INFORMATION THEORY

and

CODING

Husam Abdulldaem Mohammed

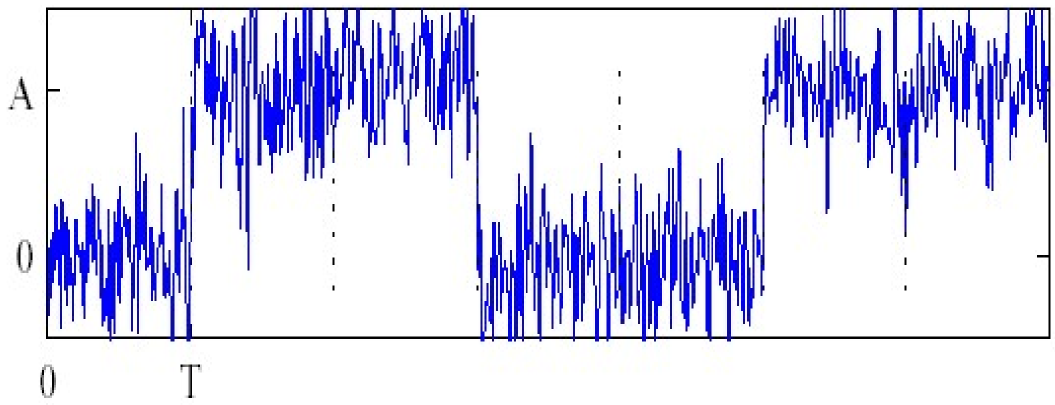
Probability of error in transmission

Here the binary levels at the receiver are nominally 0 (signal absent) and A (signal present) upon receipt of a 0 or 1 digit respectively. The function of a receiver is to distinguish the digit 0 from the digit 1. The most important performance characteristic of the receiver is the probability that an error will be made in such a determination.

Consider the received signal waveform for the bit transmitted between time 0 and time *T* . Due to the presence of noise the actual waveform *y(t)* at the receiver is

$v(t)=f(t) + n(t)$

where *v (t)* is the ideal noise-free signal.



In the case described the signal *v(t)* is

v (t) = 0 symbol 0 transmitted (signal absent)

= 1 symbol 1 transmitted (signal present):

In what follows, it is assumed that the transmitter and the receiver are synchronized, so the receiver has perfect knowledge of the arrival times of sequences of pulses. The means of achieving this synchronisation is not considered here. This means that without loss of generality we can always assume that the bit to be received lies in the interval (0,T).

Simple detection

A very simple detector could be obtained by sampling the received signal at some time instant *Ts* in the range (0,T), and using the value to make a decision. The value obtained would be one of the following:

$v(Ts) = n(Ts)$ signal absent

$v(Ts) = A + n(Ts)$ signal present:

Since the value *n(T*) is random, we cannot decide with certainty whether a signal was present or not at the time of the sample. However, a reasonable rule for the decision of whether a 0 or a 1 was received is the following:

$v(T) \leq d$	signal absent —0 received
$v(T) \geq d$	signal present— 1 received

The quantity d is a threshold which we would usually choose somewhere between 0 and A. For convenience we denote $v(T_s)$ by y.

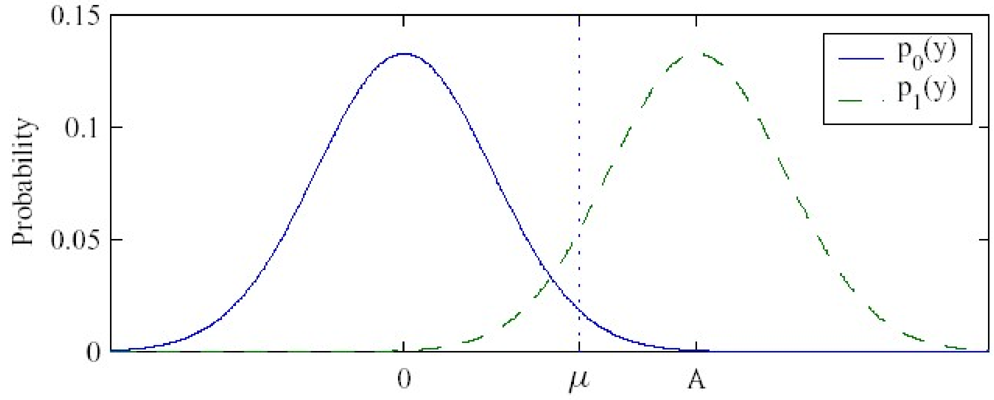
Suppose now that n(Ts) has a Gaussian distribution with a mean of zero and a variance of σ2. Under the assumption that a zero was received the probability density of v is

$$f_0(v) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(v^2)/2\sigma^2}$$

Similarly, when a signal is present, the density of v is:

$$f_1(v) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(v-A)^2/2\sigma^2}$$

These are shown below:



Using the decision rule described, it is evident that we sometimes decide that a signal is present even when it is in fact absent. The probability of such a **false alarm** occurring (mistaking a zero for a one) is

$$P_{e0} = \int_{-A/2}^{\infty} f_0(v)dv$$

Similarly, the probability of a **missed detection** (mistaking a one for a zero) is

$$P_{e1} = \int_{-\infty}^{A/2} f_1(v)dv$$

Letting *P*0 and *P*1 be the source digit probabilities of zeros and ones respectively, we can define the overall **probability of error** to be

$$P_e = P_0 P_{e0} + P_1 P_{e1}$$

In the equiprobable case this becomes

$$P_e = 1/2(P_{e0} + P_{e1})$$

The sum of these two errors will be minimised for d= A/2. This sets the decision threshold for a minimum probability of error for P0 = P1 = 1/ 2 . In that case the probabilities of each type of error are equal, so the overall probability of error is

$$P_e = \int_{A/2}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(v)^2/2\sigma^2} dv$$

Making the change of variables z = v/σ this integral becomes

$$P_e = \int_{A/2\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z)^2/2} dz = erfc\left(\frac{A}{2\sigma}\right)$$

This may be written in a more useful form by noting that the average signal power is S = A2/2, and the noise power is N =σ2. The probability of error for on-off binary is therefore

$$P_e = erfc\sqrt{\frac{S}{2N}}$$

The on-off binary signal therefore requires twice the signal power of the polar binary signal to achieve the same error rate.

The ratio of signal energy per pulse to noise power spectral density is

$$z = \frac{A^2 T}{N_o} = \frac{E_b}{N_o}$$

- where Eb is called the energy per bit because each signal pulse (+A or –A) carries one bit of information.

T = 1/B   where B is a rough measure of its bandwidth. Thus

$$z = \frac{A^2}{N_o / T} = \frac{A^2}{N_o B}$$

can be interpreted as the ration of signal power to noise power in the signal bandwidth. B also is the bit rate bandwidth.

$$Q(u) \approx \frac{e^{-u^2/2}}{u\sqrt{2\pi}} \qquad u >> 1$$

$$P_e \approx \frac{e^{-z}}{2\sqrt{z\pi}} \qquad z >> 1$$

where $$u = \frac{-\sqrt{2}\,\eta}{\sqrt{N_o T}}$$

Pe decreases exponentially with increasing z.

Optimum decision level

Probability of error

Pe = P0 Pε0 + P1 Pε1

$$P_e = P_0 \int_{d}^{\infty} P_0(y)\,dy + P_1 \int_{-\infty}^{d} P_1(y)\,dy$$

$$P_e \min imum = \frac{\partial P_e}{\partial d} = 0 \quad optimum \; decision \; level \, (d) \Rightarrow$$

$$= P_0 P_0(y) + P_1 P_1(y) = 0 \Rightarrow \qquad \frac{P_0}{P_1} = \frac{P_1(y)}{P_0(y)}$$

if P0 = P1 = 1/2 then

$$\frac{1/2}{1/2} = \frac{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-(d-v)^2/2\sigma^2}}{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-(d^2)/2\sigma^2}}$$

$$-\frac{(d-A)^2}{2\sigma^2} = -\frac{(d)^2}{2\sigma^2} \Rightarrow d^2 - 2dA + A^2 = d^2 \Rightarrow d = \frac{A}{2}$$

$$-A/2 \Rightarrow 0 \Rightarrow P_0(v) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(v-A/2)^2/2\sigma^2}$$

$$A/2 \Rightarrow 1 \Rightarrow P_1(v) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(v+A/2)^2/2\sigma^2}$$

Example1:

A polar binary signal ±A is received in the presence of additive Gaussian noise of variance N. Find the appropriate decision level if one sample of signal plus noise is taken for p1 = 0.3, 0.5, and 0.7.

Solution:

$$p_o f_o(d) = p_1 f_1(d)$$

$$p_o \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left((d+A)^2\right)}{2\sigma^2}} = p_1 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left((d-A)^2\right)}{2\sigma^2}}$$

$$\frac{p_o}{p_1} = e^{-\frac{\left((d-A)^2\right)}{2\sigma^2} + \frac{\left((d+A)^2\right)}{2\sigma^2}}$$

$$\ln \frac{p_o}{p_1} = -\frac{\left((d-A)^2\right)}{2\sigma^2} + \frac{\left((d+A)^2\right)}{2\sigma^2}$$

$$d = \frac{\sigma^2}{2A} \ln \frac{p_o}{p_1} = \frac{N}{2A} \ln \frac{p_o}{p_1}$$

$$a)\ \ for\ \ p_1 = 0.3 \Rightarrow p_0 = 0.7$$

$$d = \frac{\sigma^2}{2A} \ln \frac{p_o}{p_1} = \frac{N}{2A} \ln \frac{0.7}{0.3}$$

$$b)\ \ for\ \ p_1 = 0.5 \Rightarrow p_0 = 0.5$$

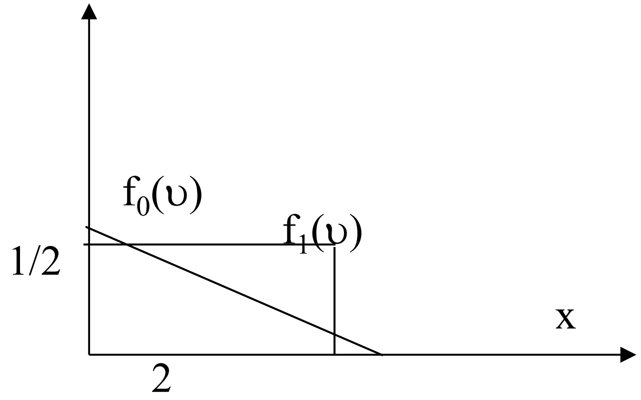
$$d = \frac{N}{2A} \ln \frac{0.5}{0.5} = 0$$

$$c)\ \ for\ \ p_1 = 0.7 \Rightarrow p_0 = 0.3$$

$$d = \frac{N}{2A} \ln \frac{0.3}{0.7}$$

Example2:

The received voltage for a binary transmission has the two conditional density functions f(u/1) = 1/2, and f(u/0) = e -u. find the optimum decision rule and minimum probability of error in the three cases p0 = 2/3.



solution:

$$p_o f_o(d) = p_1 f_1(d)$$

$$p_o \; e^{-d} = p_1 \frac{1}{2}$$

$$d = \ln \frac{2 p_o}{p_1}$$

$$a) \;\; for \;\; p_o = 2/3 \;\; \Rightarrow \;\; p_1 = 1/3$$

$$d = 1.38$$

$$p_{e=} \, p_o p_{e0} + p_1 p_{e1}$$

$$p_{e=} \, p_o p_{eo} + p_1 p_{e1}$$

$$p_e = \frac{2}{3} \int_{1.38}^{\infty} e^{-v} dv + \frac{1}{3}\left(\frac{1}{2}\right) d = 0.398$$