Trigonometric Functions

1. $\int \cos u \, du = \sin u + c$

2. $\int \sin u \, du = -\cos u + c$

3. $\int \sec^2 u \, du = \tan u + c$

4. $\int \csc^2 u \, du = -\cot u + c$

5. $\int \sec u \tan u \, du = \sec u + c$

6. $\int \csc u \cot u \, du = -\csc u + c$

7. $\int \sin ax \cos bx \, dx = \frac{1}{2}\int (\sin(a-b)x + \sin(a+b)\,x)\, dx$

8. $\int \cos ax \cos bx \, dx = \frac{1}{2}\int (\cos(a-b)x + \cos(a+b)\,x)\, dx$

9. $\int \sin ax \sin bx \, dx = \frac{1}{2}\int (\cos(a-b)x - \cos\,(a+b)\,x)\, dx$

Inverse Trigonometric Functions

1. $\int \frac{du}{\sqrt{a^2 - u^2}} = \begin{cases} \sin^{-1}\frac{u}{a} + c \\ -\cos^{-1}\frac{u}{a} + c \end{cases}$

$$Proof: \int \frac{du}{\sqrt{a^2-u^2}} = \int \frac{du}{a\sqrt{1-\left(\frac{u}{a}\right)^2}} = \int \frac{\frac{1}{a}du}{\sqrt{1-\left(\frac{u}{a}\right)^2}} = \sin^{-1}\frac{u}{a} + c$$

2. $\int \frac{du}{a^2 + u^2} = \begin{cases} \frac{1}{a}\tan^{-1}\frac{u}{a} + c \\ -\frac{1}{a}\cot^{-1}\frac{u}{a} + c \end{cases}$

$$Proof: \int \frac{du}{a^2+u^2} = \int \frac{du}{a^2(1+\left(\frac{u}{a}\right)^2)} = \frac{1}{a}\int \frac{\frac{1}{a}du}{1+\left(\frac{u}{a}\right)^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c$$

3. $$\int \frac{du}{u\sqrt{u^2-a^2}} = \begin{cases} \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + c \\ -\frac{1}{a}\csc^{-1}\left|\frac{u}{a}\right| + c \end{cases}$$

$$Proof: \int \frac{du}{u\sqrt{u^2-a^2}} = \int \frac{du}{ua\sqrt{\left(\frac{u}{a}\right)^2-1}} = \frac{a}{a}\int \frac{\frac{1}{a}du}{u\sqrt{\left(\frac{u}{a}\right)^2-1}} = \frac{1}{a}\int \frac{\frac{1}{a}du}{\frac{u}{a}\sqrt{\left(\frac{u}{a}\right)^2-1}}$$

$$= \frac{1}{a}\sec^{-1}|u/a| + c$$

EXAMPLES

Example 1: Evaluate the integral

$$\int x^2\,dx$$

Solution

$$\int x^2\,dx = \frac{x^3}{3} \div c \quad Ans.$$

Example 2: Show that

$$\int \frac{x^4+1}{x^2\sqrt{x^4-1}}dx = \frac{\sqrt{x^4-1}}{x} + c$$

Solution

$$\frac{d}{dx}\left(\frac{\sqrt{x^4-1}}{x}\right) = \frac{x\left(\frac{4x^3}{2\sqrt{x^4-1}}\right) - \sqrt{x^4-1}}{x^2} = \frac{2x^4-(x^4-1)}{x^2\sqrt{x^4-1}} = \frac{x^4+1}{x^2\sqrt{x^4-1}} \qquad O.K.$$

Example 3: *Find $f(x)$ if*

$$\int f(x)\,dx = \tan^2 \pi x + c$$

Solution

$$\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}(\tan^2 \pi x + c) \Rightarrow f(x) = 2\pi \tan \pi x \sec^2 \pi x \qquad Ans.$$

Example 4: Evaluate the integral

$$\int \sqrt{2x+1}\,dx$$

Solution

Let $u = 2x + 1 \Rightarrow du = 2dx \Rightarrow dx = \frac{1}{2}du$

$$\int u^{\frac{1}{2}}\,\frac{1}{2}du = \frac{1}{2}\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c' = \frac{1}{3}(2x+1)^{\frac{3}{2}} + c \qquad Ans.$$

Example 5: Evaluate the integral

$$\int (x^3 + 2x)^{\frac{3}{2}}(3x^2 + 2)dx$$

Solution

Let $u = x^3 + 2x \Rightarrow du = (3x^2 + 2)dx$

$$\int (u)^{\frac{3}{2}}du = \frac{(u)^{\frac{5}{2}}}{\frac{5}{2}} + c' = \frac{(x^3+2x)^{\frac{5}{2}}}{\frac{5}{2}} + c \qquad Ans.$$

Example 6: Evaluate the integral

$$\int \frac{dx}{\sqrt{1-x^2}}$$

Solution

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c \qquad Ans.$$

Example 7: Evaluate the integral

$$\int \frac{dx}{\sqrt{1-4x^2}}$$

Solution

$$\int \frac{dx}{\sqrt{1-4x^2}} = \int \frac{dx}{\sqrt{1-(2x)^2}}$$

Let $u = 2x \Rightarrow du = 2dx \Rightarrow dx = \frac{1}{2}du$

$$\frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1} u + c' = \frac{1}{2}\sin^{-1} 2x + c \qquad Ans.$$

Example 8: Evaluate the integral

$$\int \frac{x\,dx}{\sqrt{1-4x^2}}$$

Solution

Let $u = 1 - 4x^2 \Rightarrow du = -8xdx \Rightarrow xdx = -\frac{1}{8}du$

$$-\frac{1}{8}\int \frac{du}{\sqrt{u}} = -\frac{1}{8}\frac{(u)^{\frac{1}{2}}}{\frac{1}{2}} + c' = -\frac{1}{8}\frac{(1-4x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c \qquad Ans.$$

Example 9: Evaluate the integral

$$\int \frac{x\,dx}{\sqrt{4-9x^4}}$$

Solution

$$\int \frac{x\,dx}{\sqrt{4-9x^4}} = \frac{1}{6}\int \frac{6x\,dx}{\sqrt{4-(3x^2)^2}} = \frac{1}{6}\sin^{-1}\frac{3x^2}{2} + c \qquad Ans.$$

Example 10: Evaluate the integral

$$\int \frac{x^3\,dx}{\sqrt{4-9x^4}}$$

Solution

$$\int \frac{x^3\,dx}{\sqrt{4-9x^4}} = -\frac{1}{36}\int \frac{-36x^3\,dx}{(4-9x^4)^{\frac{1}{2}}} = -\frac{1}{36}\frac{(4-9x^4)^{\frac{1}{2}}}{\frac{1}{2}} + c \qquad Ans.$$

Example 11: Evaluate the integral

$$\int \frac{dx}{9 + 4x^2}$$

Solution

$$\int \frac{dx}{9 + 4x^2} = \frac{1}{2}\int \frac{2dx}{(3)^2 + (2x)^2} = \frac{1}{6}\tan^{-1}\frac{2x}{3} + c \qquad Ans.$$

Example 12: Evaluate the integral

$$\int \frac{dx}{(2x-1)\sqrt{(2x-1)^2 - 4}}$$

Solution

$$\frac{1}{2}\int \frac{2dx}{(2x-1)\sqrt{(2x-1)^2 - (2)^2}} = \frac{1}{4}\sec^{-1}\left|\frac{2x-1}{2}\right| + c \qquad Ans.$$

Example 13: Evaluate the integral

$$\int \cos 2x \, dx$$

Solution

$$\int \cos 2x \, dx = \frac{1}{2}\sin 2x + c \qquad Ans.$$

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Example 14: Evaluate the integral

$$\int x \sin\frac{x^2}{2} \, dx$$

Solution

Let $u = \frac{1}{2}x^2 \Rightarrow du = xdx$

$$\int \sin u \, du = -\cos u + c' = -\cos\frac{x^2}{2} + c \qquad Ans.$$

Example 15: Evaluate the integral

$$\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}}\, dx$$

Solution

Let $u = \sqrt{x} \Rightarrow du = \frac{1}{2}\frac{dx}{\sqrt{x}} \Rightarrow \frac{dx}{\sqrt{x}} = 2du$

$$2\int \sec^2 u\, du = 2\tan u + c' = 2\tan\sqrt{x} + c \qquad Ans.$$

Example 16: Evaluate the integral

$$\int \sin x \cos^2 x\, dx$$

Solution

Let $u = \cos x \Rightarrow du = -\sin x\, dx \Rightarrow \sin x\, dx = -du$

$$-\int u^2\, du = \frac{-u^3}{3} + c' = \frac{-\cos^3 x}{3} + c \qquad Ans.$$

Example 17: Evaluate the integral

$$\int \frac{\cos x}{\sqrt{\sin x}}\, dx$$

Solution

Let $u = \sin x \Rightarrow du = \cos x\, dx$

$$\int \frac{du}{\sqrt{u}} = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c' = \frac{\sin^{\frac{1}{2}} x}{\frac{1}{2}} + c = 2\sqrt{\sin x} + c \qquad Ans.$$

Example 18: Evaluate the integral

$$\int \frac{\cos^3 x\, dx}{(\sin x)^{\frac{1}{2}}}$$

Solution

$$\int \frac{\cos^3 x\, dx}{(\sin x)^{\frac{1}{2}}} = \int \frac{(1-\sin^2 x)\cos x}{(\sin x)^{\frac{1}{2}}} = \int \cos x\,(\sin x)^{-\frac{1}{2}}\, dx - \int \cos x\,(\sin x)^{\frac{3}{2}}\, dx$$

$$= \frac{(\sin x)^{\frac{1}{2}}}{\frac{1}{2}} - \frac{(\sin x)^{\frac{5}{2}}}{\frac{5}{2}} + c = 2\sqrt{\sin x} - \frac{2}{5}(\sin x)^{\frac{5}{2}} + c \qquad Ans.$$

Example 19: Evaluate the integral

$$\int \frac{dx}{1+\cos x}$$

Solution

$$\int \frac{dx}{1+\cos x} = \int \frac{1}{1+\cos x} \times \frac{1-\cos x}{1-\cos x}\, dx = \int \frac{1-\cos x}{\sin^2 x}\, dx = \int \frac{dx}{\sin^2 x} - \int \frac{\cos x}{\sin^2 x}\, dx$$

$$= \int \csc^2 x\, dx - \int \csc x \cot x\, dx = -\cot x + \csc x + c \qquad Ans.$$

Note: the following integrals

$$\int \frac{dx}{1-\cos x}, \int \frac{dx}{1+\sin x}, \int \frac{dx}{1-\sin x}$$

can be solved by multiplying the integrals by $\frac{1+\cos x}{1+\cos x}, \frac{1-\sin x}{1-\sin x}$ *and* $\frac{1+\sin x}{1+\sin x}$, *respectively.*

Example 20: Evaluate the integral

$$\int \sqrt{1-\sin x}\, dx$$

Solution

$$\int \sqrt{1-\sin x}\, dx = \int \sqrt{1-\sin x} \times \frac{\sqrt{1+\sin x}}{\sqrt{1+\sin x}}\, dx = \int \frac{\sqrt{1-\sin^2 x}}{\sqrt{1+\sin x}}\, dx = \int \frac{\cos x}{(1+\sin x)^{\frac{1}{2}}}\, dx$$

$$= \frac{(1+\sin x)^{\frac{1}{2}}}{\frac{1}{2}} + c = 2\sqrt{1+\sin x} + c \qquad Ans.$$

Note: the following integrals

$$\sqrt{1+\sin x}, \sqrt{1-\cos x}, \sqrt{1+\cos x}$$

can be solved by multiplying the integrals by $\frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}$, $\frac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}}$, *and* $\frac{\sqrt{1-\cos x}}{\sqrt{1-\cos x}}$, *respectively.*

Example 21 (Even power of sine): Evaluate the integral

$$\int \sin^2 \frac{x}{2}\, dx$$

Solution

Remember the following equalities

$$\boxed{\begin{aligned} \sin^2 A &= \frac{1}{2}(1-\cos 2A) \\ \cos^2 A &= \frac{1}{2}(1+\cos 2A) \end{aligned}}$$

then

$$\int \sin^2 \frac{x}{2}\, dx = \frac{1}{2}\int (1-\cos x)dx = \frac{1}{2}(x-\sin x)+c \qquad Ans.$$

Example 22 (Even power of cosine): Evaluate the integral

$$\int \cos^4 2x\, dx$$

Solution

$$\int \cos^4 2x\, dx = \int \left(\frac{1+\cos 4x}{2}\right)^2 dx = \frac{1}{4}\int (1+2\cos 4x+\cos^2 4x)dx$$

$$= \frac{1}{4}\left(x+\frac{1}{2}\sin 4x+\frac{1}{2}\int (1+\cos 8x)\, dx\right) = \frac{3}{8}x+\frac{1}{8}\sin 4x+\frac{1}{64}\sin 8x \qquad Ans.$$

Example 23 (Odd power of sine): Evaluate the integral

$$\int \sin^3 2x\, dx$$

Solution

$$\int \sin^3 2x\, dx = \int \sin^2 2x \sin 2x\, dx \int (1-\cos^2 2x)\sin 2x\, dx$$

$$= \int \sin 2x\, dx - \int \cos^2 2x \sin 2x\, dx = \frac{-1}{2}\cos 2x + \frac{1}{2}\frac{\cos^3 2x}{3} + c \qquad Ans.$$

Example 24 (Even power of secant): Evaluate the integral

$$\int \sec^4 \frac{x}{2}\, dx$$

Solution

Remember the following equalities

$$\tan^2 A + 1 = \sec^2 x$$

$$\cot^2 A + 1 = \csc^2 x$$

then

$$\int \sec^4 \frac{x}{2}\, dx = \int \sec^2 \frac{x}{2} \sec^2 \frac{x}{2}\, dx = \int (1 + \tan^2 \frac{x}{2}) \sec^2 \frac{x}{2}\, dx$$

$$= \int \sec^2 \frac{x}{2}\, dx + \int \sec^2 \frac{x}{2} \tan^2 \frac{x}{2}\, dx = 2 \tan \frac{x}{2} + \frac{2}{3} \tan^3 \frac{x}{2} + c \qquad Ans.$$

Example 25: Evaluate the integral

$$\int \frac{\sin t \cos t}{\sqrt{3 \sin t + 5}}\, dt$$

Solution

Let $u = 3 \sin t + 5 \Rightarrow du = 3 \cos t\, dt \Rightarrow \cos t\, dt = \frac{du}{3}$

and

$$\sin t = \frac{u - 5}{3}$$

$$\int \frac{\sin t \cos t}{\sqrt{3 \sin t + 5}}\, dt = \int \frac{\frac{u-5}{3} \frac{du}{3}}{\sqrt{u}} = \int \frac{1}{9} \left(u^{-\frac{1}{2}} (u - 5) \right) du$$

$$= \frac{1}{9} \int \left(u^{\frac{1}{2}} - 5u^{-\frac{1}{2}} \right) du = \frac{1}{9} \left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{5\, u^{\frac{1}{2}}}{\frac{1}{2}} \right) + c'$$

$$= \frac{2}{27} (3 \sin t + 5)^{\frac{3}{2}} - \frac{10}{9} (3 \sin t + 5)^{\frac{1}{2}} + c \qquad Ans.$$

Example 26: Evaluate the integral

$$\int \frac{dx}{x^2+2x+5}$$

Solution

$$\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{x^2+2x+1-1+5} = \int \frac{dx}{(x+1)^2+2^2} = \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+c \quad Ans$$

Example 27: Evaluate the integral

$$\int \frac{dx}{\sqrt{-4x-x^2}}$$

Solution

$$\int \frac{dx}{\sqrt{-4x-x^2}} = \int \frac{dx}{\sqrt{-(4x+x^2)}} = \int \frac{dx}{\sqrt{-(2+x)^2+4}} = \int \frac{dx}{\sqrt{2^2-(2+x)^2}}$$

$$= \sin^{-1}\left(\frac{2+x}{2}\right)+c \quad Ans.$$

Example 28: Evaluate the integral

$$\int \frac{\tan 2x}{\sqrt{\sec^2 2x-4}}dx$$

Solution

$$\int \frac{\tan 2x}{\sqrt{\sec^2 2x-4}}dx = \int \frac{\tan 2x}{\sqrt{\sec^2 2x-4}} \times \frac{\sec 2x}{\sec 2x}dx = \int \frac{\sec 2x \tan 2x}{\sec 2x\sqrt{\sec^2 2x-4}}dx$$

Let $u = \sec 2x \Rightarrow du = 2\sec 2x \tan 2x\, dx$

$$\frac{du}{2} = \sec 2x \tan 2x\, dx$$

$$\frac{1}{2}\int \frac{du}{u\sqrt{u^2-2^2}} = \frac{1}{2}\times\frac{1}{2}\sec^{-1}\left|\frac{u}{2}\right| + c' = \frac{1}{4}\sec^{-1}\left|\frac{\sec 2x}{2}\right| + c \qquad Ans.$$

Example 29: Evaluate the integral

$$\int \sin\frac{x}{2}\cos x\, dx$$

Remember

$$\int \sin ax \cos bx \; dx = \frac{1}{2}\int (\sin(a-b)x + \sin(a+b)\,x)\,dx$$

Then

$$\frac{1}{2}\int \left(\sin\left(\frac{1}{2}-1\right)x + \sin\left(\frac{1}{2}+1\right)x\right)dx = \frac{1}{2}\int \sin -\frac{x}{2}dx + \frac{1}{2}\int \sin\frac{3x}{2}dx$$

$$= -\frac{1}{2}\int \sin\frac{x}{2}dx + \frac{1}{2}\int \sin\frac{3x}{2}dx = \cos\frac{x}{2} - \frac{1}{2}\times\frac{2}{3}\cos\frac{3x}{2} + c = \cos\frac{x}{2} - \frac{1}{3}\cos\frac{3x}{2} + c \quad Ans.$$

Example 30: Find $f(x)$ if $f''(x) = \sqrt{1-\sin x}$

Solution

$$f'(x) = \int f''(x)dx \quad and \quad f(x) = \int f'(x)dx$$

$$f'(x) = \int \sqrt{1-\sin x}\,dx = \int \sqrt{1-\sin x} \times \frac{\sqrt{1+\sin x}}{\sqrt{1+\sin x}}dx = \int \frac{\sqrt{1-\sin^2 x}}{\sqrt{1+\sin x}}\,dx$$

$$= \int \frac{\sqrt{\cos^2 x}}{\sqrt{1+\sin x}}\,dx = \int \frac{\cos x}{\sqrt{1+\sin x}}\,dx = 2\sqrt{1+\sin x} + c_1$$

$$f(x) = \int \left(2\sqrt{1+\sin x} + c_1\right)dx = 2\int \sqrt{1+\sin x}\,dx + c_1\int dx$$

$$= 2\int \sqrt{1+\sin x} \times \frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}dx + c_1\int dx = 2\int \frac{\cos x}{\sqrt{1-\sin x}}dx + c_1\int dx$$

$$= -4\sqrt{1-\sin x} + c_1 x + c_2 \quad Ans.$$

Example 31: Evaluate the integral

$$\int \frac{x^4+3x^2-5}{x^2+4}dx$$

Solution

$$\int \frac{x^4+3x^2-5}{x^2+4}dx$$

One can simplify the above rational expression as follows

$$
\begin{array}{r|l}
 & x^2 - 1 \\
\hline
x^2 + 4 & x^4 + 3x^2 - 5 \\
 & -(x^4 + 4x^2) \\
\hline
 & -x^2 - 5 \\
 & -(-x^2 - 4) \\
\hline
 & -1
\end{array}
$$

Then the simplified rational expression is as follows

$$\int \left((x^2 - 1) - \frac{1}{x^2 + 4} \right) dx = \int (x^2 - 1) dx - \int \frac{dx}{x^2 + 2^2} = \frac{x^3}{3} - x - \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + c \quad \textit{Ans}$$

Example 32: Evaluate the integral

$$\int \frac{\sin x}{16 + \cos^2 x} dx$$

Solution

$$\int \frac{\sin x}{4^2 + \cos^2 x} dx = -\frac{1}{4} \tan\left(\frac{\cos x}{4}\right) + c \quad \textit{Ans.}$$

Example 33: Evaluate the integral

$$\int \frac{dx}{(1 + x^2)(1 + (\tan^{-1} x)^2)}$$

Solution

Let $u = \tan^{-1} x$

$$du = \frac{dx}{x^2 + 1}$$

$$\int \frac{du}{(1 + u^2)} = \tan^{-1} u + c' = \tan^{-1}(\tan^{-1} x) + c \quad \textit{Ans.}$$