$, $S(\infty) = \sqrt{\pi/8}$, *complementary functions*

(39) $$c(x) = \sqrt{\frac{\pi}{8}} - C(x) = \int_x^\infty \cos(t^2)\, dt$$
$$s(x) = \sqrt{\frac{\pi}{8}} - S(x) = \int_x^\infty \sin(t^2)\, dt$$

Sine integral (Fig. 555 and Table A4 in App. 5)

(40) $$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt$$

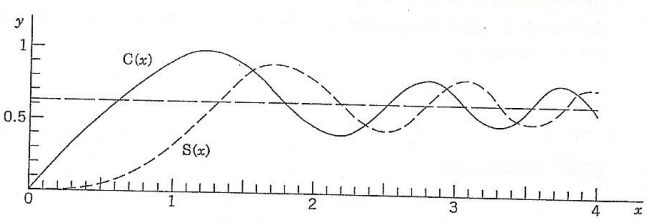

Fig. 554. Fresnel integrals

[1]AUGUSTIN FRESNEL (1788–1827), French physicist and mathematician. For tables see Ref. [GR1].

10 The Riemann Zeta Function

The Riemann Zeta Function represent the sum of the infinite Series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

, The series is Convergent for $s > 1$ and divergent for $x \leq 1$.

11 The Debye Function :

The Debye Function occurs in Thermodynamic problem (radiation), and defined by

$$D_n(x) = \int_0^x \frac{t^n}{e^t - 1}\, dt$$

For more detail see

http://mathword.wolfram.com