1 INEQUALITIES

1.1 INTERVALS

A subset of the real line is called an **interval** if it contains at least two numbers and contains all the real numbers lying between any two of its elements.

Geometrically, intervals correspond to rays and line segments on the real line, along with the real line itself. Intervals of numbers corresponding to line segments are:

1. **Finite intervals.**
2. **Infinite intervals**.

A finite interval can be classified into:

1. **Closed:** if it contains both of its endpoints.
2. **Half-open:** if it contains one endpoint but not the other.
3. **Open:** if it contains neither endpoint.

The endpoints are also called **boundary points**; they make up the interval's boundary. The remaining points of the interval are **interior points** and together comprise the interval's interior. Infinite intervals are closed if they contain a finite endpoint, and open otherwise.

Types of Intervals

	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \le x \le b\}$	Closed	
	$[a, b)$	$\{x \mid a \le x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \le b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \ge a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \le b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

1.2 INEQUALITIES

The process of finding the interval or intervals of numbers that satisfy an inequality in x is called **solving the inequality**

For example, the solution of the following simple equation

$$5x - 3 = 2$$

Is

$$x = 1$$

But the solution of the following inequality

$$5x - 3 \leq 2$$

Is

$$x \leq 1$$

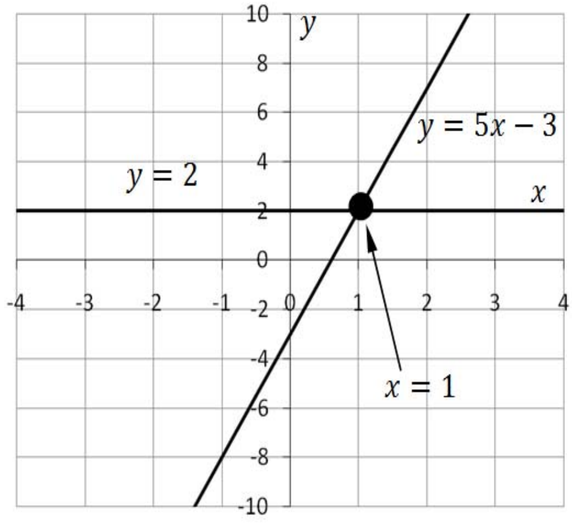
Or in a set notation

$$\{x: x \leq 1\}$$

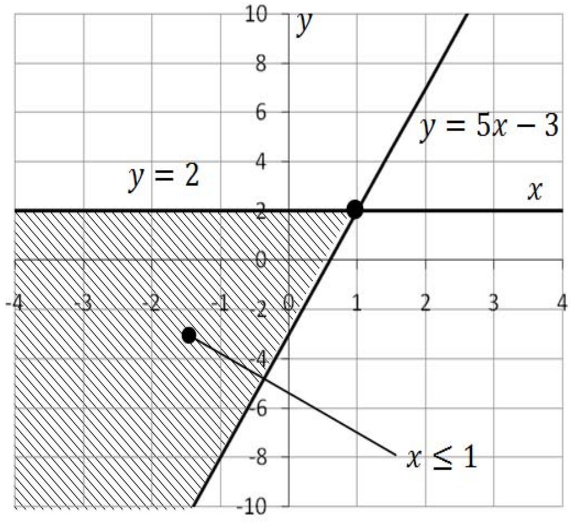
Or in interval notation

$$(-\infty, 1]$$

The following graphs illustrate the difference between solution of the equation $5x - 3 = 2$ and the solution of the inequality $5x - 3 \leq 2$



The solution of $5x - 3 = 2$ is one real number



The solution of $5x - 3 \leq 2$ is a set of real numbers

Rules for Inequalities

If a, b, and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow bc < ac$
   Special case: $a < b \Rightarrow -b < -a$
5. $a > 0 \Rightarrow \frac{1}{a} > 0$
6. If a and b are both positive or both negative, then $a < b \Rightarrow \frac{1}{b} < \frac{1}{a}$

EXAMPLES

Example 1.1

Solve the inequality

$$2x - 1 < x + 3$$

Solution

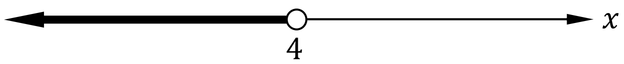
$$2x - x < 3 + 1 \Rightarrow x < 4 \quad Ans.$$

The solution can be written into the following notations:

Interval notation : $(-\infty, 4)$

Set notation: $\{x: x < 4\}$

Real line notation:



Example 1.2

Solve the inequality

$$-\frac{x}{3} < 2x + 1$$

Solution

$$-x < 6x + 3 \Rightarrow -7x < 3$$

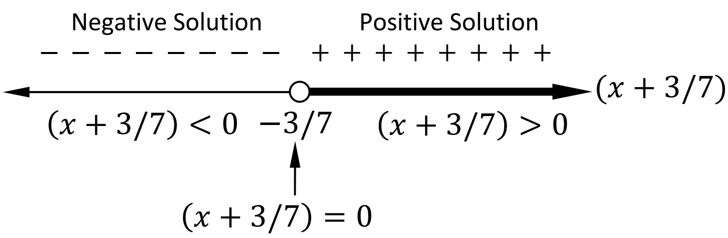
$$7x > -3 \Rightarrow x > -\frac{3}{7}\ , \quad \left(-\frac{3}{7}, \infty\right) \quad Ans.$$

Or by using the real line as follows:

$$x + \frac{3}{7} > 0$$

To find the root

$$\left(x + \frac{3}{7}\right) = 0 \;\Rightarrow\; x = -\frac{3}{7}$$



Then the solution is

$$\left(-\frac{3}{7}, \infty\right) \quad Ans.$$

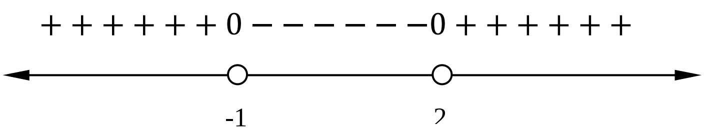
Example 1.3

Solve the inequality

$$\frac{x+1}{x-2} > 0$$

Solution

$$x + 1 = 0 \;\Longrightarrow x = -1$$
$$x - 2 = 0 \;\Longrightarrow x = 2$$



$$Ans. (-\infty\,, -2) \cup (2\,, \infty)$$

Example 1.4

Solve the inequality

$$\frac{x-1}{x+2} \le \frac{1}{x}$$

Solution

$$\frac{x-1}{x+2}-\frac{1}{x}\le 0 \quad \Rightarrow \quad \frac{x(x-1)-(x+2)}{x(x+2)}\le 0 \quad \Rightarrow \quad \frac{x^2-2x-2}{x(x+2)}\le 0$$

Find the roots for the numerator

$x^2-2x-2=0$

To find the roots of the above "Quadratic Equation" use the following general formula

$$If \qquad \boldsymbol{Ax^2+Bx+C=0} \qquad then \qquad \boldsymbol{x=\frac{-B\mp\sqrt{B^2-4AC}}{2A}}$$

Then

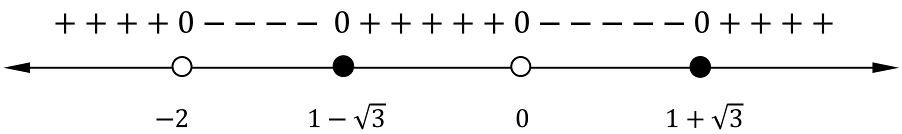
$$x=\frac{2\mp\sqrt{2^2-4(1)(-2)}}{2}=\frac{2\mp\sqrt{12}}{2}=\frac{2\mp\sqrt{4\times 3}}{2}=\frac{2\mp 2\sqrt{3}}{2}=1\mp\sqrt{3}=$$

Or $\left(x-1-\sqrt{3}\right)\left(x-1+\sqrt{3}\right)=0$

Find the roots for the denominator

$x=0$ (Denominator $\neq 0, i.e., x\neq 0$)

$x=-2$ (Denominator $\neq 0, i.e., x\neq -2$)



$$Ans.\ (-2\ ,1-\sqrt{3}\,]\ \cup (0\ ,1+\sqrt{3}\,]$$

Example 1.5

Solve the inequality

$$-1<\frac{x^2-1}{x+2}<1$$

Solution

One can rewrite the above inequality in terms of intersection notation as follows

$$-1<\frac{x^2-1}{x+2} \qquad \cap \qquad \frac{x^2-1}{x+2}<1$$

Or

$$\frac{x^2-1}{x+2}>-1 \qquad \cap \qquad \frac{x^2-1}{x+2}<1$$

$$\frac{x^2-1}{x+2} > -1 \qquad \cap \qquad \frac{x^2-1}{x+2} < 1$$

$$\frac{x^2-1}{x+2} + 1 > 0 \qquad \cap \qquad \frac{x^2-1}{x+2} - 1 < 0$$

$$\frac{x^2-1+x+2}{x+2} > 0 \qquad \cap \qquad \frac{x^2-1-x-2}{x+2} < 0$$

$$\frac{x^2+x+1}{x+2} > 0 \qquad \cap \qquad \frac{x^2-x-3}{x+2} < 0$$

$$x^2+x+1=0$$

$$x = \frac{-1 \mp \sqrt{1-4}}{2}$$

$$= \frac{-1 \mp \sqrt{3}\boldsymbol{i}}{2} \quad where \quad \boldsymbol{i} = \sqrt{-1}$$

Then the roots are complex

(i.e., there is no real root)

And $x^2+x+1>0$ for any value of x

$x+2=0 \Rightarrow x=-2$

$$x^2-x-3=0$$

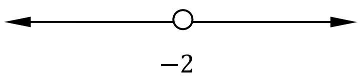
$$x = \frac{1 \mp \sqrt{1-4\times 3}}{2}$$

$$= \frac{1 \mp \sqrt{13}}{2}$$

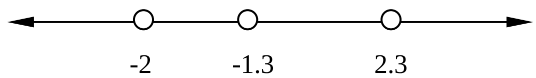
$x_1 = -1.3 \quad and \quad x_2 = 2.3$

$x+2=0 \Rightarrow x=-2$

$- - - - - 0 + + + +$



$- - - 0 + + + 0 - - - - 0 + + +$



$$x > -2 \qquad \cap \qquad \{(-\infty, -2) \cup (-1.3, 2.3)\}$$

$Ans.\ (-1.3, 2.3)$

HOMEWORK

Solve the following inequalities and show their solution sets on the real line.

HW 1.1

$$5x - 3 \leq 7 - 3x$$

HW 1.2

$$2x - \frac{1}{2} \geq 7x + \frac{7}{6}$$

HW1. 3

$$\frac{6 - x}{4} < \frac{3 - 4}{2}$$

HW 1.4

$$\frac{1}{9} < x^2 < \frac{1}{4}$$

HW 1.5

$$-1 < \frac{1 + x}{1 - x} \leq 1$$

HW 1.6

$$\frac{3x - 1}{x} \leq 2x$$

HW 1.7

$$(x + 3)^2 < 2$$

HW 1.8

$$x^2 - x - 2 \geq 0$$

HW 1.9

$$-\frac{x + 5}{2} \leq \frac{12 + 3x}{4}$$

HW 1.10

$$-1 < \frac{x^2 - 2}{x + 1} < 2$$

1.3 ABSOLUTE VALUE

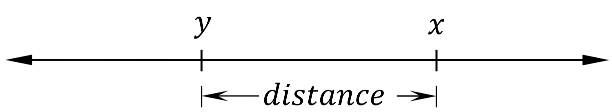
The **absolute value** of a number x, denoted by $|x|$, is defined by the formula

$$|x| = \begin{cases} x & , x \geq 0 \\ -x & x < 0 \end{cases}$$

Geometrically, the absolute value of x is the distance from x to 0 on the real number line. Since **distances are always positive** or **0**, we see that $|x| \geq 0$ for every real number x, and $|x| = 0$ if and only if $x = 0$. Also,

$$|x - y| = the\ distance\ between\ x\ and\ y$$

on the real line



Definition of the absolute value by using the square root

$$|x| = \sqrt{x^2}$$

Note: Remember that $\sqrt{a^2} = \mp a$, if $a \geq 0$, then $\sqrt{a^2} = a$.

The absolute value has the following properties

Absolute Value Properties

1. $|-a| = |a|$ — A number and its additive inverse or negative have the same absolute value.
2. $|ab| = |a||b|$ — The absolute value of a product is the product of the absolute values.
3. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ — The absolute value of a quotient is the quotient of the absolute values.
4. $|a + b| \leq |a| + |b|$ — The **triangle inequality**. The absolute value of the sum of two numbers is less than or equal to the sum of their absolute values.

The following statements are all consequences of the definition of absolute value and are often helpful when solving equations or inequalities involving absolute values

Absolute Values and Intervals

If a is any positive number, then

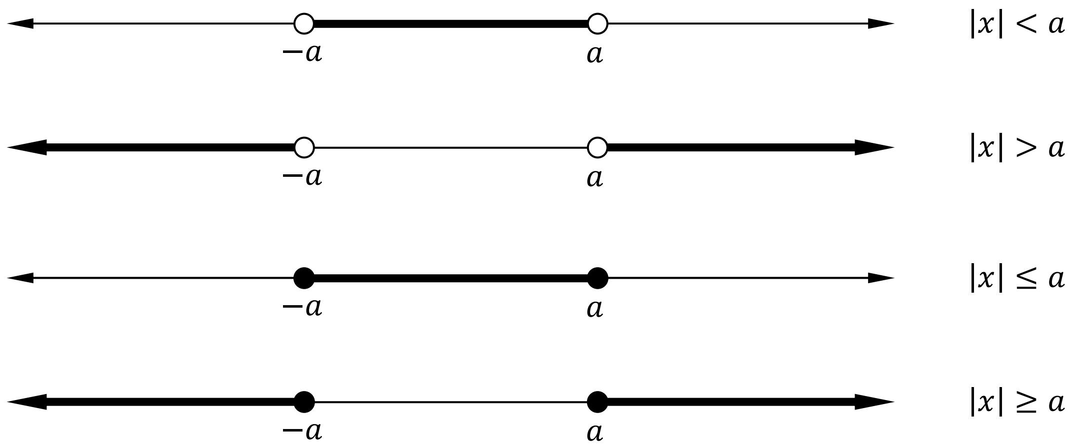
5. $|x| = a$ if and only if $x = \pm a$

6. $|x| < a$ if and only if $-a < x < a$

7. $|x| > a$ if and only if $x > a$ or $x < -a$

8. $|x| \leq a$ if and only if $-a \leq x \leq a$

9. $|x| \geq a$ if and only if $x \geq a$ or $x \leq -a$



EXAMPLES

Example 2.1

Solve the equation

$$|2x - 3| = 7$$

Solution

$$2x - 3 = \mp 7 \quad \Leftrightarrow \quad |x| = a \ \ then \ \ x = \mp a$$

Either $2x - 3 = +7 \quad \Rightarrow \quad x = 5$ or $2x - 3 = -7 \quad \Rightarrow \quad x = -2$

Then the solutions are $x = 5$ and $x = -2 \quad Ans.$

Example 2.2

Solve the equation

$$x^2 - 2|x| - 3 = 0$$

Solution

The solution will be

$$x^2 - 2x - 3 = 0 \quad for \quad x \geq 0$$

$$x^2 + 2x - 3 = 0 \quad for \quad x < 0$$

Simplify the above equations, yields

$$(x-3)(x+1) = 0 \quad for \quad x \geq 0$$

Either $x = 3$

Or $x = -1$ this solution should be ignored because $x \geq 0$

$$(x+3)(x-1) = 0 \quad for \quad x < 0$$

Either $x = -3$

Or $x = 1$ this solution should be ignored because $x < 0$

Then the solutions are $x = 3$ and $x = -3$ $Ans.$

Example 2.3

Solve the inequality

$$\left|5 - \frac{2}{x}\right| < 1$$

Solution

Use property 6, $|x| < a \Leftrightarrow -a < x < a$

$$-1 < 5 - \frac{2}{x} < 1 \quad \Leftrightarrow \quad -6 < -\frac{2}{x} < -4 \quad \Leftrightarrow \quad -3 < -\frac{1}{x} < -2$$

$$3 > \frac{1}{x} > 2 \quad \Leftrightarrow \quad \frac{1}{3} < x < \frac{1}{2}$$

Then the solution is

$$\left(\frac{1}{3}, \frac{1}{2}\right) \; Ans.$$

Example 2.4

Solve the inequality

$$\frac{|x+2|}{|x-1|} < 1$$

Solution

$$\left|\frac{x+2}{x-1}\right| < 1$$

$$-1 < \frac{x+2}{x-1} < 1$$

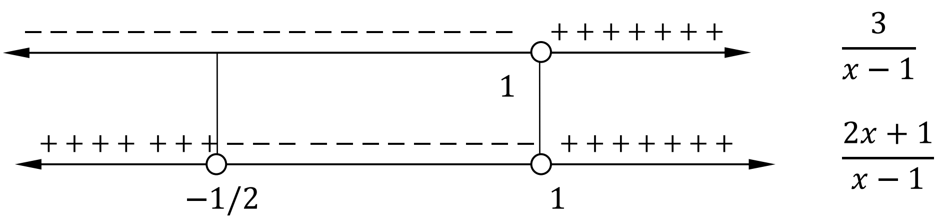
Or

$$-1 < \frac{x+2}{x-1} \quad \cap \quad \frac{x+2}{x-1} < 1$$

$$\frac{x+2}{x-1} < 1 \quad \cap \quad \frac{x+2}{x-1} > -1$$

$$\frac{x+2}{x-1} - 1 < 0 \quad \cap \quad \frac{x+2}{x-1} + 1 > 0$$

$$\frac{3}{x-1} < 0 \quad \cap \quad \frac{2x+1}{x-1} > 0$$



Then the solution is $(-\infty, 1,) \cap \{(-\infty, -1/2) \cup (1, \infty)\}$

Or

$Ans. \left(-\infty, -\frac{1}{2}\right)$

Example 2.5

Solve the inequality

$$\frac{|x-1|}{x+2} \leq 1$$

Solution

$$|x-1| = \begin{cases} x-1 & when \quad x-1 \geq 0 \Rightarrow x \geq 1 \\ -(x-1) & when \quad x-1 < 0 \Rightarrow x < 1 \end{cases}$$

$$\frac{|x-1|}{x+2} - 1 \leq 0$$

$$when \quad x-1 \geq 0 \Rightarrow x \geq 1 \qquad\qquad when \quad x-1 < 0 \Rightarrow x < 1$$

$$\left\{\left(\frac{x-1}{x+2} - 1 \leq 0\right) \cap (x \geq 1)\right\} \quad \cup \quad \left\{\left(\frac{-(x-1)}{x+2} - 1 \leq 0\right) \cap (x < 1)\right\}$$

$$\left\{\left(\frac{x-1-(x+2)}{x+2} \leq 0\right) \cap (x \geq 1)\right\} \quad \cup \quad \left\{\left(\frac{-(x-1)-(x+2)}{x+2} \leq 0\right) \cap (x < 1)\right\}$$

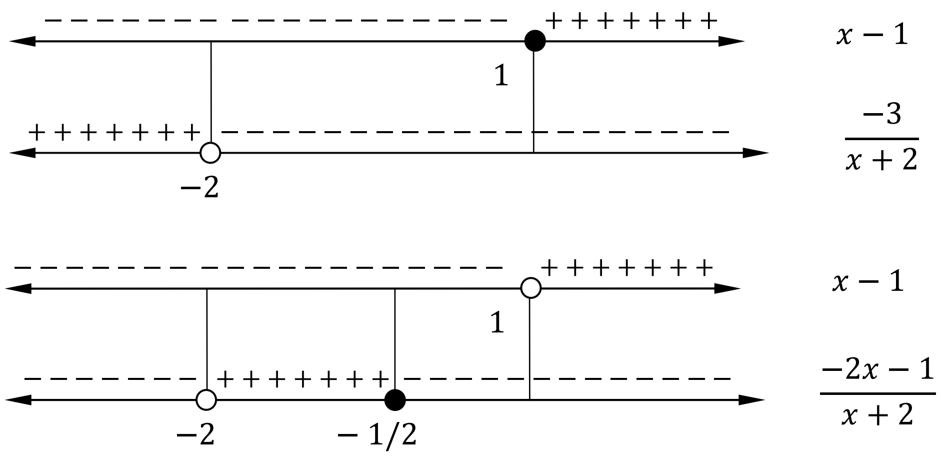
$$\left\{\left(\frac{x-1-x-2}{x+2} \leq 0\right) \cap (x \geq 1)\right\} \quad \cup \quad \left\{\left(\frac{-x+1-x-2}{x+2} \leq 0\right) \cap (x < 1)\right\}$$

$$\left\{\left(\frac{-3}{x+2} \leq 0\right) \cap (x-1) \geq 0\right\} \quad \cup \quad \left\{\left(\frac{-2x-1}{x+2} \leq 0\right) \cap (x-1) < 0\right\}$$

The solution will be

$$[1, \infty) \quad \cup \quad (-\infty, -2) \cup [-1/2\ ,1)$$

or $[1, \infty) \cup \{(-\infty, -2) \cup [-1/2\ ,1)\} = (-\infty, 2) \cup [-1/2\ , \infty)$ $Ans.$



Example 2.6

Solve the inequality

$$\left|\frac{2x+5}{x}\right| \geq 1$$

Solution

$$\frac{2x+5}{x} \geq 1 \qquad \cup \qquad \frac{2x+5}{x} \leq -1$$

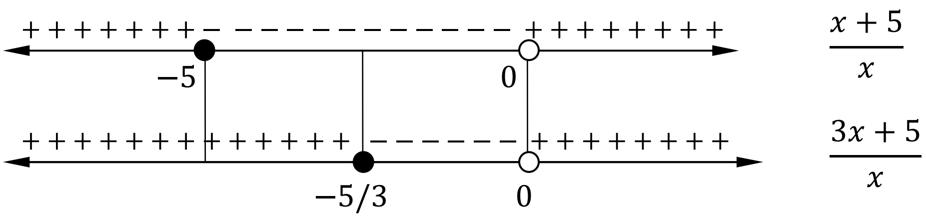
$$\frac{2x+5}{x} - 1 \geq 0 \qquad \cup \qquad \frac{2x+5}{x} + 1 \leq 0$$

$$\frac{2x+5-x}{x} \geq 0 \qquad \cup \qquad \frac{2x+5+x}{x} \leq 0$$

$$\frac{x+5}{x} \geq 0 \qquad \cup \qquad \frac{3x+5}{x} \leq 0$$

The roots are

$$x = -5, \quad x \neq 0 \qquad \cup \qquad x = -\frac{5}{3}, \quad x \neq 0$$



The solution will be

$$\{(-\infty, -5] \cup (0, \infty)\} \qquad \cup \qquad [-5/3, 0)$$

or

$$(-\infty, -5] \cup [-5/3, 0) \cup (0, \infty) \qquad Ans.$$

HOMEWORK

Solve the following inequalities and show their solution sets on the real line.

HW 2.1

$$\left|\frac{3x}{5} - 1\right| > \frac{2}{5}$$

HW 2.2

$$\left|\frac{2}{x} - 4\right| < 3$$

HW 2.3

$$\frac{2}{|x-1|} > 1$$

HW 2.4

$$\frac{|x-2|}{x} < |x+2|$$

HW 2.5

$$|2+x| + |x+1| < 16$$

HW 2.6

$$\frac{5}{|x-2x^2|} > \frac{1}{x}$$

HW 2.7

$$\left|\frac{x+3}{x+1}\right| < 2|x-2|$$

HW 2.8

$$\left|\frac{3}{1+x} - \frac{2x}{1+x}\right| \le 4$$

HW 2.9

$$\frac{|2x+6|}{x-1} > 1$$

HW 2.10

$$\left|\frac{1}{x^3+3} + \frac{1}{|x|+2}\right| \le \frac{5}{6}$$

2 FUNCTIONS

2.1 FUNCTIONS AND THEIR GRAPHS

Functions are the major objects we deal with in calculus because they **are key to describing the real world in mathematical terms**. This section reviews the ideas of functions, their graphs, and ways of representing them.

Functions; Domain and Range

If the value of one variable y is completely determined by the value of x (that is, y depends on the value of x) we say that y is a function of x. Often the value of y is given by a **rule** or **formula** that says how to calculate it from the variable x. For instance, the equation $A = \pi r^2$ is a rule that calculates the area A of a circle from its radius r.

A symbolic way to say y is a function of x is by writing

$$y = f(x) \qquad (y\ equals\ f\ of\ x)$$

In this notation:

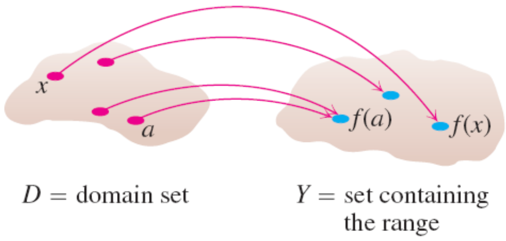
- The symbol f represents the function.
- The letter x, called **the independent variable**, represents the input value of f.
- y, **the dependent variable**, represents the corresponding output value of f at x.

Definition of a Function

> **DEFINITION Function**
> A **function** from a set D to a set Y is a rule that assigns a *unique* (single) element $f(x) \in Y$ to each element $x \in D$.

The set D of all possible input values is called the **domain** of the function. The set of all values of $f(x)$ as x varies throughout D is called the **range** of the function. The range may not include every element in the set Y.

A function can also be pictured as an arrow diagram. Each arrow associates an element of the domain D to a **unique** or **single** element in the set Y.



For real-valued domains and ranges, the following points should be satisfied:

1. We cannot divide by zero. The denominator cannot be zero.
2. Any value under the square root cannot be negative.

EXAMPLES

Example 2.1

Verify the domains and ranges of the following functions

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Solution

Example 2.1a

$$y = x^2$$

The domain is $D_f = (-\infty, \infty)$

One can substitute the domain into the function to find the range. Or on can find x as a function of y to find the range:

$$x = \sqrt{y} \qquad , y \geq 0$$

y cannot be negative $R_f = [0, \infty)$

Example 2.1b

$$y = \frac{1}{x}$$

$$x \neq 0$$

Then the domain is $D_f = (-\infty, 0) \cup (0, \infty)$

$$x = \frac{1}{y}$$

$$y \neq 0$$

Then the range is $R_f = (-\infty, 0) \cup (0, \infty)$

Example 2.1c

$$y = \sqrt{x}$$

$$x \geq 0$$

Then the domain is $D_f = [0, \infty)$

By substituting the domain into y one can find the range

The range is $R_f = [0, \infty)$

Example 2.1d

$$y = \sqrt{4 - x}$$

$$4 - x \geq 0 \quad \Rightarrow \quad x \leq 4 \quad \Rightarrow \quad y \geq 0$$

Then the domain is $D_f = (-\infty, 4]$

By substituting the domain into y one can find the range

The range is $R_f = [0, \infty)$

Or

$$y^2 = 4 - x \quad \Rightarrow \quad x = 4 - y^2 \quad but \quad y \geq 0$$

Then $R_f = (-\infty, \infty) \cap [0, \infty) = [0, \infty)$

Example 2.1e

$$y = \sqrt{1 - x^2}$$

$$1 - x^2 \geq 0 \quad \Rightarrow \quad x^2 \leq 1 \quad \Rightarrow \quad \sqrt{x^2} \leq \sqrt{1} \quad \Rightarrow \quad |x| \leq 1 \quad \Rightarrow -1 \leq x \leq 1$$

The domain is $D_f = [-1,1]$

From the above function

$$y \geq 0 \quad (Generally)$$

But

$$y^2 = 1 - x^2 \quad \Rightarrow \quad x^2 = 1 - y^2 \quad \Rightarrow \quad \sqrt{x^2} = \sqrt{1 - y^2}$$

$$1 - y^2 \geq 0 \Rightarrow \ -1 \leq y \leq 1$$

Then $R_f = \{y \geq 0\} \cap \{-1 \leq y \leq 1\} = [0,1]$

Example 2.2

Find the domain and range of the function

$$y = \frac{1}{x - 2}$$

Solution

$$x - 2 \neq 0 \quad \Rightarrow \quad x \neq 2$$

The domain is $D_f = (-\infty, 2) \cup (2, \infty)$

From the above function

$$-\infty < y < \infty \quad (Generally)$$

But

$$x - 2 = \frac{1}{y} \quad \Rightarrow \quad x = \frac{1}{y} + 2$$

$$y \neq 0$$

The range is $R_f = (-\infty, 0) \cup (0, \infty)$

Example 2.3

Find the domain and range of the function

$$y = \sqrt{\sqrt{x} - 1}$$

Solution

$$\sqrt{x} - 1 \geq 0 \quad \Rightarrow \quad \sqrt{x} \geq 1 \quad \Rightarrow \quad x \geq 1$$

The domain is $D_f = [1, \infty)$

From the above function

$$y \geq 0 \quad (Generally)$$

But

$$y^2 = \sqrt{x} - 1 \quad \Rightarrow \quad \sqrt{x} = y^2 + 1 \quad \Rightarrow \quad x = (y^2 + 1)^2$$

$$-\infty < y < \infty$$

Then $R_f = \{y \geq 0\} \cap \{-\infty < y < \infty\} = [0, \infty)$

Example 2.4

Find the domain and range of the function

$$x^2 + y^2 = 1$$

Solution

$$y^2 = 1 - x^2 \quad \Rightarrow \quad \sqrt{y^2} = \sqrt{1 - x^2} \quad \Rightarrow \quad y = \pm\sqrt{1 - x^2}$$

$$1 - x^2 \geq 0 \quad \Rightarrow \quad -1 \leq x \leq 1$$

The domain is $D_f = [-1,1]$

$$x^2 = 1 - y^2 \quad \Rightarrow \quad \sqrt{x^2} = \sqrt{1 - y^2} \quad \Rightarrow \quad x = \pm\sqrt{1 - y^2}$$

$$1 - y^2 \geq 0 \quad \Rightarrow \quad -1 \leq y \leq 1$$

The range is $R_f = [-1,1]$

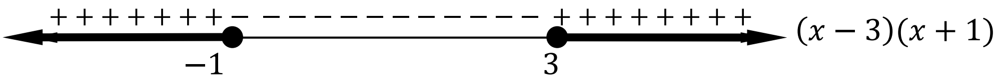
Example 2.5

Find the domain and range of the function

$$y = \sqrt{x^2 - 2x - 3}$$

Solution

$$x^2 - 2x - 3 \geq 0 \quad \Rightarrow \quad (x-3)(x+1) \geq 0$$



The domain is $D_f = (-\infty, -1] \cup [3, \infty)$

From the above function

$$y \geq 0 \quad (Generally)$$

But

$$y^2 = x^2 - 2x - 3 \quad \Rightarrow \quad x^2 - 2x - (3 + y^2) = 0$$

$$x = \frac{2 \pm \sqrt{4 + 4(3 + y^2)}}{2} = 1 \pm \sqrt{4 + y^2}$$

$$4 + y^2 \geq 0 \quad or \quad 4 + y^2 \neq -ve$$

$$-\infty < y < \infty$$

Then $R_f = \{y \geq 0\} \cap \{-\infty < y < \infty\} = [0, \infty)$

Example 2.6

Find the domain and range of the function

$$y = \frac{x^2 + 1}{x^2 - 1}$$

Solution

$$x^2 - 1 \neq 0 \quad \Rightarrow \quad x \neq \pm 1$$

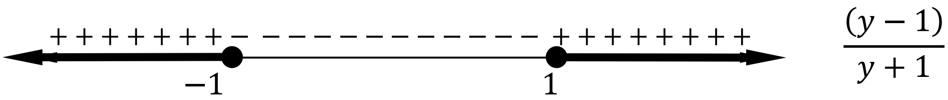
The domain is $D_f = \mathbb{R} \setminus \{-1, 1\}$

Where $\mathbb{R} = (-\infty, \infty)$, the set of real numbers

$$y(x^2-1)=x^2+1 \quad \Rightarrow \quad yx^2-y-x^2-1=0$$

$$x^2(y-1)=y+1 \quad \Rightarrow \quad x^2=\frac{y-1}{y+1} \quad \Rightarrow \quad x=\sqrt{\frac{y-1}{y+1}}$$

$$\frac{y-1}{y+1}\geq 0$$



The range is $R_f=(-\infty,-1]\cup[1,\infty)$

HOMEWORK

Find the domains and ranges of following functions

HW 2.1

$$x^2-y^2=1$$

HW 2.2

$$y=\sqrt{x}-\sqrt{x-3}$$

HW 2. 3

$$\sqrt{y+5}\times\sqrt{x-3}=4$$

HW 2. 4

$$\sqrt{x+5}-\sqrt{y-3}=4$$

HW 2. 5

$$y=\sqrt[3]{x}+\frac{1}{\sqrt{x-1}}$$

HW 2. 6

$$y=\sqrt{\frac{x-3}{4-x^2}}$$

2.2 IDENTIFYING FUNCTIONS AND MATHEMATICAL MODELS

One can classify the ordinary functions into the followings:

1. Algebraic Functions. These functions can be subdivided into the following functions:
   a. Linear Functions.
   b. Nonlinear Functions. This family of functions includes; Power Functions, Polynomials and Rational Functions.
2. Transcendental Functions. These functions are not algebraic they include:
   a. Trigonometric Functions.
   b. Exponential Functions.
   c. Logarithmic Functions.

Algebraic Functions

An algebraic function is a function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots).

Linear Functions

A function of the form

$$f(x) = m\,x + \;b,$$

for constants m and b, is called a linear function.

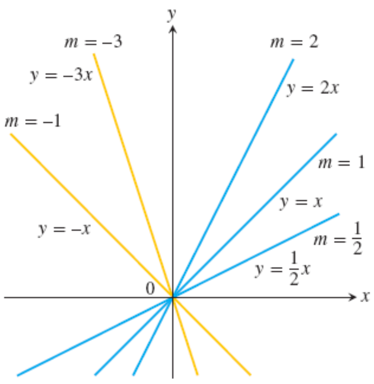


FIG e collection of lines $y = mx$ has slope m and all lines pass through the origin.

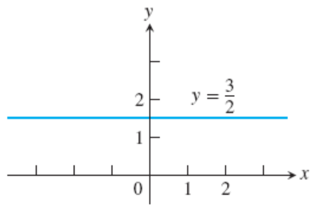


FIG onstant function has slope $m = 0$.

Power Functions

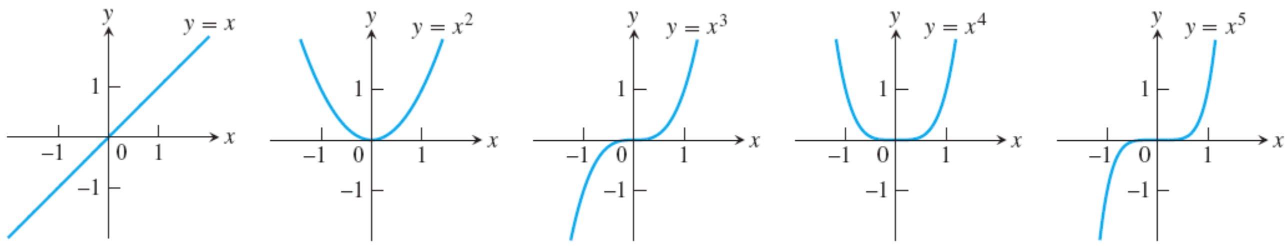
A function

$$f(x) = x^a$$

where a is a constant, is called a power function. There are several important cases to consider.

a. $a = n$, a positive integer.

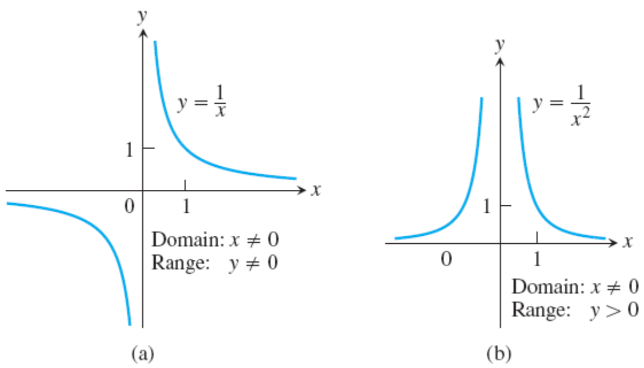
The graphs of $f(x) = x^n$, for $n = 1, 2, 3, 4, 5$, are displayed in figure shown. These functions are defined for all real values of x. Notice that as the power gets larger, the curves tend to flatten toward the x −axis on the interval $(1, 1)$, and also rise more steeply for $|x| > 1$. Each curve passes through the point $(1, 1)$ and through the origin.



Graphs of $f(x) = x^n, n = 1, 2, 3, 4, 5$ defined for $-\infty < x < \infty$.

b. $a = -1 \quad or \quad a = -2$

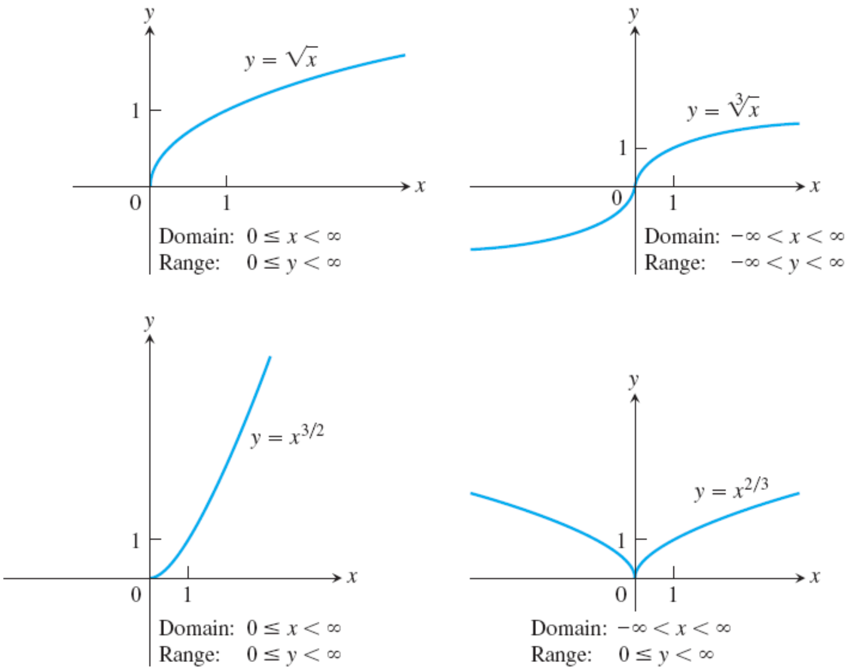
The graphs of the functions $f\ (x) = x^{-1} = 1/x$ and $g(x) = \ x^{-2} = 1/x^2$ are shown in the figure. Both functions are defined for all $x \neq 0$ (you can never divide by zero). The graph of $y = 1/x$ is the hyperbola $xy = 1$ which approaches the coordinate axes far from the origin. The graph of $y = 1/x^2$ also approaches the coordinate axes.



FI aphs of the power functions $f(x) = x^a$ for part (a) $a = -1$ and for part (b) $a = -2$.

a. $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$ and $\frac{2}{3}$

The functions $f(x) = x^{1/2} = \sqrt{x}$ and $g(x) = x^{1/3} = \sqrt[3]{x}$ are the **square root** and **cube root** functions, respectively. The domain of the square root function is $[0, \infty)$, but the cube root function is defined for all real x. Their graphs are displayed in the following figures.



Polynomials

A function p is a polynomial if

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_o$$

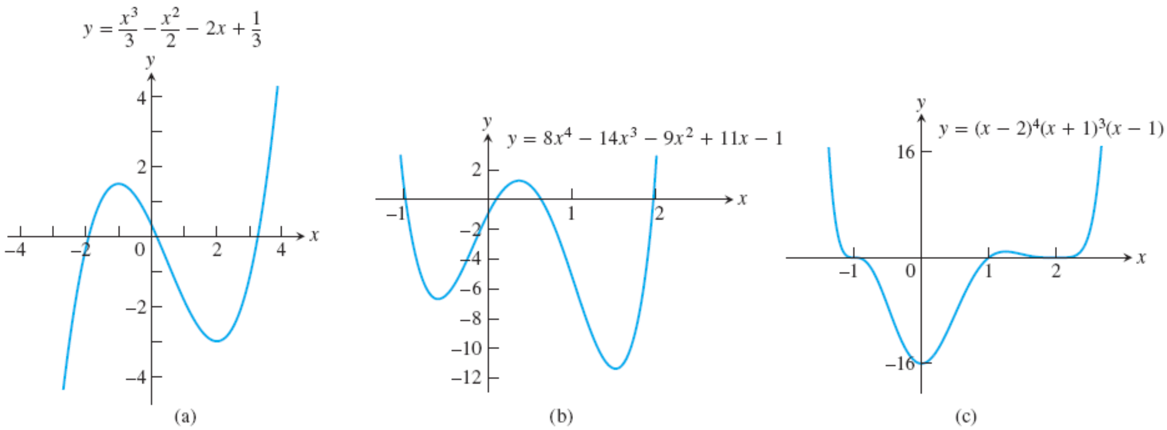
where n is a nonnegative integer and the numbers a_o , $a_1, a_2,$ … , a_n are real constants (called the **coefficients of the polynomial**). All polynomials have domain $(-\infty, \infty)$. If the leading coefficient $a_n \neq 0$ and $n > 0$, then n is called the **degree of the polynomial**.

Polynomials of degree 1, $p(x) = a_1 x + a_o$, are called **Linear functions**.

Polynomials of degree 2, $p(x) = ax^2 + bx + c$, are called **Quadratic functions**.

Polynomials of degree 3, $p(x) = ax^3 + bx^2 + ex + d$, are called **Cubic functions**.

The figure below shows the graphs of three polynomials.



Graph of three Polynomial Functions

Rational Functions

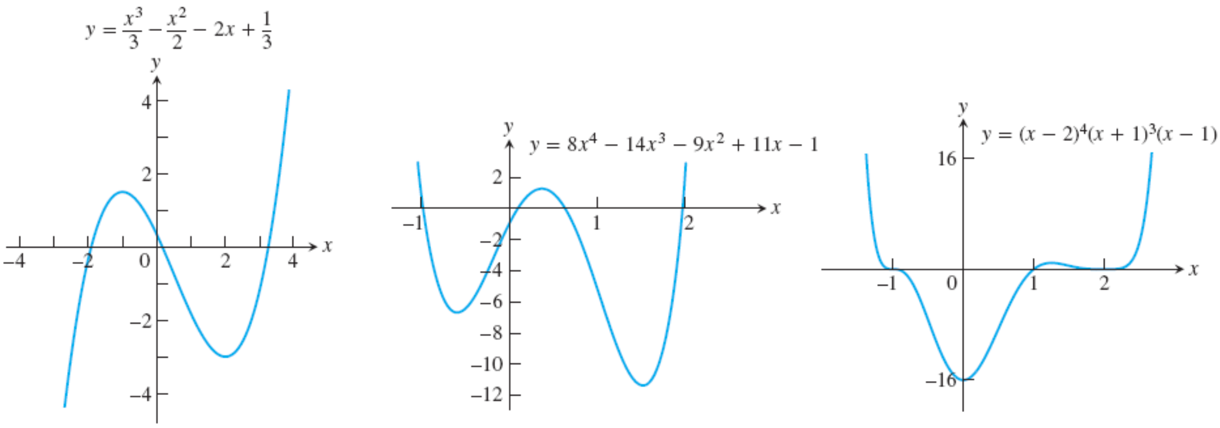
A rational function is a quotient or ratio of two polynomials:

$$f(x) = \frac{P(x)}{q(x)}$$

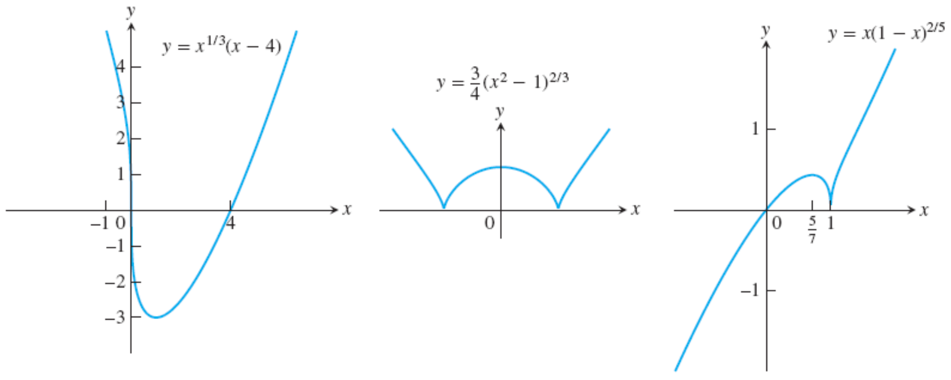
where p and q are polynomials. The domain of a rational function is the set of all real x for which $q(x) \neq 0$. For example, the function

$$f(x) = \frac{2x^2 - 3}{7x + 4}$$

is a rational function with domain$\{x : x \neq 4/7\}$. The graph of these type types of functions are shown below



Also, the following figure shows general algebraic functions



Transcendental Functions

These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions.

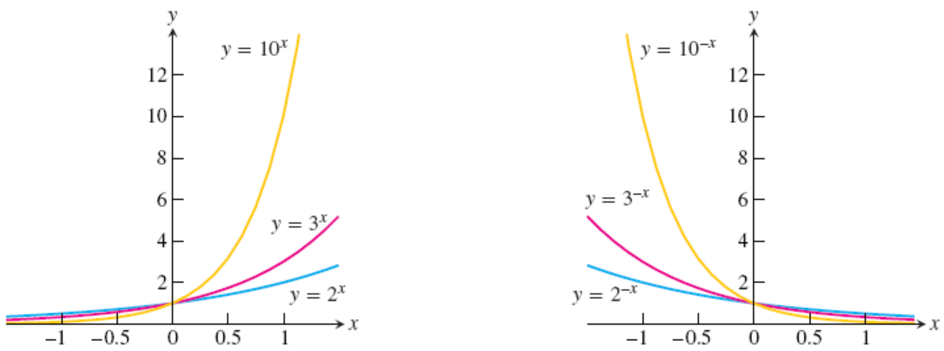
Exponential Functions

Functions of the form

$$f\,(x) = a^x,$$

where the base $a\ > 0$ is a positive constant and $a \neq 1$, are called exponential functions.

All exponential functions have domain $D_f = (-\infty, \infty)$ and range $R_f = (0, \infty)$. so an exponential function never assumes the value 0. The graphs of some exponential functions are shown in the figure below.

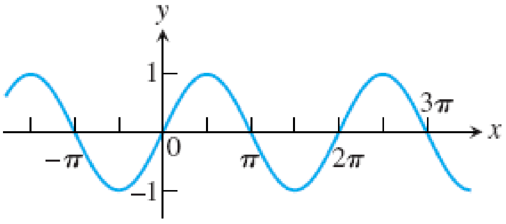


(a) $y = 2^x, y = 3^x, y = 10^x$

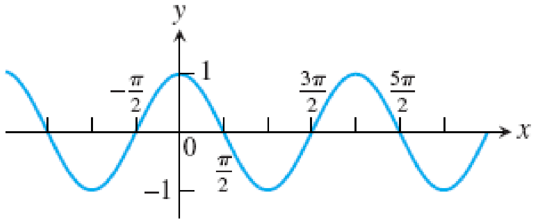
(b) $y = 2^{-x}, y = 3^{-x}, y = 10^{-x}$

Trigonometric Functions

These functions include $\sin x$, $\cos x$, $\tan x$, ... See the figure below



(a) $f(x) = \sin x$



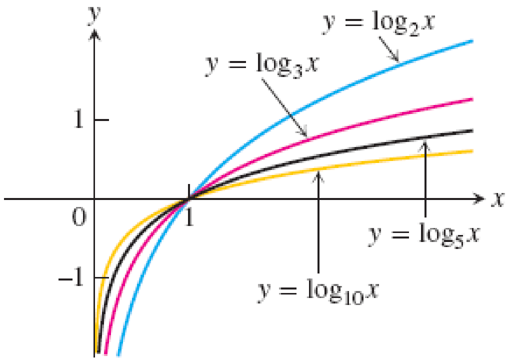
(b) $f(x) = \cos x$

Logarithmic Functions

These are the functions

$$f(x) = \log_a x\,,$$

where the base $a \neq 1$ is a positive constant. They are the inverse functions of the exponential functions. See the figure below



Even and Odd Functions

DEFINITIONS Even Function, Odd Function

A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,
odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

EXAMPLES

Example 1

Recognize if the functions are even or odd:

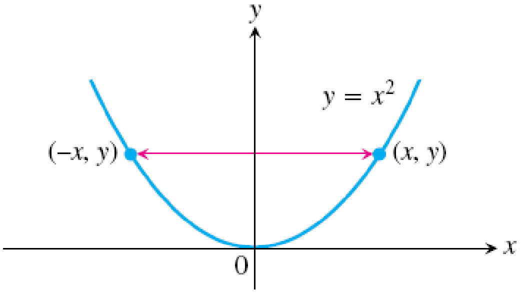
a. $f(x) = x^2$.
b. $f(x) = x^2 + 1$.
c. $f(x) = x^3$
d. $f(x) = x$
e. $f(x) = x + 1$

Solution

Solution a:

$$f(-x) = (-x)^2 = x^2 = f(x) \quad for\ all\ x$$

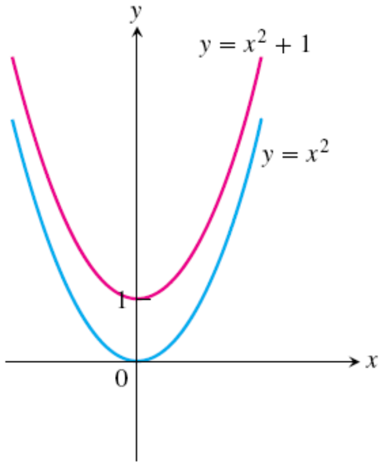
Then the function is Even and it is Symmetric about $y - axis$.



Solution b:

$$f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x) \quad for\ all\ x$$

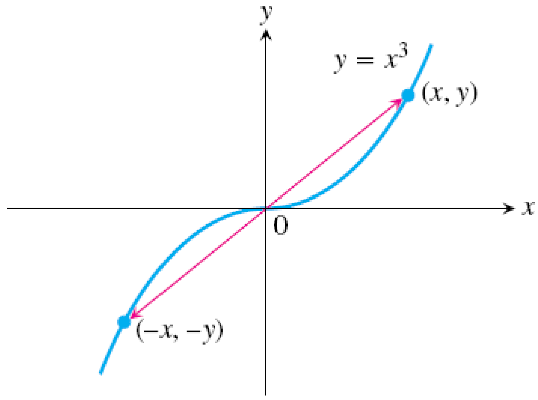
Then the function is Even and it is Symmetric about $y - axis$.



Solution c:

$$f(-x) = (-x)^3 = -x^3 = -f(x) \quad for\ all\ x$$

Then the function is Even and it is Symmetric about the origin.



Solution d:

$$f(-x) = (-x) = -x = -f(x) \quad for\ all\ x$$

Then the function is Even and it is Symmetric about the origin.

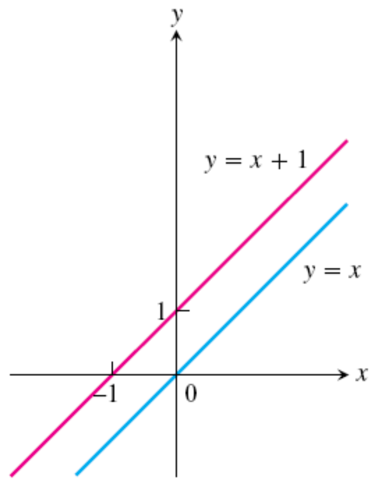
Solution e:

$$f(-x) = (-x) + 1 = -x + 1 \neq f(x) = x + 1 \text{ Not even}$$

$$f(-x) = (-x) + 1 = -x + 1 \neq -f(x) = -x - 1 \text{ Not odd}$$

Then the function is not Even nor odd.

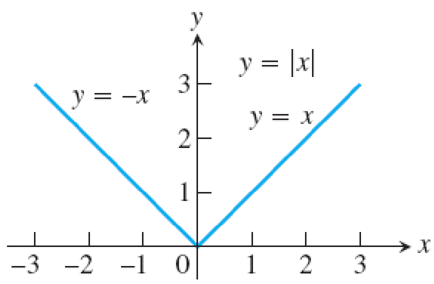
Then the function is not Even nor odd.



Piecewise-Defined Functions

Sometimes a function is described by using different formulas on different parts of its domain. One example is the **absolute value function**

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



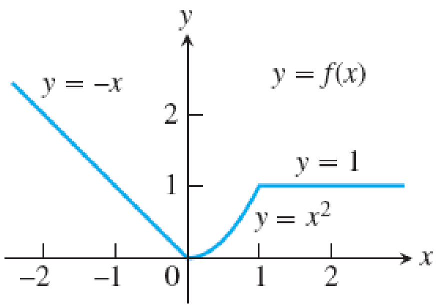
$$D_f = [-\infty, \infty], \qquad R_f = [0, \infty]$$

Example 2.7

Graph the following function

$$|x| = \begin{cases} -x, & x < 0 \\ x^2, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

Solution

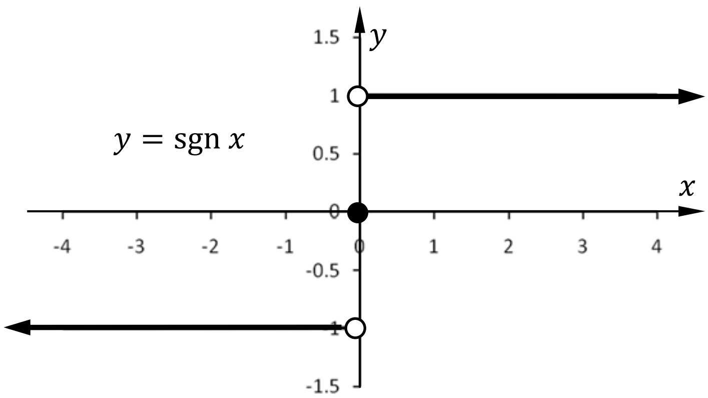


Signum Function

It derives its name from the Latin word for "sign". The definition of this function sgn x is as follows

$$\operatorname{sgn} x = \begin{cases} -1, & if\ x < 0 \\ 0, & if\ x = 0 \\ 1, & if\ x > 0 \end{cases}$$

therefore the domain of the function is $[-\infty, \infty]$ and the range of the function is $\{-1, 0, 1\}$.



Example 2.8

Graph the following functions:

a. $y = \text{sgn}\,(x^2 - 1)$.

b. $y = x\,\text{sgn}\,(x + 2)$.

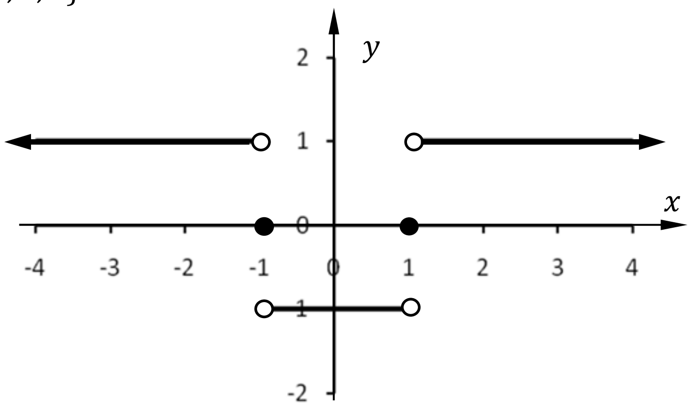
c. $y = \sqrt{x}\,\text{sgn}\,\left(\frac{x^2+1}{x}\right)$.

Solution 8.a:

$$\text{sgn}\,(x^2 - 1) = \begin{cases} -1, & if\ x^2 - 1 < 0 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1 \\ 0, & if\ x^2 - 1 = 0 \Rightarrow x = \mp 1 \\ 1, & if\ x^2 - 1 > 0 \Rightarrow |x| > 1 \Rightarrow (x < -1) \cup (x > 1) \end{cases}$$

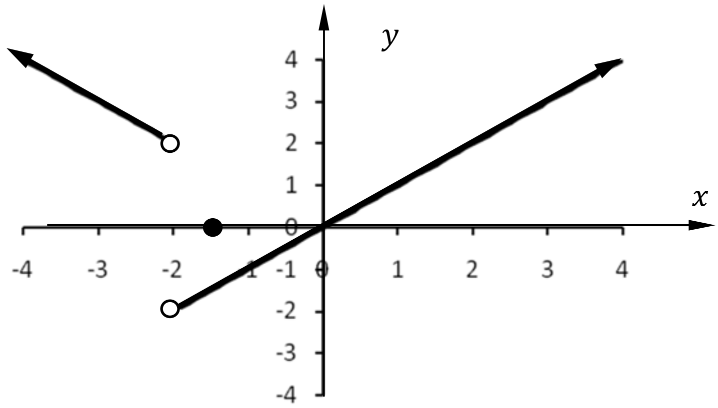
then

$D_f = (-\infty, \infty),\ R_f = \{-1, 0, 1\}$



Solution8.b:

$$x\,\text{sgn}\,(x + 2). = \begin{cases} (-1)\times x = -x, & if\ x + 2 < 0 \Rightarrow x < -2 \\ 0, & if\ x + 2 = 0 \Rightarrow x = -2 \\ (1)\times x = x, & if\ x + 2 > 0 \Rightarrow x > -2 \end{cases}$$



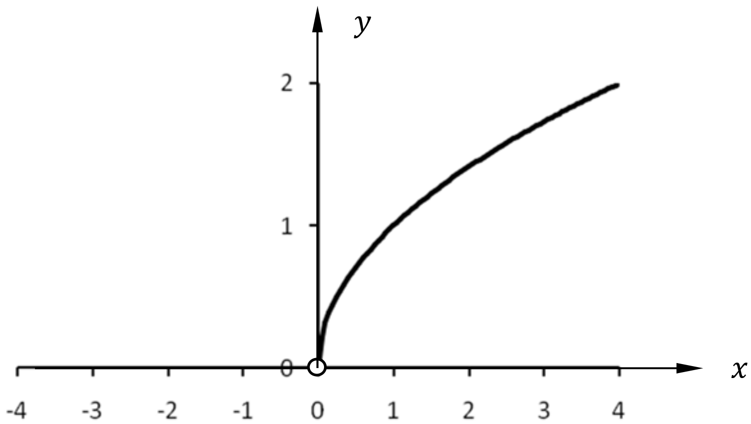
Then from the graph

$D_f = (-\infty, \infty), \qquad R_f = (-2, \infty)$

Solution8.c:

$$\sqrt{x}\,\mathrm{sgn}\left(\frac{x^2+1}{x}\right)$$

$$= \begin{cases} -\sqrt{x}, & if\ \frac{x^2+1}{x} < 0 \cap x \geq 0 \Rightarrow x^2+1 \neq 0\,, x \neq 0 \Rightarrow x < 0 \cap x \geq 0 \Rightarrow \emptyset \\ 0, & if\ \frac{x^2+1}{x} = 0 \Rightarrow x^2+1 \neq 0\,, x \neq 0 \Rightarrow \emptyset \\ \sqrt{x}, & if\ \frac{x^2+1}{x} > 0 \cap x \geq 0 \Rightarrow x^2+1 \neq 0\,, x \neq 0 \Rightarrow x > 0 \cap x \geq 0 \Rightarrow x > 0 \end{cases}$$



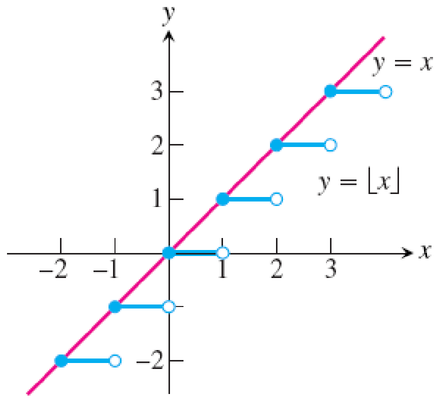
$D_f = (0, \infty), \qquad R_f = (0, \infty)$

The Greatest Integer Function

The function whose value at any number x ***is the greatest integer less than or equal to*** x is called the greatest integer function or the integer floor function. It is denoted $\lfloor x \rfloor$, or, in some books, $[x]$ or $[[x]]$ or int x. See the figure shown. Observe that

$$[2.4] = 2, \quad [1.9] = 1, \quad [0] = 0, \quad [-1.2] = -2,$$

$$[2] = 2, \quad [0.2] = 0, \quad [-0.3] = -1, \quad [-2] = -2$$



Properties of the Greatest Integer Function

1. $\left[[[x]]\right] = [x]$.
2. $[x \mp n] = [x] \mp n, \quad where\ n\ is\ an\ integer\ number.$

Example 2.9

Find the value of x for the following greatest integer functions:

f. $[x^2 + 1] = 2$.

g. $[2x + 1] = -1$.

h. $[x^2 - 2] = 3$.

Solution

By using the definition of $[x]$:

x	$[x]$
$-3 \le x < -2$	-3
$-2 \le x < -1$	-2
$-1 \le x < 0$	-1
$0 \le x < 1$	0
$1 \le x < 1$	1
$2 \le x < 3$	2

Then

Solution 9.a:

$$2 \le x^2 + 1 < 3 \quad \Rightarrow \quad 1 \le x^2 < 2 \quad \Rightarrow \quad 1 \le \sqrt{x^2} < \sqrt{2} \quad \Rightarrow \quad 1 \le |x| < \sqrt{2}$$

$$1 \le x < \sqrt{2} \ \cup \ -\sqrt{2} < x \le -1 \quad Ans.$$

Solution 9.b:

$$-1 \le 2x + 1 < 0 \quad \Rightarrow \quad -2 \le 2x < -1 \quad \Rightarrow \quad -1 \le x < -\frac{1}{2} \quad Ans.$$

Solution 9.c:

$$3 \le x^2 - 2 < 4 \quad \Rightarrow \quad 5 \le x^2 < 6 \quad \Rightarrow \quad \sqrt{5} \le \sqrt{x^2} < \sqrt{6} \quad \Rightarrow \quad \sqrt{5} \le |x| < \sqrt{6}$$

$$\sqrt{5} \le x < \sqrt{6} \ \cup \ -\sqrt{6} < x \le -\sqrt{5} \quad Ans.$$

Example 10:

Graph the following functions:

a. $y = 3x\,[2x], \quad [-2, 2[$

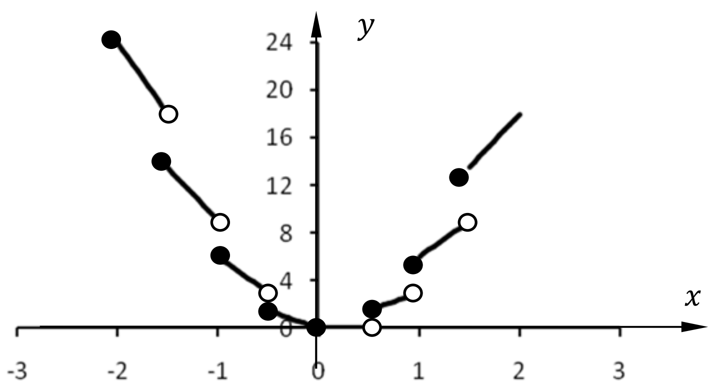
b. $y = \frac{[x/3]}{|x+1|}, \quad [-6,6[$

Solution 10.a:

By using the definition of $[x]$:

x	$[2x]$	$y = 3x\,[2x]$
$-2 \le x < -1.5$	-4	$-12x$
$-1.5 \le x < -1$	-3	$-9x$
$-1 \le x < -0.5$	-2	$-6x$
$-0.5 \le x < 0$	-1	$-3x$
$0 \le x < 0.5$	0	0
$0.5 \le x < 1$	1	$3x$
$1 \le x < 1.5$	2	$6x$
$1.5 \le x < 2$	3	$9x$

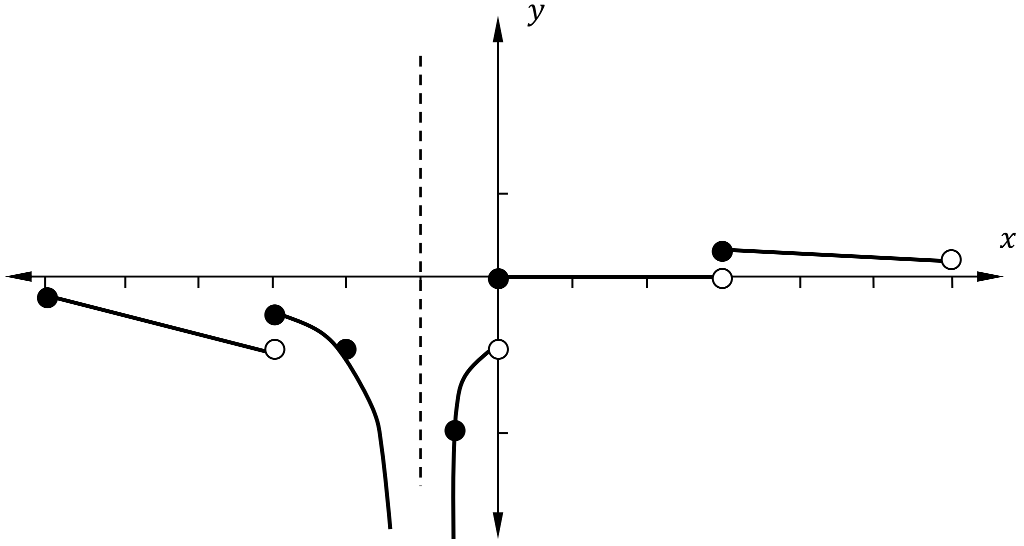
Then



Solution 10.b:

$$y = \frac{[x/3]}{|x+1|}, \quad [-6, 6]$$

x	$\left[\frac{x}{3}\right]$	$y = 3x\ [2x]$
$-6 \le x < -3$	-2	$\frac{-2}{-(x+1)} = \frac{2}{x+1}$
$-3 \le x < 0 \begin{cases} -3 \le x < -1 \\ -1 < x < 0 \end{cases}$	-1	$\frac{-1}{-(x+1)} = \frac{1}{x+1}$; $\frac{-1}{(x+1)}$
$0 \le x < 3$	0	0
$3 \le x < 6$	1	1



x	y
-6 -3	-2/5 -1
-3 -2 -1	-1/2 -1 -∞
-1 -0.5 0	-∞ -2 -1
0 3	0 0
3 6	1/4 1/7

H.W.: $y = x^2 \left[\frac{x}{3} + 1\right], \qquad [-6\ , 6[$

SHIFTING AND SCALING GRAPHS

SHIFTING GRAPHS

To shift the graph of a function $y = f\ (x)$ **straight up**, add a **positive constant** to the right-hand side of the formula $y = \ f(x)$.

To shift the graph of a function $y = f(x)$ **straight down**, add a **negative constant** to the right-hand side of the formula $y = f(x)$.

To shift the graph of $y = f(x)$ to the **left**, add a **positive constant** to x.

To shift the graph of $y = f(x)$ to the **right**, add a **negative constant** to x

---

Shift Formulas

Vertical Shifts

$y = f(x) + k$	Shifts the graph of f *up* k units if $k > 0$
	Shifts it *down* $\|k\|$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$	Shifts the graph of f *left* h units if $h > 0$
	Shifts it *right* $\|h\|$ units if $h < 0$

EXAMPLES

Example 1

Graph the following functions:

a. $y = (x - 2)^2$
b. $y = (x + 3)^2$

Solution

See the figure to the right hand side.

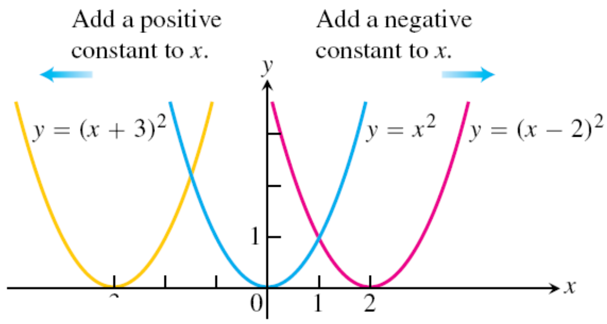


FIGURE 1.55 To shift the graph of $y = x^2$ to the left, we add a positive constant to x. To shift the graph to the right, we add a negative constant to x

Example 2

Graph the following functions:

a. $y = x^2 + 1$
b. $y = x^2 + 2$
c. $y = x^2 - 2$

Solution

See the figure to the right hand side.

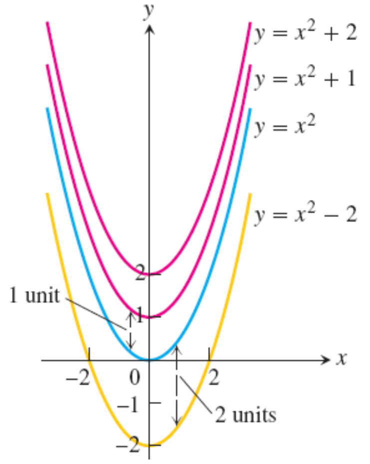


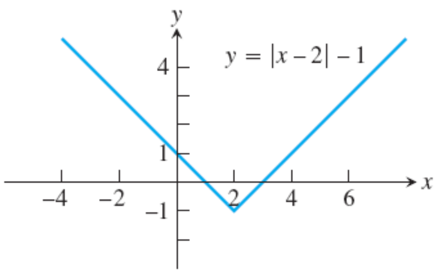
FIGURE 1.54 To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for f (Example 4a and b).

Example 3

Graph the following function:

$$y = |x - 2| - 1$$

Solution



FI(hifting the graph of $y = |x|$ 2 units to the right and 1 unit down (Ex

SCALING GRAPHS

To scale the graph of a function $y = f(x)$ is to **stretch** or **compress** it, **vertically** or **horizontally**. This is accomplished by multiplying the function $f(x)$, or the independent variable x, by an appropriate constant c.

Reflections across the coordinate axes are special cases where $c = -1$.

Vertical and Horizontal Scaling and Reflecting Formulas

For $c > 1$,

$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c} f(x)$	Compresses the graph of f vertically by a factor of c.
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c.

For $c = -1$,

$y = -f(x)$	Reflects the graph of f across the x-axis.
$y = f(-x)$	Reflects the graph of f across the y-axis.

EXAMPLES

Example 1

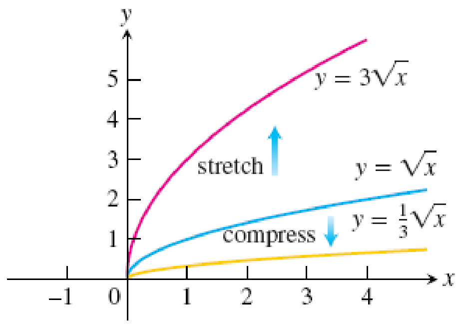
Graph the following functions:

a. $y = 3\sqrt{x}$

b. $y = \frac{1}{3}\sqrt{x}$

Solution

See the figure to the right hand side.



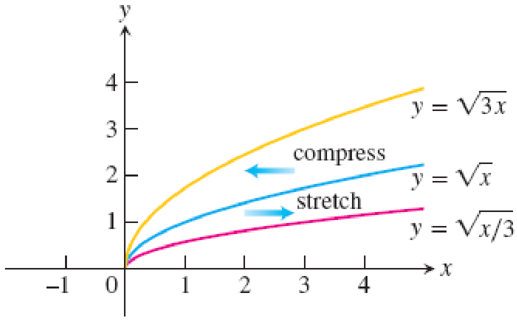
Vertically stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3

Example 2

Graph the following functions:

a. $y = \sqrt{3x}$

b. $y = \sqrt{x/3}$



Horizontally stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3

Solution

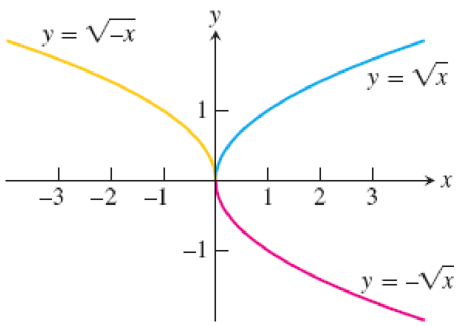
See the figure to the right hand side.

Example 3

Graph the following functions:

a. $y = \sqrt{-x}$

b. $y = -\sqrt{x}$



Reflections of the graph $y = \sqrt{x}$ across the coordinate axes

Solution

See the figure to the right hand side.

ADDITIONAL EXAMPLES

Example 1

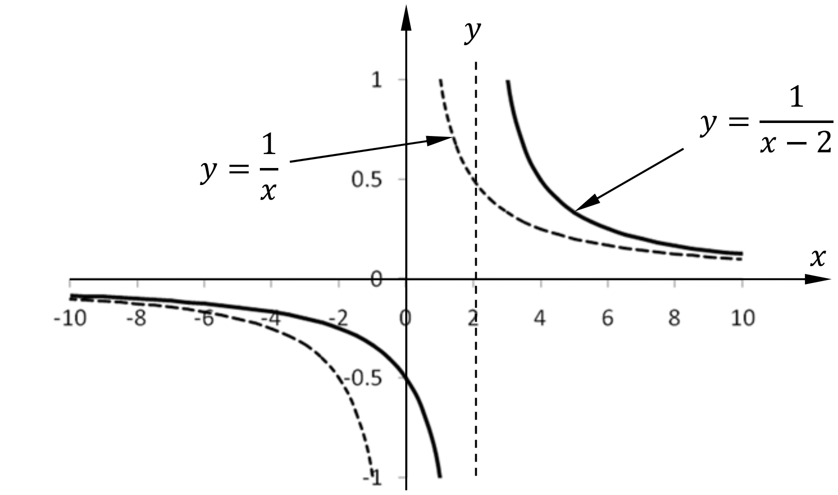
Graph the following functions:

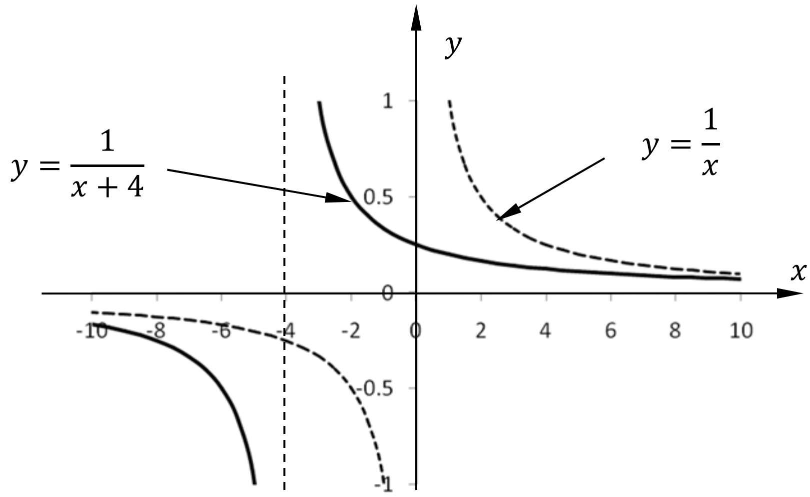
a. $y = \frac{1}{x-2}$

b. $y = \frac{1}{x+4}$

Solution

See the figures below.





Example 2

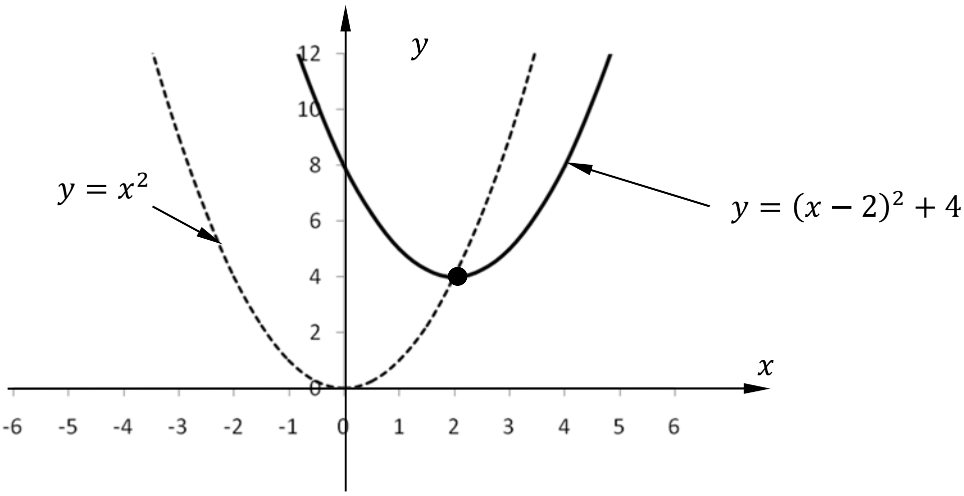
Graph the following functions:

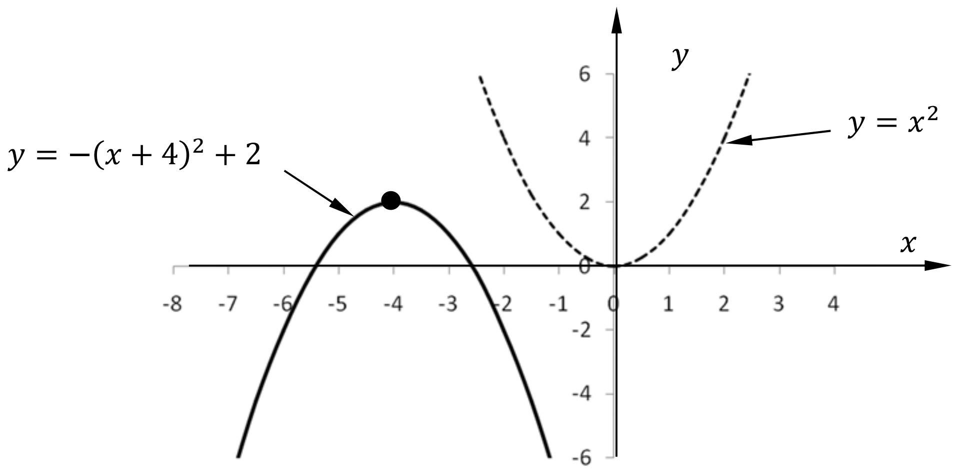
a. $y = (x - 2)^2 + 4$

b. $y = -(x + 4)^2 + 2$

Solution

See the figures below.





Example 3

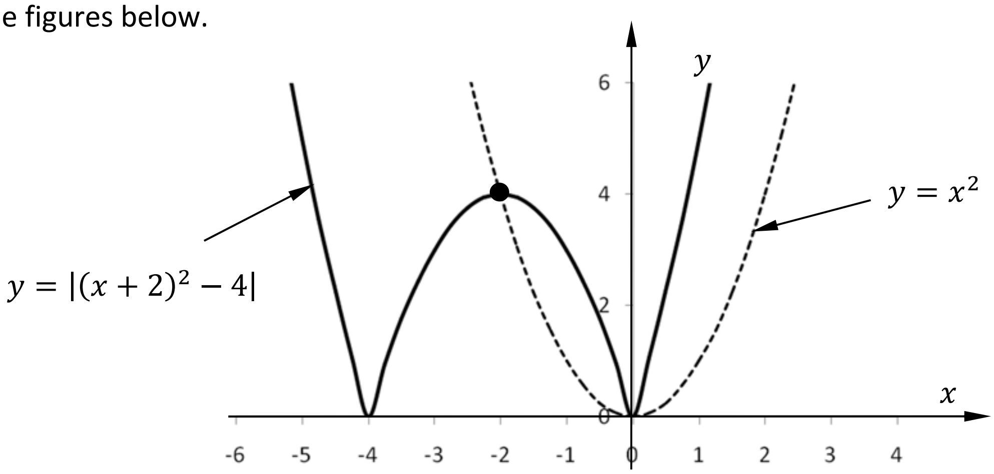
Graph the following functions:

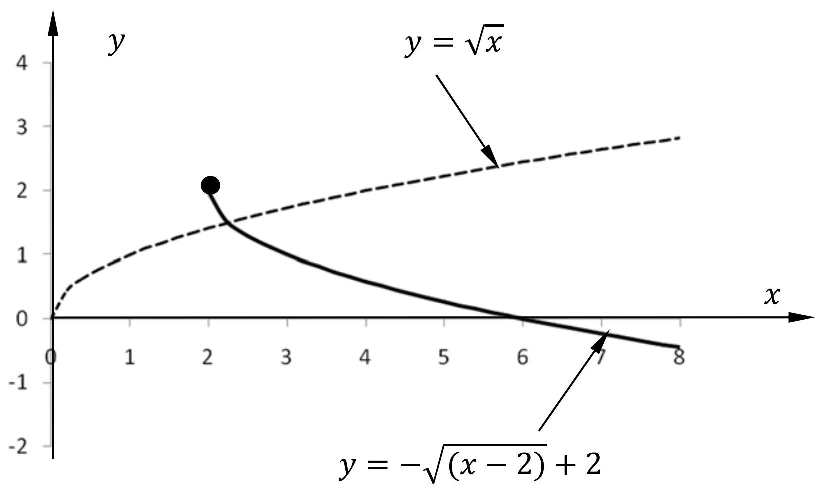
a. $y = |(x+2)^2 - 4|$

b. $y = -\sqrt{(x-2)} + 2$

Solution

See the figures below.





Example 4

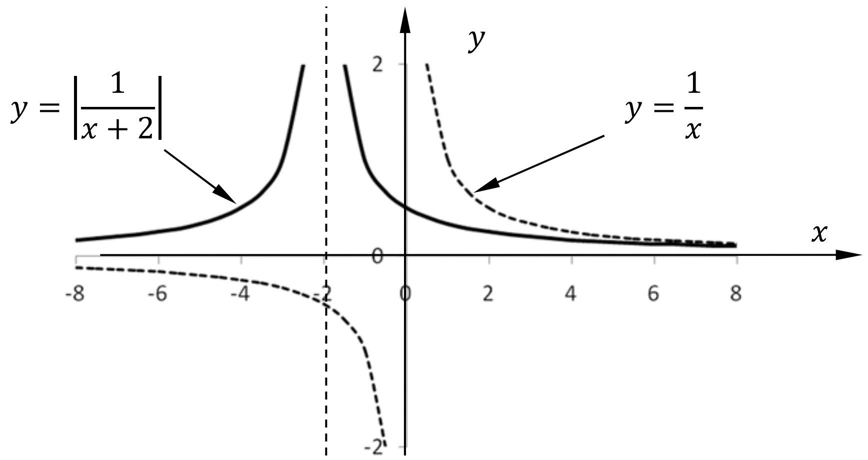
Graph the following functions:

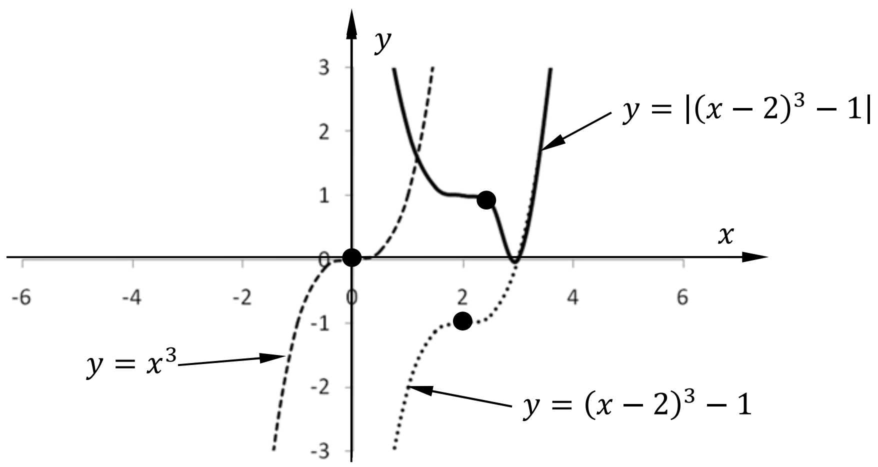
a. $y = \left|\frac{1}{x+2}\right|$

b. $y = |(x-2)^3 - 1|$

Solution

See the figures below.



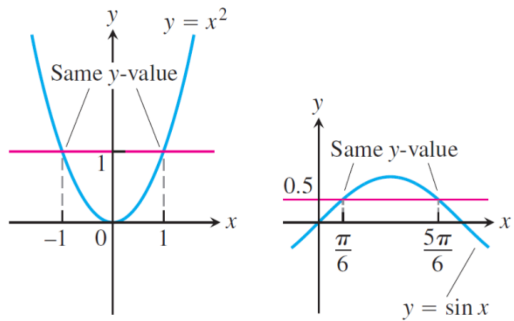


One – to One Functions

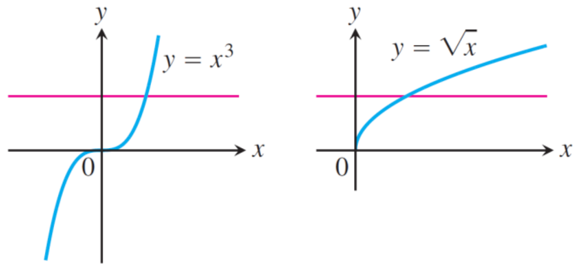
Functions like $y = \sin x \ \ and \ y = x^2$ can assign the same output to different inputs,

$$\sin(0) = 0, \qquad \sin(\pi) = 0$$
$$f(x) = x^2, \qquad f(1) = 1\,, \qquad f(-1) = 1$$

Other functions , however, like $y = \sqrt{x}$, and $y = x^3$, never assign an output number more than once. Functions like these are called one –to –one (1 – 1).



Not one-to-one: Graph meets one or more horizontal lines more than once.



One-to-one: Graph meets each horizontal line at most once.

The function defined by reversing a one to one continuous function $f(x)$ is called the inverse of f. The symbol for the inverse is f^{-1}, read "f inverse".

$$y = f(x), \qquad D_f, \qquad R_f$$
$$f^{-1}(x), \qquad D_{f^{-1}} = R_f, \qquad R_{f^{-1}} = D_f$$

To express f^{-1} as a function of x:

1. Switch x and y,
2. Solve the equation $x = f(y)$ for y in terms of x

The resulting formula will be $y = f^{-1}(x)$

$$f^{-1}\big(f(x)\big) = x$$

To draw the graph of f^{-1},we have to reconstruct it from the graph of f in the following way, we reflect the graph of $y = f(x)$ across the line $y = x$.

Example

Find the inverse of the function

$$y = \sqrt{x}$$

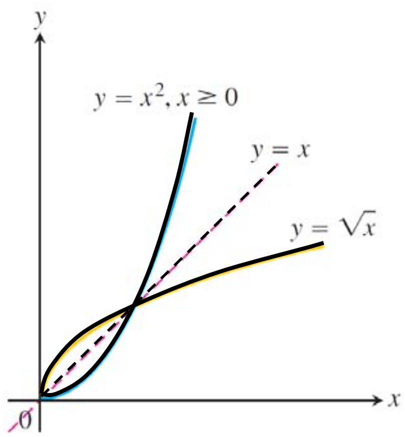
Solution

$$f(x) = \sqrt{x}\,, \quad D_f : x \geq 0, \quad R_f : y \geq 0$$
$$f^{-1}(x), \quad D_{f^{-1}} = R_f = x \geq 0$$
$$R_{f^{-1}} = D_f = y \geq 0$$

To find f^{-1}:

$$y = \sqrt{x} \quad \Rightarrow \quad x = \sqrt{y} \quad \Rightarrow \quad y = x^2 \quad \Rightarrow \quad f^{-1}(x) = x^2$$

$$f\left(f^{-1}(x)\right) = \sqrt{x^2} = x$$

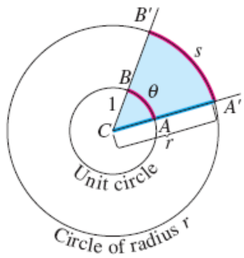


3 TRIGONOMETRIC FUNCTIONS

3.1 TRIGONOMETRIC FUNCTIONS

Radian Measure

The **radian measure** of the angle ACB at the center of the unit circle (see the figure shown) equals the length of the arc that $\widehat{ACB}$ cuts from the unit circle.



The figure shows that

$$s = r\theta$$

is the **length of arc** cut from a circle of radius r when the subtending angle θ producing the arc is measured in radians.

Since the circumference of the circle is 2π and one complete revolution of a circle is $360°$, the relation between radians and degrees is given by the following conversion formula

Conversion Formulas

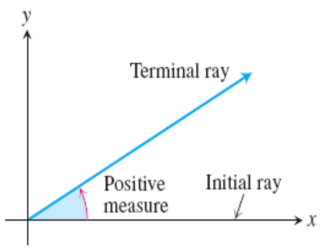
$$1 \text{ degree} = \frac{\pi}{180} (\approx 0.02) \text{ radians}$$

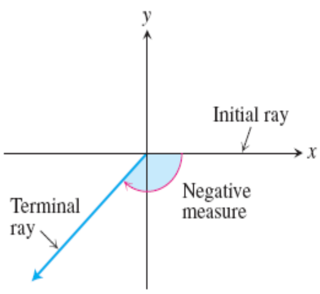
Degrees to radians: multiply by $\frac{\pi}{180}$

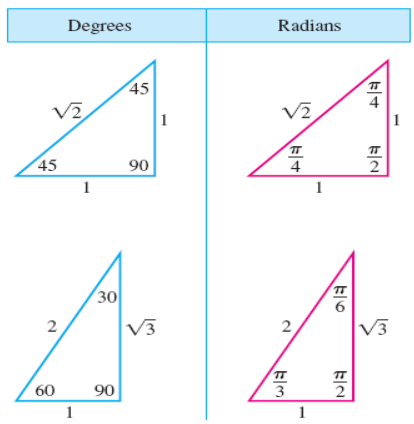
$$1 \text{ radian} = \frac{180}{\pi} (\approx 57) \text{ degrees}$$

Radians to degrees: multiply by $\frac{180}{\pi}$

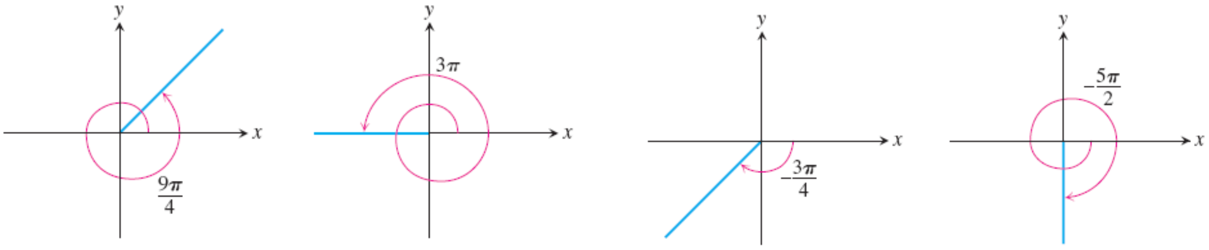
An angle in the xy-plane is said to be in **standard position** if its vertex lies at the origin and its initial ray lies along the positive x-axis (see the figure below). Angles measured **counter-clockwise** from the **positive** x-axis are assigned positive measures; angles measured clock¬wise are assigned negative measures.







When angles are used to describe counterclockwise rotations, our measurements can go arbitrarily far beyond 2π radians or $360°$. Similarly, angles describing clockwise rotations can have negative measures of all sizes (see the figure below).



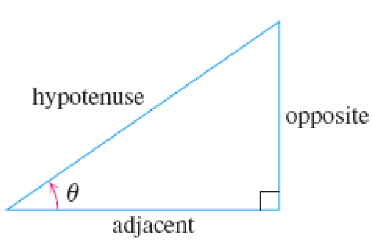
The Six Basic Trigonometric Functions

The trigonometric functions of an acute angle in terms of the sides of a right triangle is shown in the figure below

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

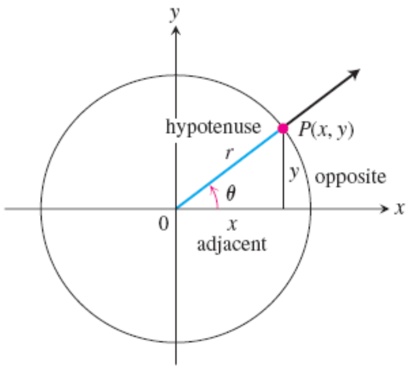


We extend this definition to obtuse and negative angles by first placing the angle in standard position in a circle of radius r. We then define the trigonometric functions in terms of the coordinates of the point $P(x, y)$ where the angle's terminal ray intersects the circle.

sine: $\sin\theta = \frac{y}{r}$ **cosecant:** $\csc\theta = \frac{r}{y}$

cosine: $\cos\theta = \frac{x}{r}$ **secant:** $\sec\theta = \frac{r}{x}$

tangent: $\tan\theta = \frac{y}{x}$ **cotangent:** $\cot\theta = \frac{x}{y}$



and from the above definition on can get the following

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

Periodicity

When an angle of measure θ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric function values:

$$\cos(\theta + 2\pi) = \cos\theta \qquad \sin(\theta + 2\pi) = \sin\theta \qquad \tan(\theta + 2\pi) = \tan\theta$$
$$\sec(\theta + 2\pi) = \sec\theta \qquad \csc(\theta + 2\pi) = \csc\theta \qquad \cot(\theta + 2\pi) = \cot\theta$$

DEFINITION **Periodic Function**

A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

Identities

The coordinates of any point $P(x, y)$ in the plane can be expressed in terms of the point's distance from the origin and the angle that ray OP makes with the positive x-axis.

$$\cos^2\theta + \sin^2\theta = 1.$$

$$1 + \tan^2\theta = \sec^2\theta.$$
$$1 + \cot^2\theta = \csc^2\theta.$$

Addition Formulas

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$
$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

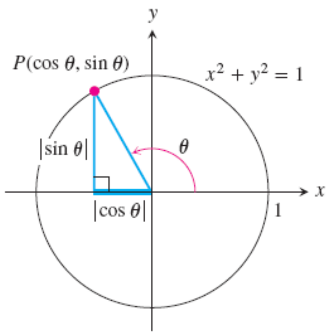
Double-Angle Formulas

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$

Half-Angle Formulas

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$



Additional Trigonometric Identities

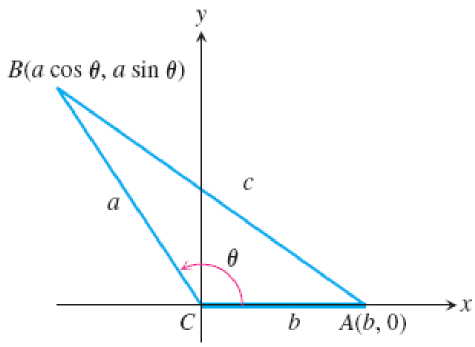
$$\cos\left(x - \frac{\pi}{2}\right) = \sin x \qquad \cos\left(x + \frac{\pi}{2}\right) = -\sin x$$

$$\sin\left(x + \frac{\pi}{2}\right) = \cos x \qquad \sin\left(x - \frac{\pi}{2}\right) = -\cos x$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

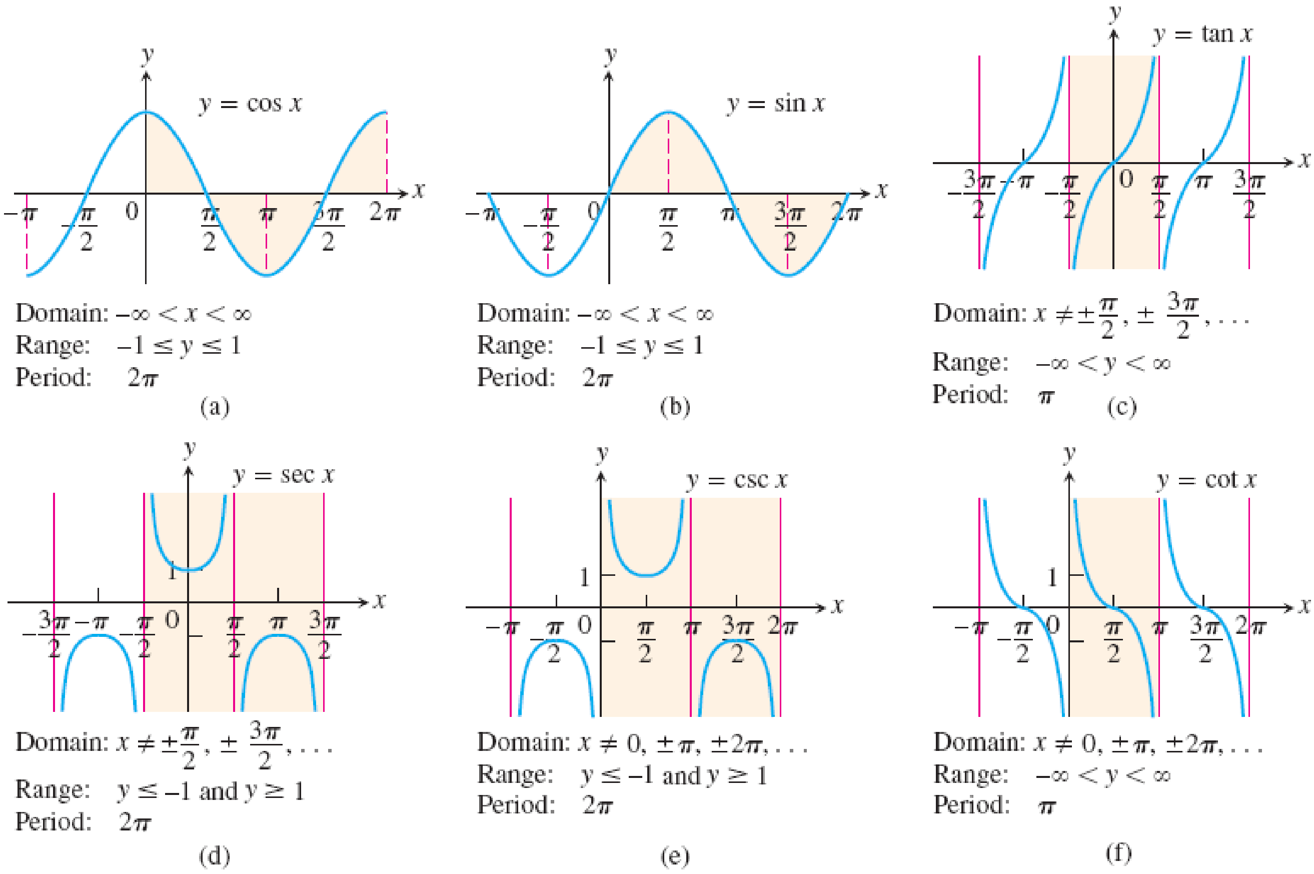


$\sin A \cos B = \frac{1}{2}(\sin(A - B) + \sin(A + B))$

$\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$

$\cos A \cos B = \frac{1}{2}(\cos(A - B) + \cos(A + B))$

Graphs of the Trigonometric Functions



Graphs of the (a) cosine, (b) sine, (c) tangent, (d) secant, (e) cosecant, and (f) cotangent functions using radian measure. The shading for each trigonometric function indicates its periodicity.

The above functions have the following properties

Periods of Trigonometric Functions

Period π:
$\tan(x + \pi) = \tan x$
$\cot(x + \pi) = \cot x$

Period 2π:
$\sin(x + 2\pi) = \sin x$
$\cos(x + 2\pi) = \cos x$
$\sec(x + 2\pi) = \sec x$
$\csc(x + 2\pi) = \csc x$

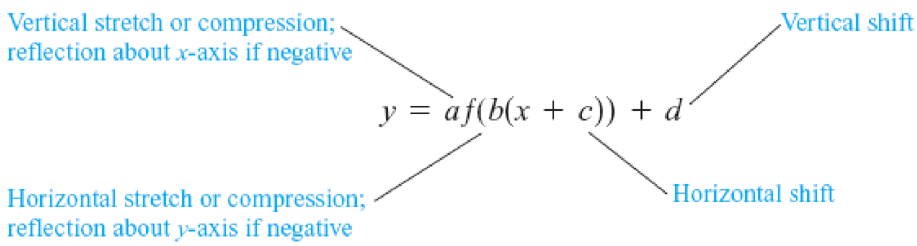
Even	Odd
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$

Transformation of Trigonometric Functions

The rules for shifting, stretching, compressing, and reflecting the graph of a function apply to the trigonometric functions. The following diagram will remind you of the controlling parameters

Transformations of Trigonometric Graphs

The rules for shifting, stretching, compressing, and reflecting the graph of a function apply to the trigonometric functions. The following diagram will remind you of the controlling parameters.



EXAMPLES

Example 1

Graph the following functions

$$y = 2\sin\left(3x + \frac{\pi}{4}\right)$$

Solution

1. Find the roots of the functions ($y = 0$):

$$y = 0\,, \qquad \sin\left(3x + \frac{\pi}{4}\right) = 0 \Leftrightarrow \boldsymbol{\sin\theta = 0}$$

when

$$3x + \frac{\pi}{4} = \pm n\pi, \qquad \Leftrightarrow \qquad \boldsymbol{\theta = \pm n\pi} \qquad n = 0, 1, 2, 3, \dots$$

Then

$$x = -\frac{\pi}{12} \pm \frac{n\pi}{3} = -\frac{\pi}{12} \pm \frac{4n\pi}{12}, \qquad n = 0, 1, 2, 3, \dots$$

or

$$x = \cdots, -\frac{9\pi}{12}, -\frac{5\pi}{12}, -\frac{\pi}{12}, \frac{3\pi}{12}, \frac{7\pi}{12}, \dots$$

2. Find the maximum point of the functions ($y = 2$):

$$y = 2\,, \qquad \sin\left(3x + \frac{\pi}{4}\right) = +1 \Leftrightarrow \mathbf{\sin\boldsymbol{\theta} = +1}$$

when

$$3x + \frac{\pi}{4} = \frac{\pi}{2} \pm 2n\pi, \qquad \Leftrightarrow \qquad \boldsymbol{\theta = \frac{\pi}{2} \pm 2n\pi} \qquad n = 0, 1, 2, 3, \dots$$

Then

$$x = \frac{\pi}{12} \pm \frac{2n\pi}{3} = \frac{\pi}{12} \pm \frac{8n\pi}{12}, \qquad n = 0, 1, 2, 3, \dots$$

or

$$x = \cdots, -\frac{7\pi}{12}, \frac{\pi}{12}, \frac{9\pi}{12}, \dots$$

3. Find the minimum point of the functions ($y = -2$):

$$y = -2\,, \qquad \sin\left(3x + \frac{\pi}{4}\right) = -1 \Leftrightarrow \mathbf{\sin\boldsymbol{\theta} = -1}$$

when

$$3x + \frac{\pi}{4} = \frac{3\pi}{2} \pm 2n\pi, \qquad \Leftrightarrow \qquad \boldsymbol{\theta = \frac{3\pi}{2} \pm 2n\pi} \ \left(or\ \boldsymbol{-\frac{\pi}{2} \pm 2n\pi}\right) \ n = 0, 1, 2, 3, \dots$$

Then

$$x = \frac{5\pi}{12} \pm \frac{2n\pi}{3} = \frac{5\pi}{12} \pm \frac{8n\pi}{12}, \qquad n = 0, 1, 2, 3, \dots$$

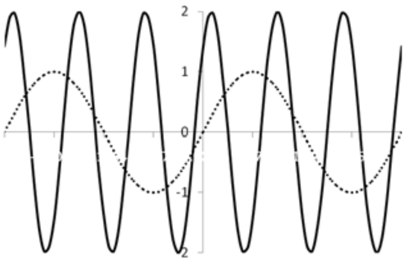
or

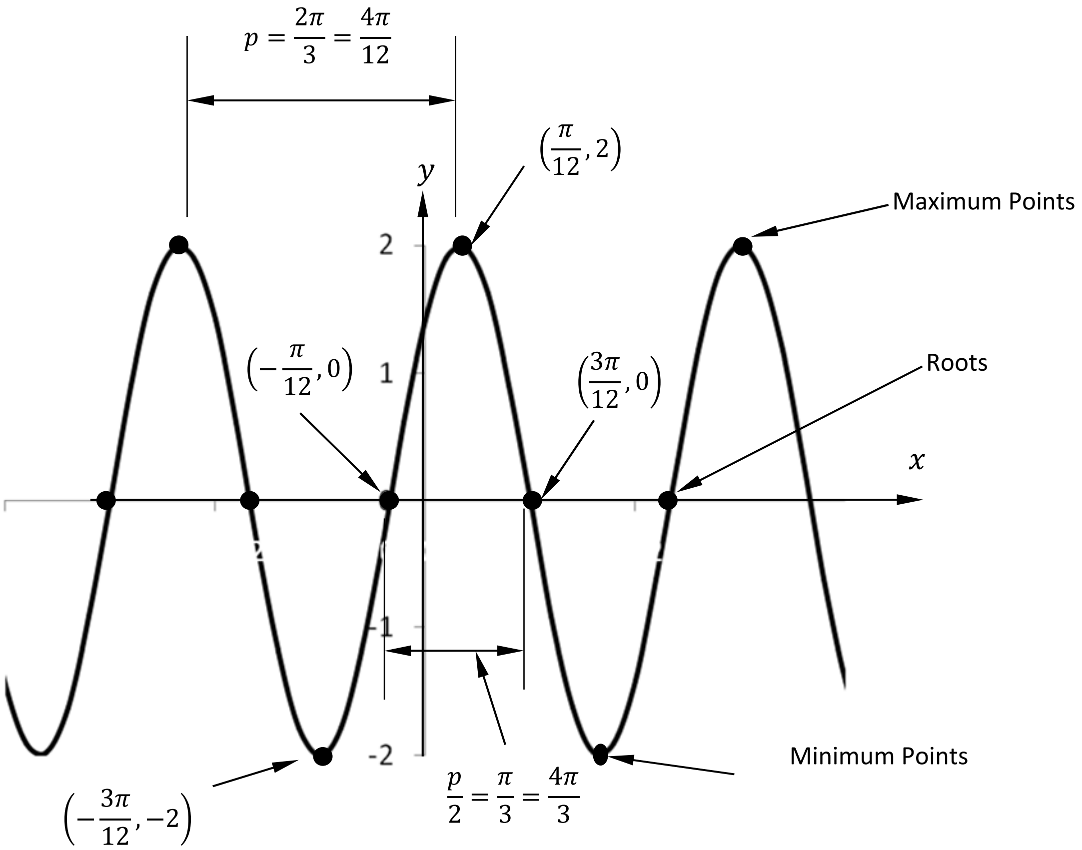
$$x = \cdots, -\frac{3\pi}{12}, \frac{5\pi}{12}, \dots$$

From the above calculations

The period of the function will be $2\pi/3$ or

$$p = \frac{2\pi}{Coeff.\ of\ x}$$





Example 2

Graph the following functions

$$y = 3\cos\left(2x + \frac{\pi}{3}\right)$$

Solution

1. Find the roots of the functions ($y = 0$):

$$y = 0\,, \qquad \cos\left(2x + \frac{\pi}{3}\right) = 0 \Leftrightarrow \boldsymbol{\cos\theta = 0}$$

when

$$2x + \frac{\pi}{3} = \frac{\pi}{2} \pm n\pi, \qquad \Leftrightarrow \qquad \boldsymbol{\theta = \frac{\pi}{2} \pm n\pi} \qquad n = 0, 1, 2, 3, \ldots$$

Then

$$x = \frac{\pi}{12} \pm \frac{n\pi}{2} = \frac{\pi}{12} \pm \frac{6n\pi}{12}, \qquad n = 0, 1, 2, 3, \ldots$$

or

$$x = \cdots, -\frac{11\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \ldots$$

2. Find the maximum point of the functions ($y = 3$):

$$y = 3\,, \qquad \cos\left(2x + \frac{\pi}{3}\right) = +1 \Leftrightarrow \boldsymbol{\cos\theta = +1}$$

when

$$2x + \frac{\pi}{3} = \pm 2n\pi, \qquad \Leftrightarrow \quad \boldsymbol{\theta = \pm 2n\pi} \qquad n = 0, 1, 2, 3, \ldots$$

Then

$$x = -\frac{\pi}{6} \pm n\pi = -\frac{\pi}{6} \pm \frac{6n\pi}{6}, \qquad n = 0, 1, 2, 3, \ldots$$

or

$$x = \cdots, -\frac{7\pi}{6}, -\frac{\pi}{6}, \frac{5\pi}{6}, \ldots$$

3. Find the minimum point of the functions ($y = -3$):

$$y = -3\,, \qquad \cos\left(2x + \frac{\pi}{3}\right) = -1 \Leftrightarrow \boldsymbol{\cos\theta = -1}$$

when

$$2x + \frac{\pi}{3} = \pi \pm 2n\pi, \qquad \Leftrightarrow \quad \boldsymbol{\theta = \pi \pm 2n\pi} \qquad n = 0, 1, 2, 3, \ldots$$

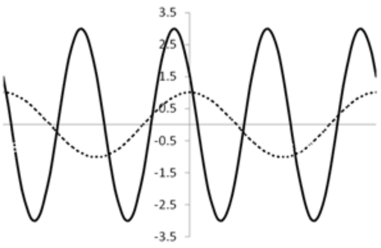
Then

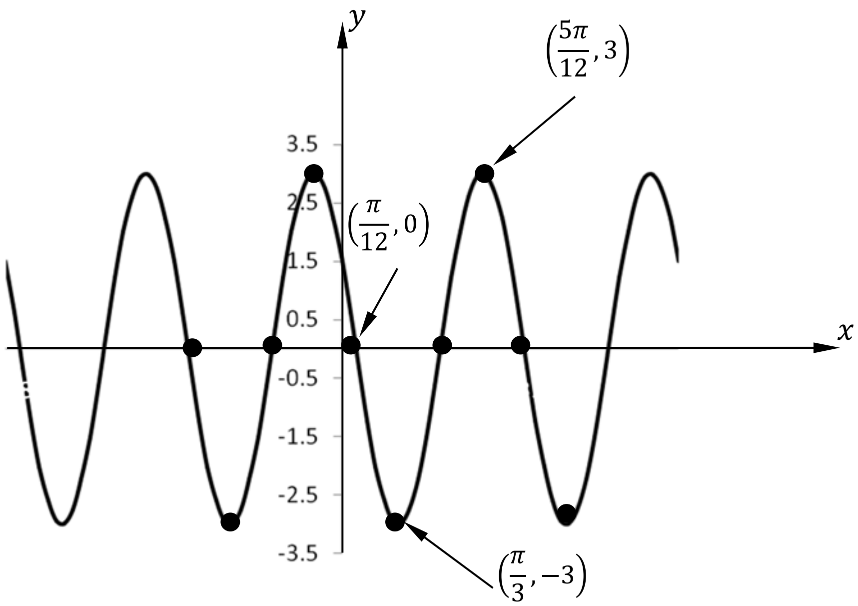
$$x = \frac{\pi}{3} \pm n\pi = \frac{\pi}{3} \pm \frac{3n\pi}{3}, \qquad n = 0, 1, 2, 3, \ldots$$

or

$$x = \cdots, -\frac{2\pi}{3}, \frac{\pi}{3}, \frac{4\pi}{3}, \ldots$$

The period of the function will be π





Example 3

Graph the following functions

$$y = 2\tan\left(3x - \frac{\pi}{2}\right)$$

Solution

4. Find the roots of the functions ($y = 0$):

$$y = 0\,, \qquad \tan\left(3x - \frac{\pi}{2}\right) = 0 \Leftrightarrow \boldsymbol{\tan\theta = 0}$$

when

$$3x - \frac{\pi}{2} = \pm n\pi, \qquad \Leftrightarrow \qquad \boldsymbol{\theta = \pm n\pi} \qquad n = 0, 1, 2, 3, \ldots$$

Then

$$x = \frac{\pi}{6} \pm \frac{n\pi}{3} = \frac{\pi}{6} \pm \frac{2n\pi}{6}, \qquad n = 0, 1, 2, 3, \ldots$$

$$x = \cdots, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{3\pi}{6}\,, \ldots$$

5. Find the asymptotes of functions ($y = \pm\infty$):

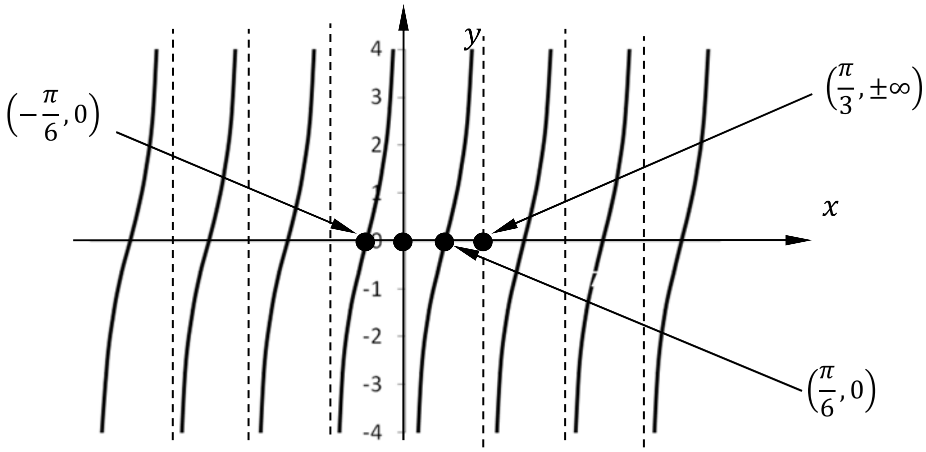
$$y = \pm\infty\,, \qquad \tan\left(3x - \frac{\pi}{2}\right) = \pm\infty \Leftrightarrow \boldsymbol{\tan\theta} = \pm\infty$$

when

$$3x - \frac{\pi}{2} = \frac{\pi}{2} \pm n\pi, \qquad \Leftrightarrow \qquad \boldsymbol{\theta = \frac{\pi}{2} \pm n\pi} \qquad n = 0, 1, 2, 3, \ldots$$

Then

$$x \neq \frac{\pi}{3} \pm \frac{n\pi}{3}, \qquad n = 0, 1, 2, 3, \dots \qquad or \qquad x \neq \cdots -\frac{\pi}{3}, 0, \frac{\pi}{3}, \frac{2\pi}{3}, \dots$$

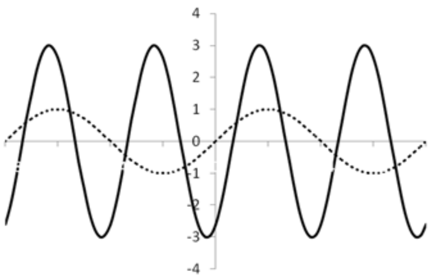


HOMEWORK

Graph the following trigonometric functions.

HW 1:

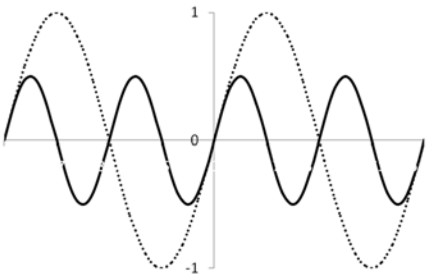
$$y = 3\sin\left(2x - \frac{\pi}{3}\right)$$



HW 2:

$$y = \sin x \cos x$$

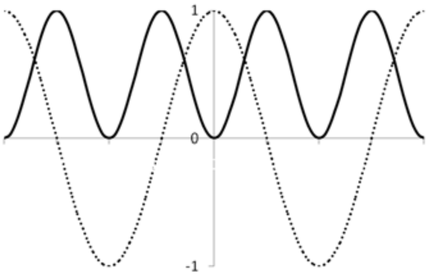
Hint: Use the identity $\sin x \cos x = \frac{1}{2}\sin 2x$



HW 3:

$$y = \sin^2 x$$

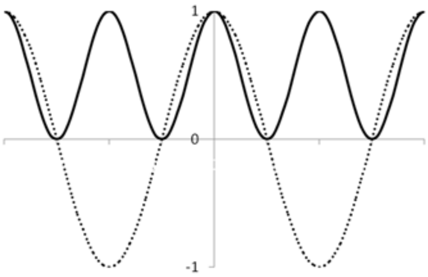
Hint: Use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$



HW 4:

$$y = \cos^2 x$$

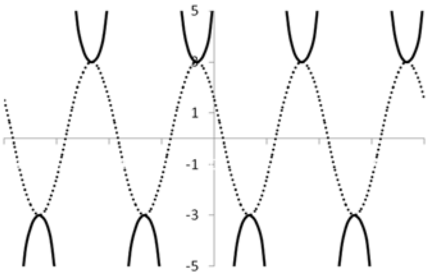
Hint: Use the identity $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$



HW 5:

$$y = 3\sec\left(2x + \frac{\pi}{3}\right)$$

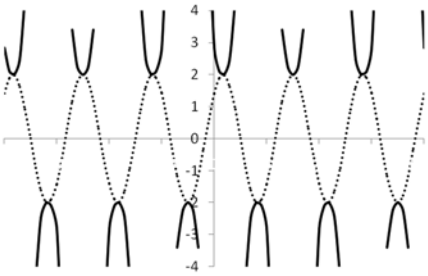
Hint: Try to use the graph $y = 3\cos\left(2x + \frac{\pi}{3}\right)$



HW 6:

$$y = 2\csc\left(3x + \frac{\pi}{4}\right)$$

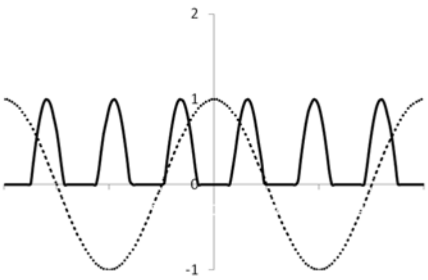
Hint: Try to use the graph $y = 2\sin\left(3x + \frac{\pi}{4}\right)$



HW 7:

$$y = \frac{1}{2}(|\cos\pi x| - \cos\pi x)$$

Hint: Use the definition of the absolute value $|\cos\pi x| = \begin{cases} \cos\pi x\,, & \text{when } \cos\pi x \geq 0 \\ -\cos\pi x\,, & \text{when } \cos\pi x < 0 \end{cases}$



3.2 Inverse Trigonometric Functions

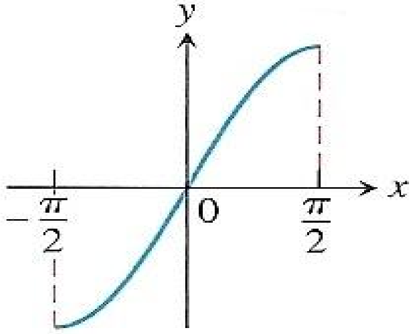
The six basic trigonometric functions are not one to one (their values repeat periodically). However we can restrict their domains to intervals on which they are one to one. The sine function increase from -1 at x = -π/2 to +1 at x = π/2. By restricting its domain to the interval$[-\frac{\pi}{2},\frac{\pi}{2}]$ we make it one-to-one, so that it has an inverse $\sin^{-1}x$. Similar domain restrictions can be applied to all six trigonometric functions.

The Arc Sine:

y = arcsin x or $y = \sin^{-1} x$

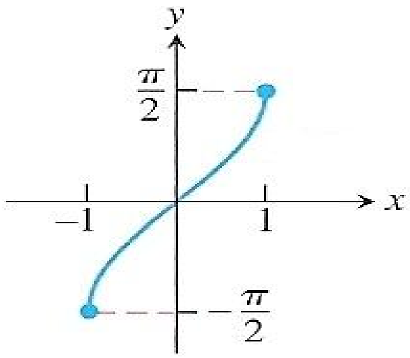
y= sin x

$D_f : \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ $R_f : [-1,1]$



$y = \sin^{-1} x$

$D_f^{-1} : [-1,1]$ $R_f^{-1} : \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

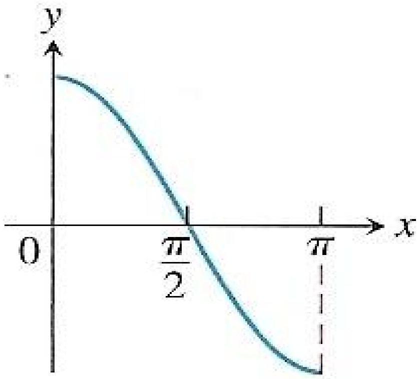


The Arc Cosine:

y = arccos x or $y = \cos^{-1} x$

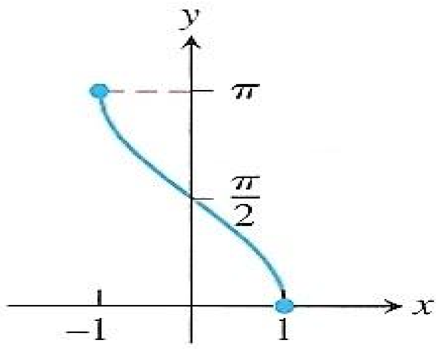
y= cos x

$D_f : [0,\pi]$ $R_f : [-1,1]$



$y = \cos^{-1} x$

$D_f^{-1} : [-1,1]$ $R_f^{-1} : [0,\pi]$

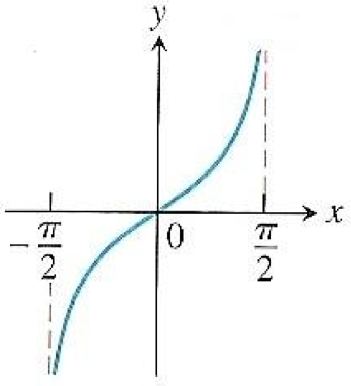


The Arc Tangent:

y = arctan x or y = $\tan^{-1}$ x

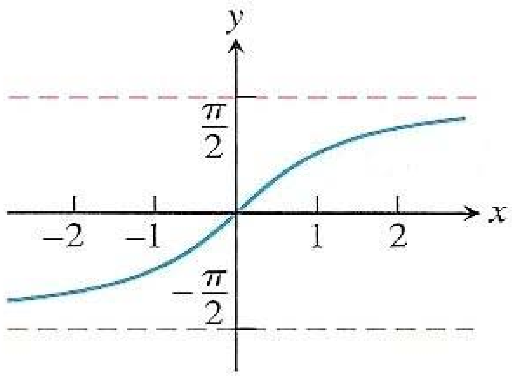
y = tan x

$D_f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $R_f : [-\infty, \infty]$



y = $\tan^{-1}$ x

$D_f^{-1} : [-\infty, \infty]$ $R_f^{-1} : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



The Arc Cotangent:

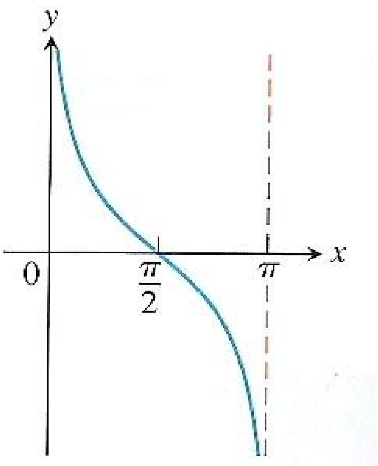
Y = arccot x or y = $\cot^{-1}$ x

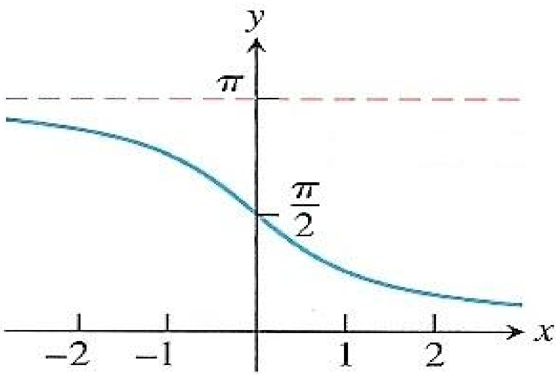
Y= cot x

$D_f : (0, \pi)$ $R_f : [-\infty, \infty]$

y = $\cot^{-1}$ x

$D_f^{-1} : [-\infty, \infty]$ $R_f^{-1} : (0, \pi)$



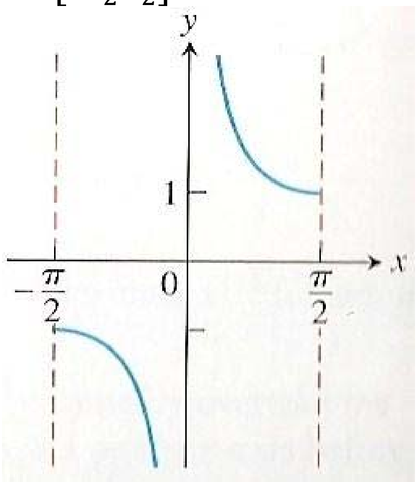


The Arc Cosecant:

y = arccsc x   or   y = $\csc^{-1}$ x

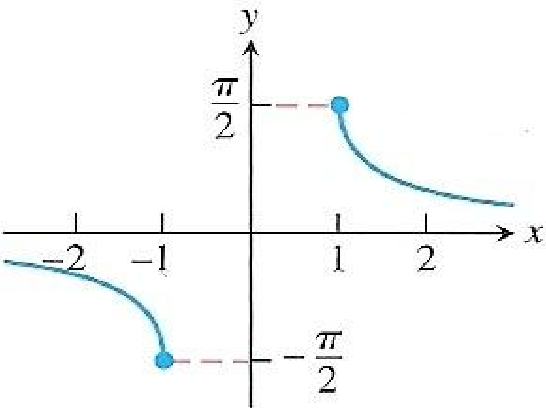
y= csc x

D_f : $\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \cup x \neq 0$ R_f :$|y| \geq 1$



y = $\csc^{-1}$ x

D_f^{-1} :$|x| \geq 1$ R_f^{-1} :$\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \cup y \neq 0$

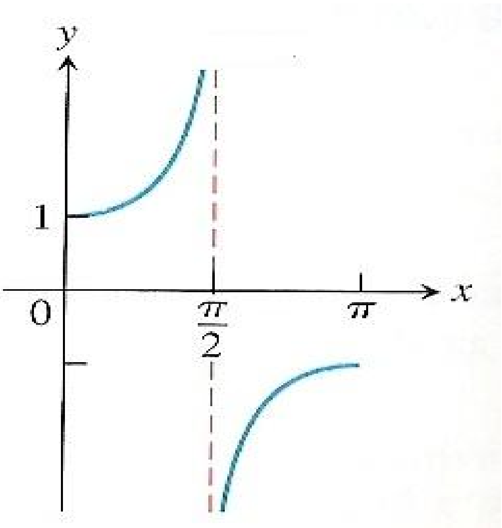


The Arc Secant:

Y = arcsec x   or   y = $\sec^{-1}$ x

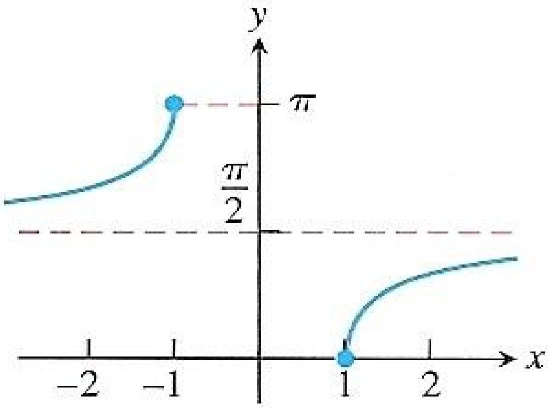
Y= sec x

D_f : $[0\,,\pi] \cup x \neq \frac{\pi}{2}$ R_f :$|y| \geq 1$



y = $\sec^{-1}$ x

D_f^{-1} :$|x| \geq 1$ R_f^{-1} : $[0\,,\pi] \cup y \neq \frac{\pi}{2}$



The properties:

1. $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

   $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

   $\sec^{-1} + \csc^{-1} x = \frac{\pi}{2}$

2. $\sin^{-1}(-x) = -\sin^{-1} x$ odd function
   $\tan^{-1}(-x) = -\tan^{-1} x$ odd function
   $\csc^{-1}(-x) = -\csc^{-1} x$ odd function

3. $\cos^{-1}(-x) + \cos^{-1} x = \pi$
   $\cot^{-1}(-x) + \cot^{-1} x = \pi$
   $\sec^{-1}(-x) + \sec^{-1} x = \pi$

4. $\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1} x$

   $\sec^{-1}\left(\frac{1}{x}\right) = \sin^{-1} x$

   $\cot^{-1}(x) = \left(\frac{\pi}{2}\right) - \tan^{-1} x$

Example 1

Draw the function

$$y = 2sin^{-1}(3x - 1).$$

Solution

D_f : $-1 \le 3x - 1 \le 1$

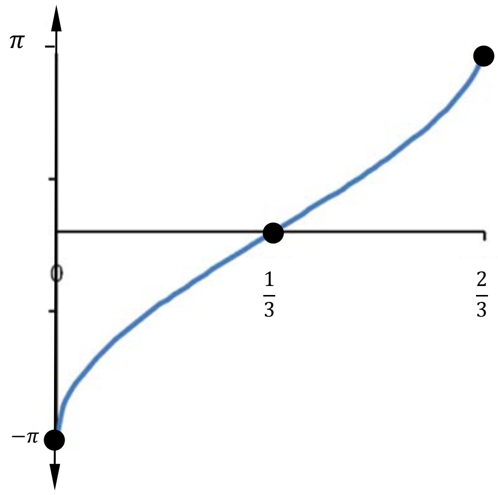
$0 \le 3x \le 2$

$0 \le x \le \frac{2}{3}$

R_f : $-\frac{\pi}{2} \le \sin^{-1}(3x - 1) \le \frac{\pi}{2}$

$-\pi \le 2\sin^{-1}(3x - 1) \le \pi$

x	y
0	$-\pi$
$\frac{2}{3}$	π
$\frac{1}{3}$	0



Example 2

Draw the function

$$y = \frac{1}{2}\cos^{-1}(x+1).$$

Solution

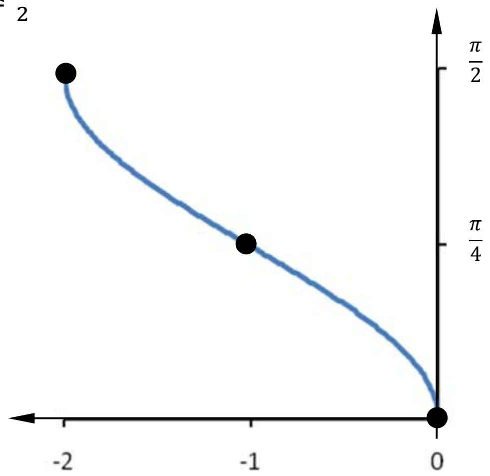
D_f : $\quad -1 \le x+1 \le 1$

$\quad -2 \le x \le 0$

R_f : $0 \le \cos^{-1}(x+1) \le \pi$

$\quad 0 \le \frac{1}{2}\cos^{-1}(x+1) \le \frac{\pi}{2}$

x	y
0	0
-1	$\frac{\pi}{4}$
-2	$\frac{\pi}{2}$



Example 3

Draw the functions

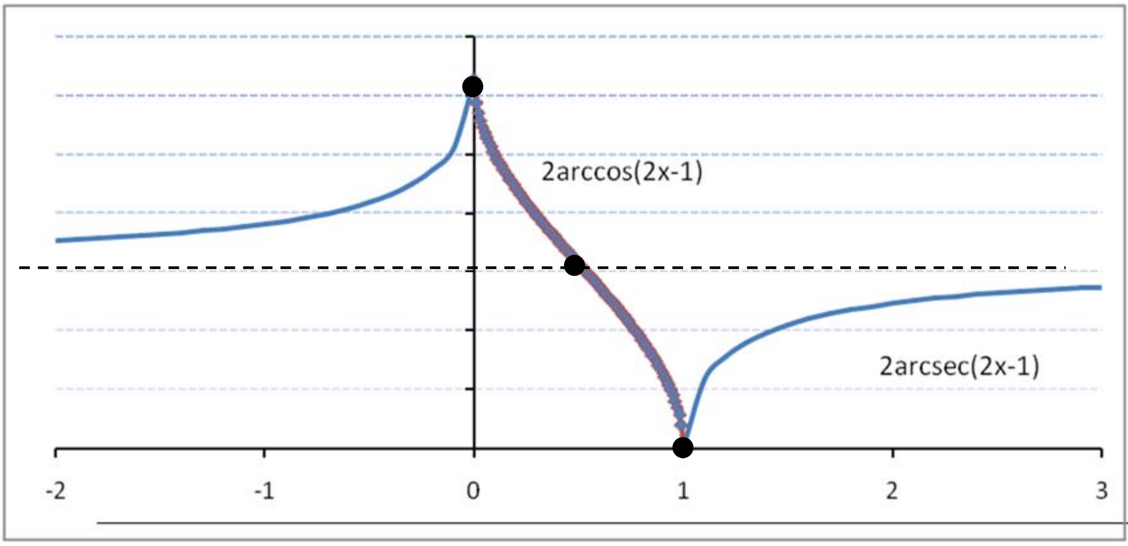
$$y = 2\cos^{-1}(2x-1) \;\; and \;\; y = 2\sec^{-1}(2x-1).$$

Solution

D_f: $-1 \le 2x-1 \le 1 \;\Rightarrow\; 0 \le 2x \le 2 \;\Rightarrow\; 0 \le x \le 1$

R_f : $0 \le \cos^{-1}(2x-1) \le \pi \;\Rightarrow\; 0 \le 2\cos^{-1}(2x-1) \le 2\pi$

x	y
1	0
$\frac{1}{2}$	π
0	2π



Example 4

Draw the function

$$y = 3\sin^{-1}\frac{2}{x+1}$$

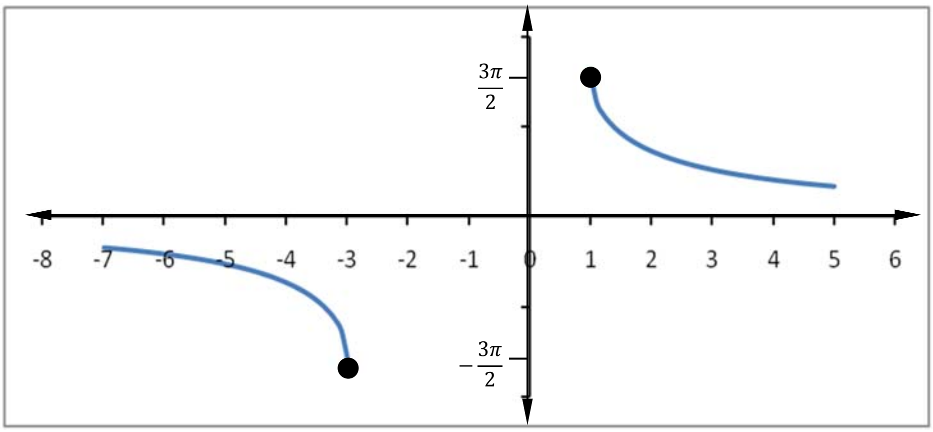
Solution $y = 3\sin^{-1}(\frac{2}{x+1}) = 3\csc^{-1}(\frac{x+1}{2})$

Df: $x \neq -1$

Draw the function $y = 3\csc^{-1}\frac{x+1}{2}$ *by drawing* $y = 3\sin^{-1}\frac{x+1}{2}$

Df : $-1 \leq \frac{(x+1)}{2} \leq 1 \implies -2 \leq x+1 \leq 2 \implies -3 \leq x \leq 1$

Rf : $-\frac{\pi}{2} \leq \sin^{-1}\frac{x+1}{2} \leq \frac{\pi}{2} \implies -\frac{3\pi}{2} \leq 3\sin^{-1}\frac{x+1}{2} \leq \frac{3\pi}{2}$



x	y
-3	$-\frac{3\pi}{2}$
-1	0
1	$\frac{3\pi}{2}$

Example 5

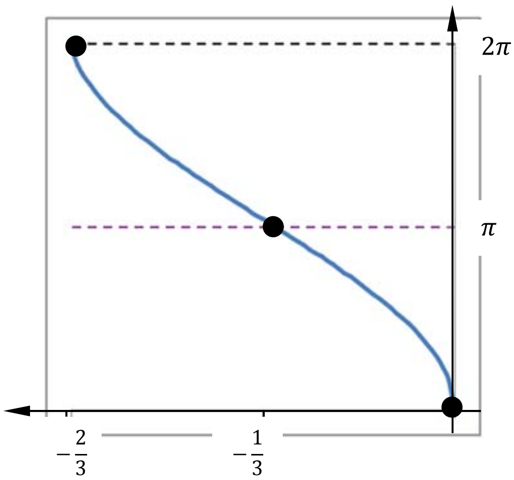
Draw the function

$$y = 2\sec^{-1}\frac{1}{3x+1}.$$

Solution

Df : $-1 \leq 3x+1 \leq 1 \implies -\frac{2}{3} \leq x \leq 0 \cup x \neq -\frac{1}{3}$

Rf : $0 \leq \cos^{-1}(3x+1) \leq \pi \implies 0 \leq 2\cos^{-1}(3x+1) \leq 2\pi$



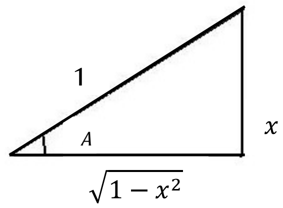
x	y
0	0
$-\frac{1}{3}$	π
$-\frac{2}{3}$	2π

Example 6

Prove that

$\cos(\sin^{-1} x) = \sqrt{1-x^2}$

Solution



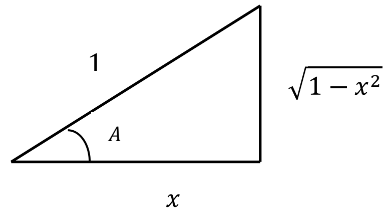
Let $A = \sin^{-1} x \Rightarrow \sin A = x \Rightarrow \cos A = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

Example 7

Prove that

$$\sec(2\cos^{-1} x) = \frac{1}{2x^2 - 1}.$$

Solution



Let $A = \cos^{-1} x \Rightarrow \cos A = x$

$$\sec 2A = \frac{1}{\cos 2A} = \frac{1}{COS^2 A - \sin^2 A}$$

$$= \frac{1}{\left(\frac{x}{1}\right)^2 - \left(\frac{\sqrt{1-x^2}}{1}\right)} = \frac{1}{x^2 - 1 + x^2} = \frac{1}{2x^2 - 1}$$

Example 8

Find the value of x: (with using calculator)

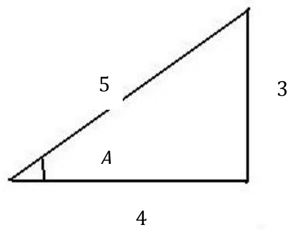
Solution

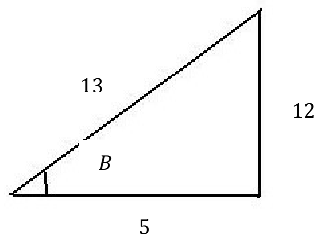
1. $x = \sin^{-1}\left(\frac{1}{2}\right) = 30$
2. $x = \tan^{-1}\sqrt{3} = 60$
3. $x = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = 135$
4. $x = \sec^{-1}\sqrt{2} = \cos^{-1}\frac{1}{\sqrt{2}} = 45$
5. $x = \csc^{-1} 2 = \sin^{-1}\frac{1}{2} = 30$
6. $x = \csc^{-1}\left(-\frac{2}{\sqrt{3}}\right) = \sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = -60$
7. $x = \cot^{-1}\sqrt{3} = \frac{\pi}{2} - \tan^{-1}\sqrt{3} = 90 - 60 = 30$
8. $x = \tan(\sin^{-1}\frac{-1}{2}) = \frac{-\sqrt{3}}{2} = -0.577$
9. $\tan(\sec^{-1} 1) + \sin(\csc^{-1} -2) = \frac{-1}{2}$
10. $x = \sec(\cot^{-1}\sqrt{3} + \csc^{-1}(-1)) = \frac{1}{\cos((\frac{\pi}{2} - \tan^{-1}\sqrt{3}) + \sin^{-1}(-1))} = 2$

Example 9

Find the value of x. (without using calculator)

$$x = \cos(\cos^{-1}\frac{4}{5} + \sin^{-1}\frac{12}{13})$$





Solution

$$let\ A = \cos^{-1}\frac{4}{5} \implies cosA = \frac{4}{5}$$

$$let\ B = \sin^{-1}\frac{12}{13} \implies \sin B = \frac{12}{13}$$

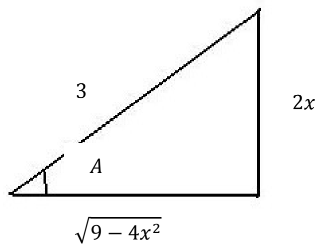
$$x = \cos(A + B) = cosA\ cosB - \sin A \sin B$$

$$= \frac{4}{5} * \frac{15}{13} - \frac{3}{5} * \frac{12}{13}$$

Example 10

Evaluate the following expression:

$$\cos(\sin^{-1}\frac{2x}{3})$$



Solution

$$let\ A = \sin^{-1}\frac{2x}{3} \implies \cos A = \frac{\sqrt{9-4X^2}}{3}$$

Example 11

Sketch the graph

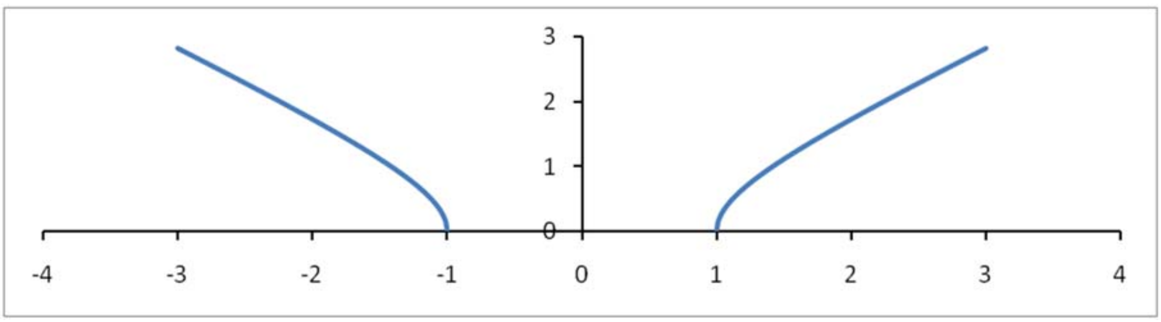
$$y = \cot(\csc^{-1} x)$$

Solution

Let $A = \csc^{-1} x \implies csc\ A = x$

$$y = \cot A = \frac{\sqrt{X^2-1}}{1} = \sqrt{x^2-1}$$

Df: $x^2 - 1 \geq 0 \implies x^2 \geq 1 \implies |x| \geq 1$



HOMEWORK

Find the value of x to the following:

1. $x = \sin(\cos^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{3}{\sqrt{10}})$
2. $x = \tan^{-1}\frac{1}{3} + \tan^{-1} 3$
3. $x = \sec^{-1}\frac{5}{3} + \cos^{-1}\frac{3}{5}$
4. $\sin(\tan^{-1}\frac{x}{\sqrt{x^2+1}})$ (Evaluate the expression)
5. $x = \sin 2(\cos^{-1}\frac{1}{3})$
6. $x = \tan^{-1}\frac{1}{3} + \cot^{-1} 3$ $\quad Ans. x = \sin^{-1}\frac{3}{5}$
7. $x = \sin^{-1}\frac{2}{\sqrt{5}} + \cos^{-1}\frac{1}{\sqrt{10}}$
8. $x = \cos(2\tan^{-1}\frac{1}{3})$
9. $x = \sin(\cos^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{3}{\sqrt{10}})$
10. $x = \tan^{-1}\frac{1}{3} + \tan^{-1} 3$

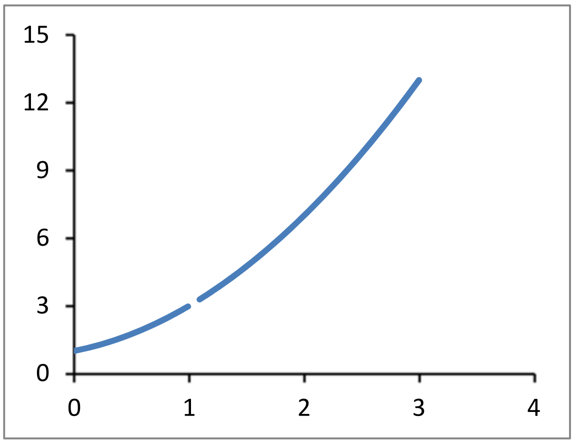
4 LIMITS

One of the important things to know about a function f is how its outputs will change when the inputs change. If the inputs get closer and closer to some specific value c, for example, will the outputs get closer to some specific value L? If they do, we want to know that, because it means we can control the outputs by controlling the inputs. To talk sensibly about this, we need the language of limits.

Definition:

If the values of a function f(x) approach the value L as x approaches c, we say f has limit L as x approaches c and we write $\lim_{x \to c} f(x) = L$ (Read " the limit of f of x as approaches c equals L").

For example $f(x) = \frac{x^3-1}{x-1}$, $x \neq 1$, $f(1) = ?$



x	y
1.25	3.813
1.1	3.310
1	?
.999	2.997
.99	2.97

From the fig. $\lim_{x \to 1} \frac{x^3-1}{x-1} = 3$

Or we can simplify the formula by factoring the numerator and canceling common factors:

$$\lim_{x \to 1} \frac{x^3-1}{x-1} = \lim_{x \to 1} \frac{(x-1)(x^2+x+1)}{(x-1)} = \lim_{x \to 1}(x^2 + x + 1) = 1 + 1 + 1 = 3$$

Ex: $\lim_{x \to 1} \frac{x+2}{2x-1} = \frac{3}{1} = 3$

After substitution if f has limit without using factorial analysis $\Longrightarrow$ f(x) is defined at x = c

Ex: $\lim_{x \to 2} \frac{x^2-4}{x-2} = \frac{0}{0} = \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)} = \lim_{x \to 2}(x+2) = 4$

If f(x) is not defined at x = c, but after factorial analysis we can find the limit $\Longrightarrow$ f(x) is hole at x = c

Ex: $\lim_{x \to 2} \frac{x^2-4}{(x-2)^2} = \frac{0}{0} = \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)(x-2)} = \lim_{x \to 2} \frac{x+2}{x-2} = \frac{4}{0} = \infty$

If *f*(x) is not defined at x = c and after factorial analysis the limit not exist ⟹ x=c is asymptote

The properties:

If $\lim_{x\to c} f_1(x) = L_1$ and $\lim_{x\to c} f_2(x) = L_2$, then:

1. $\lim_{x\to c} K = K$ $\qquad (K = any\ number)$

2. $\lim_{x\to c} K\, f_1(x) = K \lim_{x\to c} f_1(x) = KL_1$

3. $\lim_{x\to c}(f_1(x) + f_2(x)) = \lim_{x\to c} f_1(x) + \lim_{x\to c} f_2(x) = L_1 + L_2$

4. $\lim_{x\to c}(f_1(x) - f_2(x)) = \lim_{x\to c} f_1(x) - \lim_{x\to c} f_2(x) = L_1 - L_2$

5. $\lim_{x\to c}(f_1(x) * f_2(x)) = \lim_{x\to c} f_1(x) * \lim_{x\to c} f_2(x) = L_1 * L_2$

6. $\lim_{x\to c} \frac{f_1(x)}{f_2(x)} = \frac{\lim_{x\to c} f_1(x)}{\lim_{x\to c} f_2(X)} = \frac{L_1}{L_2}$ $\qquad If\ L_2 \neq 0$

7. $\lim_{x\to c} \sqrt[n]{f_1(x)} = \sqrt[n]{\lim_{x\to c} f_1(x)} = L_1^{\frac{1}{n}}$ $\qquad L_1 > 0, n = even\ number$

8. $\lim_{x\to c} | f_1(x)| = \left|\lim_{x\to c}(f_1(x)\right| = |L_1|$

Ex: $\lim_{x\to 4} \frac{x^2+2x-24}{x-4} = \lim_{x\to 4} \frac{(x+6)(x-4)}{x-4} = \lim_{x\to 4}(x+6) = 10$

Ex: $\lim_{x\to 1} \frac{\frac{1}{x+1}-\frac{1}{2}}{x-1} = \lim_{x\to 1} \frac{\frac{2-(x+1)}{2(x+1)}}{x-1} = \lim_{x\to 1} \frac{2-x-1}{2(x-1)(x+1)} = \lim_{x\to 1} \frac{1-x}{2(x-1)(x+1)} =$
+

$\lim_{x\to 1} \frac{-(x-1)}{2(x-1)(x+1)} = \lim_{x\to 1} \frac{-1}{2(x+1)} = \frac{-1}{4}$

Ex: $\lim_{x\to 3} \frac{x^3-27}{x-3} = \lim_{x\to 3} \frac{(x-3)(x^2+3x+9)}{x-3} = \lim_{x\to 3}(x^2+3x+9) = 27$

Ex: $\lim\limits_{x\to 0} \dfrac{\frac{1}{\sqrt{2x+1}}-1}{x} = \lim\limits_{x\to 0} \dfrac{\frac{1-\sqrt{2x+1}}{\sqrt{2x+1}}}{x} = \lim\limits_{x\to 0} \dfrac{1-\sqrt{2x+1}}{x\sqrt{2x+1}} * \dfrac{1+\sqrt{2x+1}}{1+\sqrt{2x+1}} =$

$\lim\limits_{x\to 0} \dfrac{1-(2x+1)}{x\sqrt{2x+1}\,(1+\sqrt{2x+1})} = \lim\limits_{x\to 0} \dfrac{-2x}{x\sqrt{2x+1}\,(1+\sqrt{2x+1})} = \lim\limits_{x\to 0} \dfrac{-2}{x\sqrt{2x+1}\,(1+\sqrt{2x+1})} =$

-1

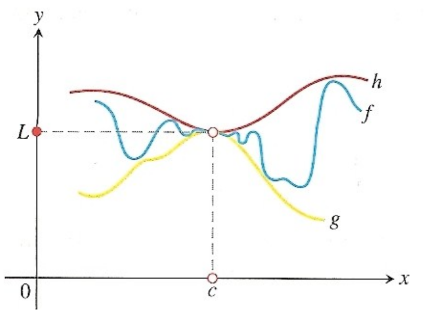
Ex: $\lim\limits_{x\to 1} \dfrac{x^2-1}{\sqrt[3]{x}-1} = \lim\limits_{x\to 1} \dfrac{x^2-1}{\sqrt[3]{x}-1} * \dfrac{(x^{2/3}+x^{1/3}+1)}{(x^{2/3}+x^{1/3}+1)} = \lim\limits_{x\to 1} \dfrac{(x^2-1)(x^{2/3}+x^{1/3}+1)}{x-1} =$

$\lim\limits_{x\to 1} \dfrac{(x-1)(x+1)(x^{2/3}+x^{1/3}+1)}{x-1} = 6$

Ex: $\lim\limits_{h\to 0} \dfrac{(1+h)^{\frac{2}{3}}-1}{h} = \lim\limits_{h\to 0} \dfrac{\left((1+h)^{\frac{1}{3}}-1\right)\left((1+h)^{\frac{1}{3}}+1\right)}{h} *$

$\dfrac{(1+h)^{\frac{2}{3}}+(1+h)^{\frac{1}{3}}+1}{(1+h)^{\frac{2}{3}}+(1+h)^{\frac{1}{3}}+1} = \lim\limits_{h\to 0} \dfrac{(1+h-1)\left((1+h)^{\frac{1}{3}}+1\right)}{h\left((1+h)^{\frac{2}{3}}+(1+h)^{\frac{1}{3}}+1\right)} =$

$\lim\limits_{h\to 0} \dfrac{h((1+h)^{\frac{1}{3}}+1)}{h\left((1+h)^{\frac{2}{3}}+(1+h)^{\frac{1}{3}}+1\right)} = \dfrac{2}{3}$



The Sandwich Theorem:

Suppose that g(x) ≤ f(x) ≤ h(x)

for all x ≠ c in some interval about c and that,

$\lim\limits_{x\to c} g(x) = \lim\limits_{x\to c} h(x) = L$

Then: $\lim\limits_{x\to c} f(x) = L$

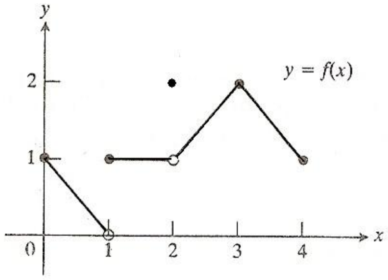
Ex: $\lim\limits_{x\to\infty} \dfrac{\sin x}{x}$

$-1 \le \sin x \le 1 \quad \Longrightarrow \quad -\dfrac{1}{x} \le \dfrac{\sin x}{x} \le \dfrac{1}{x}$

$$\left.\begin{aligned}\lim_{x\to\infty}\frac{1}{x}=0\\ \lim_{x\to\infty}\frac{-1}{x}=0\end{aligned}\right\}\Longrightarrow \lim_{x\to\infty}\frac{\sin x}{x}=0$$

Right – hand Limits and Left – hand Limits:

Sometimes the values of a function *f(x)* tend to different limits as x approaches a number c from different sides.



Right limit (RL):

$$RL = \lim_{x\to a^+} f(x) = \lim_{\epsilon\to 0} f(a+\epsilon)$$

Left limit (LL):

$$LL = \lim_{x\to a^-} f(x) = \lim_{\epsilon\to 0} f(a-\ \epsilon)$$

A function *f(x)* has a limit as x approaches a if and only if the right-hand and left-hand limits at a exist and are equal.

Ex: For the function *f(x)* shown in figure find the following limits.

At x = 0 : $\lim_{x\to 0^+} f(x) = 1$

At x = 1 : $\lim_{x\to 1^-} f(x) = 0 \quad even\ though\ f(1) = 1$

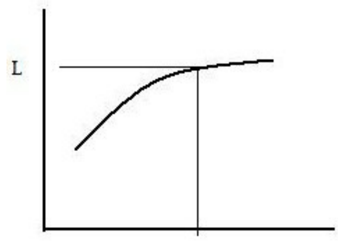
$\lim_{x\to 1^+} f(x) = 1$

f(x) has no limit as x → 1 (RL ≠ LL)

At x = 2 : $\lim_{x\to 2^-} f(x) = 1$

$\lim_{x\to 2^+} f(x) = 1$

$\lim_{x\to 2} f(x) = 1 \quad even\ though\ f(2)\ =\ 2$



At x = 3 : $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3} f(x) = f(3) = 2$

At x = 4 : $\lim_{x\to 4^-} f(x) = 1$

Ex: $\lim_{x\to 1^+}\frac{x^2-1}{\sqrt{x}-1} = \lim_{x\to 1^+}\frac{(x-1)(x+1)}{\sqrt{x}-1} = \lim_{x\to 1^+}\frac{(\sqrt{x}-1)(\sqrt{x}+1)(x+1)}{\sqrt{x}-1} =$

$\lim_{x\to 1^+}\left(\sqrt{x+1}\right)(x+1) = \lim_{\epsilon\to 0}\left(\sqrt{1+\epsilon}\ +\ 1\right)(1+\epsilon+1) = 4$

Ex: $f(x) = \begin{cases} sinx & x \geq \frac{\pi}{4} \\ cosx & x < \frac{\pi}{4} \end{cases}$

Required: $\lim_{x \to \frac{\pi}{4}} f(x)$, $\lim_{x \to \frac{\pi}{2}} f(x)$.

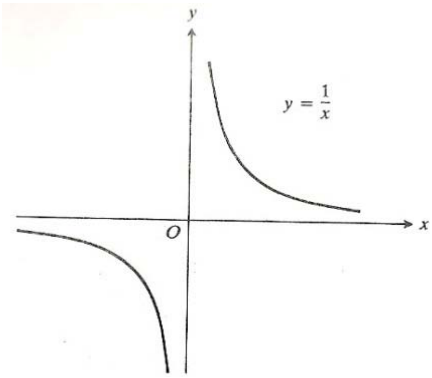
solution

RL: $\lim_{x \to \frac{\pi}{4}^+} f(x) = \lim_{x \to \frac{\pi}{4}} sinx = \frac{1}{\sqrt{2}}$

LL: $\lim_{x \to \frac{\pi}{4}^-} f(x) = \lim_{x \to \frac{\pi}{4}} cosx = \frac{1}{\sqrt{2}}$

LL = RL = $\frac{1}{\sqrt{2}} \Longrightarrow \lim_{x \to \frac{\pi}{4}} f(x) = \frac{1}{\sqrt{2}}$

$\lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} sinx = 1$



Ex: $\lim_{z \to 0.5} \frac{z^2 - 0.25}{|0.5 - z|} = \lim_{z \to 0.5} \frac{(z-0.5)(z+0.5)}{|0.5-z|}$

$|0.5 - z| =$

$\begin{cases} 0.5 - z & 0.5 - z \geq 0 \Rightarrow -z \geq -0.5 \Rightarrow z \leq 0.5 \\ -(0.5 - z) & 0.5 - z < 0 \Rightarrow -z < -0.5 \Rightarrow z > 0.5 \end{cases}$

RL: $\lim_{Z \to 0.5^+} \frac{(z-0.5)(z+0.5)}{-(0.5-Z)} = \lim_{Z \to 0.5^+} \frac{-(0.5-z)(z+0.5)}{-(0.5-Z)} = \lim_{Z \to 0.5^+} (z + 0.5) = 1$

LL: $\lim_{Z \to 0.5^-} \frac{(Z-0.5)(Z+0.5)}{(0.5-Z)} = \lim_{Z \to 0.5^-} \frac{(Z-0.5)(Z+0.5)}{-(Z-0.5)} = \lim_{Z \to 0.5^-} -(z + 0.5) = -1$

RL $\neq$ LL

Limits Involving Infinity:

$f(x) = \frac{1}{x}$

$\lim_{x \to 0^+} \frac{1}{x} = \infty$ or $\lim_{x \to 0} \frac{1}{x} = \infty$ $x > 0$

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$ or $\lim_{x \to 0} \frac{1}{x} = -\infty$ $x < 0$

$\lim_{x \to \infty} \frac{1}{x} = 0$

$\lim_{x\to-\infty}\frac{1}{x}=0$

Properties of ∞ :

1) $\infty+\infty=\infty$

2) $\infty^n=\infty$

3) $\infty*\infty=\infty$

4) $a+\infty=\infty$

5) $a-\infty=-\infty$

6) $a*\infty=\infty$ $\quad a>0$

$\quad =-\infty$ $\quad a<0$

7) $\frac{a}{\infty}=0$

8) $\frac{\infty}{a}=\infty$ $\quad a>o$

$\quad =-\infty$ $\quad a<o$

Note: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty-\infty$, $\infty*0$, 1^{∞} $\Longrightarrow$ $undefined\ quantities$

Note: To find the limit of rational function as x→±∞ we divide the numerator and denominator by the highest power of x.

Ex: $\lim_{x\to\infty}\frac{2x^3-x+1}{x^3+1}=\lim_{x\to\infty}\frac{2-\frac{1}{x^2}+\frac{1}{x^3}}{1+\frac{1}{x^3}}=2$

Ex: $\lim_{x\to\infty}(\sqrt{x^2+1}-x)=\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)*\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}=\lim_{x\to\infty}\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}=$

$\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}=0$

Ex: $\lim_{n\to\infty}(\sqrt{n^2+10}-\sqrt{n^2+n})=$

$\lim_{n\to\infty}\left(\sqrt{n^2+10}-\sqrt{n^2+n}\right)\cdot\frac{\sqrt{n^2+10}+\sqrt{n^2+n}}{\sqrt{n^2+10}+\sqrt{n^2+n}}=\lim_{n\to\infty}\frac{n^2+10-(n^2+n)}{\sqrt{n^2+10}+\sqrt{n^2+n}}=$

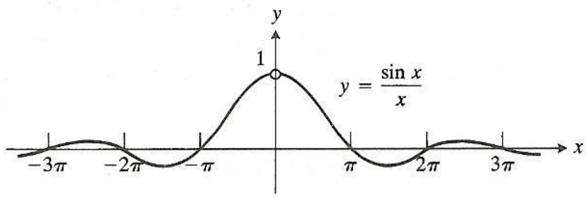
$\lim_{n\to\infty}\frac{10-n}{\sqrt{n^2+10}+\sqrt{n^2+n}}=\lim_{n\to\infty}\frac{\frac{10}{n}-1}{\sqrt{1+\frac{10}{n^2}}+\sqrt{1+\frac{1}{n}}}=\frac{-1}{2}$

Note: $\lim_{x\to a} f(x) = \infty \quad \Longrightarrow \quad x = a \quad vertical\ asymptote$

$\lim_{x\to\infty} f(x) = b \quad \Longrightarrow \quad x = b \quad horizontal\ asymptote$

Theorem: $\lim_{\theta\to 0} \frac{sin\theta}{\theta} = 1$

θ	$\frac{sin\theta}{\theta}$
1	0.84
0.5	0.95
0.1	0.998
0.01	0.9995
↓	↓
0	?
↑	↑
-0.01	0.9995
-0.1	0.998



Ex: $\lim_{x\to 0} \frac{sin2x}{x} = \lim_{x\to 0} \frac{2sin2x}{2x} = 2$

Ex: $\lim_{x\to 0} \frac{sin2x}{sin3x} = \lim_{x\to 0} \frac{\frac{2\sin 2x}{2x}}{\frac{3\sin 3x}{3x}} = \frac{2}{3}$

Ex: $\lim_{x\to 0} \frac{\sqrt{1-\cos x}}{x} = \lim_{x\to 0} \frac{\sqrt{2\sin^2\left(\frac{x}{2}\right)}}{x} = \sqrt{2}\lim_{x\to 0} \frac{\sin(\frac{x}{2})}{x} = \sqrt{2}\ \lim_{x\to 0} \frac{\sin(\frac{x}{2})}{\frac{2x}{2}} = \frac{1}{\sqrt{2}}$

Example: Evaluate the following limits:

1. $\lim_{y\to\frac{\pi}{2}} \frac{\cos y}{\frac{\pi}{2} - y} = \lim_{x\to 0} \frac{\cos(\frac{\pi}{2} - x)}{x} = \lim_{x\to 0} \frac{\cos(\frac{\pi}{2})\cos 0 - \sin(\frac{\pi}{2})\ sinx}{x} = \lim_{x\to 0} \frac{\sin x}{x} = 1$

let $x = \frac{\pi}{2} - y \quad \Longrightarrow \quad y = \frac{\pi}{2} - x$

$y \to \frac{\pi}{2} \Longrightarrow x \to 0$

2. $\lim_{y\to\infty} 2y\tan\frac{\pi}{y} = \lim_{y\to\infty} 2\frac{\pi}{x}\tan x = \lim_{y\to\infty} 2\pi\,\frac{\sin x}{x\cos x} = 2\pi$

Let $x = \frac{\pi}{y} \implies y = \frac{\pi}{x}$

$y \to \infty \implies x \to 0$

3. $\lim_{x\to-\frac{\pi}{8}} \frac{\sin 2x+\cos 2x}{x+\frac{\pi}{8}} = \lim_{y\to 0} \frac{\sin 2\left(y-\frac{\pi}{8}\right)+\cos 2\left(y-\frac{\pi}{8}\right)}{y} =$

Let $y = x + \frac{\pi}{8} \implies x = y - \frac{\pi}{8}$

$x \to -\frac{\pi}{8} \implies y \to 0$

$$\lim_{y\to 0} \frac{\sin\left(2y-\frac{\pi}{4}\right)\cos\left(2y-\frac{\pi}{8}\right)}{y} = \lim_{y\to 0} \frac{\sin 2y\cos\frac{\pi}{4} - \cos 2y\sin\frac{\pi}{4} + \cos 2y\cos\frac{\pi}{4} + \sin 2y\sin\frac{\pi}{4}}{y} =$$

$$\lim_{y\to 0} \frac{\frac{1}{\sqrt{2}}\sin 2y - \frac{1}{\sqrt{2}}\cos 2y + \frac{1}{\sqrt{2}}\cos 2y + \frac{1}{\sqrt{2}}\sin 2y}{y} = \lim_{y\to 0} \frac{\frac{2}{\sqrt{2}}\sin 2y}{y} = \frac{2}{\sqrt{2}}\lim_{y\to 0} \frac{\sin 2y}{\frac{2y}{2}} = \frac{4}{\sqrt{2}} =$$

$2\sqrt{2}$

4. $\lim_{x\to 0} \frac{\sin(1-\cos x)}{x} = \lim_{x\to 0} \frac{\sin(1-\cos x)}{x} * \frac{1-\cos x}{1-\cos x} = \lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{1-\cos x}{x} *$
$\frac{1+\cos x}{1+\cos x} = \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)} =$

$$\lim_{x\to 0} \frac{\sin x}{x} * \lim_{x\to 0} \frac{\sin x}{(1+\cos x)} = 1 * \frac{0}{2} = 0$$

5. $\lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0} \frac{\sin x\cos h+\sin h\cos x-\sin x}{h} = \lim_{h\to 0} \frac{\sin x\,(\cos h-1)+\sin h\cos x}{h} =$

$$\lim_{h\to 0} \frac{\sin x\,(\cos h-1)}{h} + \lim_{h\to 0} \frac{\sin h\cos x}{h} = \sin x \lim_{h\to 0} \frac{\cos h-1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} =$$

$$\left(\sin x \lim_{h\to 0} \frac{\cos h-1}{h} * \frac{\cos h+1}{\cos h+1}\right) + \cos x = \left(\sin x \lim_{h\to 0} \frac{-\sin^2 h}{h(\cos h+1)}\right) + \cos x =$$

$$\left(\sin x \lim_{h\to 0} \frac{\sin h}{h} * \lim_{h\to 0} \frac{-\sin h}{(\cos h+1)}\right) + \cos x = \sin x + 1 * \frac{0}{2} + \cos x = \cos x$$

Ex: $\lim_{x\to\frac{\pi}{2}} (\tan x - \sec x) = \lim_{x\to\frac{\pi}{2}} \left(\frac{\sin x}{\cos x} - \frac{1}{\cos x}\right) = \lim_{x\to\frac{\pi}{2}} \frac{\sin x-1}{\cos x} * \frac{\sin x+1}{\sin x+1} =$

$$\lim_{x\to\frac{\pi}{2}} \frac{-\cos^2 x}{\cos x\,(\sin x+1)} = \lim_{x\to\frac{\pi}{2}} \frac{-\cos x}{\sin x+1} = \frac{0}{2} = 0$$

H.W:

1. $\lim_{x \to \frac{\pi}{2}} (\frac{\pi}{2} - x) \tan x$

2. $\lim_{x \to \infty} (\sqrt{x^2 + 2x} - x)$

3. $\lim_{x \to \infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x}$

4. $\lim_{x \to \infty} f(x)$ if $\frac{3x-4}{x} < f(x) < \frac{3x^2+5x}{x^2}$

5. $\lim_{x \to 2} \frac{4-x^2}{3-\sqrt{x^2+5}}$

6. $\lim_{x \to a} \frac{x^2-(a+b)x+ab}{x-a}$

7. $\lim_{x \to 0} \frac{\cot 4x}{\cot 3x}$

8. $\lim_{x \to \frac{\pi}{4}} \frac{\cos x - \sin x}{x - \frac{\pi}{4}}$

9. $\lim_{x \to 0} \frac{\tan(x - \sin x)}{x}$

10. $If\ f(x) = 5x^2\ find\ \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$

12. Find the vertical and horizontal asymptotes to the following function.

$a)\ y = \frac{x+1}{\sqrt{x^2-4}}$

$b)\ y = \frac{2x^2 - x + 5}{6x^2 + 5x - 1}$

Continuous Functions :

The function $y = f(x)$ is continuous at $x = c$ if and only if all three of the following statements are true:

1. $f(c)$ exists (c lies in the domain of *f)*

2. $\lim_{x \to c} f(x)$ exists (f has a limit as x⟶c)

3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

(The limit in the continuity test is to be two sided if c an interior point of the domain of f , it is to be the appropriate one-sided limit if c is an end point of the domain).

For example: for the function shown in the Figure.

a) f is continuous at x = 0 because

i) *f(0)* exists (it equals 1)

ii) $\lim_{x\to 0^+} f(x) = 1$ (f has a limit as x⟶0^+)

iii) $\lim_{x\to 0^+} f(x) = f(0)$ (the limit equals the function value)

b) f is discontinuous at x = 1 because $\lim_{x\to 1} f(x)$ does not exist.(LL≠RL)

c) f is discontinuous at x = 2 because $\lim_{x\to 2} f(x) \neq f(2)$

d) f is continuous at x = 3 because :

i) *f(3)* exists (it equals 2)

ii) $\lim_{x\to 3} f(x) = 2$ (f has a limit as x⟶3)

iii) $\lim_{x\to 3} f(x) = f(3)$ (the limit equals the function value)

e) f is continuous at x=4 because

i) *f(4)* exists (it equals 1)

ii) $\lim_{x\to 4^-} f(x) = 1$ (f has a limit as x→ 4^-)

iii) $\lim_{x\to 4^-} f(x) = f(4)$ (the limit equals the function value)

Theorem:

If the following *f* and *g* are continuous at x=c, then the following combinations are continuous at x=c

f + g , *f - g* , *f * g* , *k * g* (*k*=any number) , *f / g* (*g(c)≠0)* , *f o g* , *g o f* , $\sqrt[n]{f}$ *(f(c) >0, n= even).*

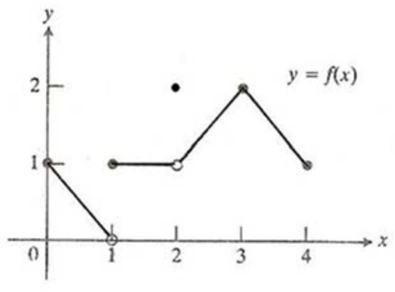
Some functions are always continuous:

1. polynomials functions

$$F(x) = a_0\, x_n + a_1\, x_{n-1} + \ldots + a_n$$

2. The functions $cos\ x$ and $sin\ x$.

3. Rational functions (a rational function is continuous everywhere except at the point where the denominator is zero).



Theorem:

If f is continuous on $[a, b]$, and if *f(a)* and *f(b)* have opposite signs, then there is at least one solution of the equation *f(x)=0* in the interval (a , b).

$F(a)>0$

a $F(x) =$ b

$F(b) < 0$

Example:

If the function *f(x)* is continuous in all points in the interval, find a, and b.

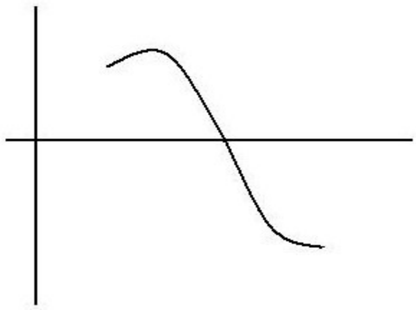
$$f(x) = \begin{cases} -1 & x \le 0 \\ ax + b & 0 < x < 1 \\ 1 & x \ge 1 \end{cases}$$

Solution.

at x=0 $\Longrightarrow$ *f(o)* = -1

LL = $\lim_{x \to 0^-} f(x) = \lim_{x \to 0} -1 = -1$

RL = $\lim_{x \to 0^+} f(x) = \lim_{x \to 0} ax + b = b$



$\because$RL = LL $\Longrightarrow$ b = -1

at x = 1 $\Longrightarrow$ *f*(1) = 1

LL = $\lim_{x \to 1^-} f(x) = \lim_{x \to 1} ax + b = a + b$

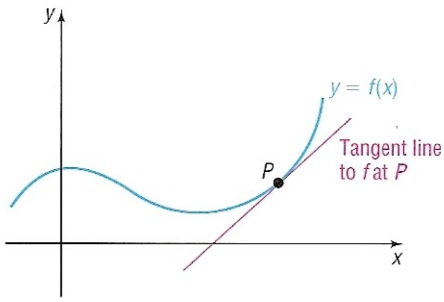
RL = $\lim_{x \to 1^+} f(x) = \lim_{x \to 1} 1 = 1$

$\because$ RL = LL $\Longrightarrow$ a + b =1 $\Longrightarrow$ a – 1 =1 $\Longrightarrow$ a = 2

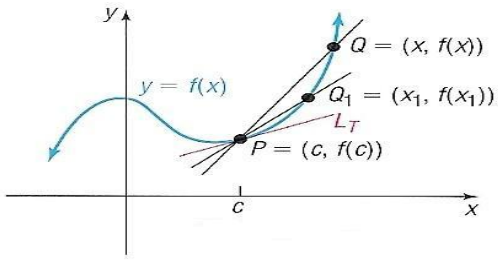
5 DIFFERENTIATION

Tangent lines and rates of change:

On question that motivated the development of calculus was a geometry problem, the tangent problem. This problem asks, "What is the slope of the tangent line to the graph of a function y=f(x) at a point P on its graph?"



The tangent line L_t to the graph of a function $y = f(x)$ at a point P necessarily contains the point P. To find an equation for L_t using the point –slope from of the equation of a line, it remains to find the slope $m_{\tan}$ of the tangent line.

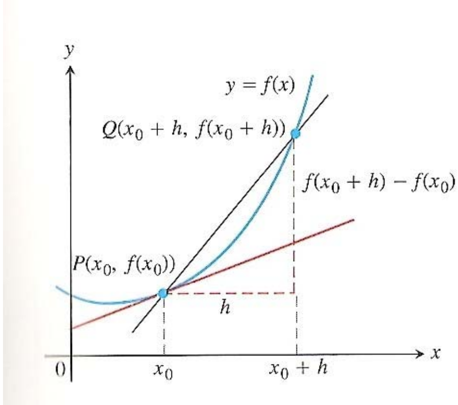


Suppose that the coordinates of the point P are $(c\,, f(c))$.Locate another point $Q = (x\,, f(x))$ on the graph of f. The line containing P and Q is a secant line. The slope of $m_{\sec}$ of the secant line is

$$m_{sec} = \frac{f(x) - f(c)}{x - c}$$

Now look at figure. As we move along the graph of f from Q toward P, we obtain a succession of secant lines. The closer we get to P, the closer the secant line is to the tangent line L_t. The limiting position of these secant lines is to the tangent line L_t. Therefore, the limiting value of the slopes of these secant lines equals the slope of the tangent line. But, as we move from Q towards P, the value of x get closer to c . Therefore,

$$m_{tan} = \lim_{x\to c} m_{sec} = \lim_{x\to c} \frac{f(x) - f(c)}{x-c}$$



Or $\quad m_{tan} = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$

Or we can say $f''''(x) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$

Where $f'(x)$is read " prime of x "

Ex: Let f(x) = x² + 1

a) Find f\`(x) b) f\`(2) , f\`(0) , f\`(-2)

Sol.

$$f`(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{((x+h)^2 + 1) - (x^2 + 1)}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h}$$

$$= \lim_{h\to 0} (2x+h) = 2x$$

$f'(2) = 2 * 2 = 4$

$f'(0) = 0$

$f'(-2) = -4$

H.W. Find the tangent slope to the curve $f(x) = \frac{1}{2x-1}$ at x=1

The Derivative

Definition: The function f' defined by the formula

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

Is called the derivative with respect to x of the function f . The domain of f' consists of all x for which the limit exists.

f' is the function whose value at x is the slope of the tangent line to the graph of f at x.

or f' is the function whose value at x is the instantaneous rate of change of y with respect to x at the point x.

Differentiable function: a function that is differentiable at every point of its domain.

Differentiable at the point : a function that has a derivative at a point x.

The most common notations for the derivative of a function y=*f*(x), beside f' (x), are:

y' , $\frac{dy}{dx}$, $\frac{df}{dx}$, $D_x(f)$, $\frac{d}{dx}(f)$

Ex: a) Find the derivative of $y = \sqrt{x}$ for x > 0.

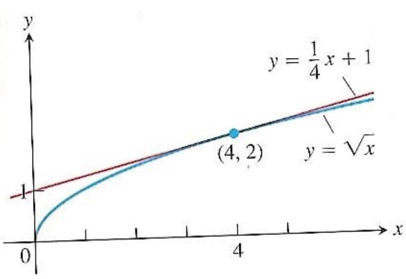
b) Find the tangent line to the curve $y = \sqrt{x}\ at\ x = 4$.

Sol.

a) $f'(x) = \lim_{h\to 0}\frac{f(x+h)- f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0}\frac{\sqrt{x+h} - \sqrt{x}}{h} * \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} =$
$\lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \frac{1}{2\sqrt{x}}$

b)the slope of the curve at x = 4 is $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

the tangent line is the line through the point (4 , 2) with slope 1/4 :



$y = 2 + \frac{1}{4}(x-4) \Rightarrow y = \frac{1}{4}x + 1$

Differentiation Rules:

If u and v are differentiable functions at x.

1. $\frac{d}{dx}\, k = 0$ (k = constant number)

2. $\frac{d}{dx}(x^n) = n\, x^{n-1}$

3. $\frac{d}{dx}\,(k\, u) = k\, \frac{du}{dx}$

4. $\frac{d}{dx}\,(c\, x^n) = c\, n\, x^{n-1}$

5. $\frac{d}{dx}\,(u + v) = \frac{du}{dx} + \frac{dv}{dx}$

6. $\frac{d}{dx}(u - v) = \frac{du}{dx} - \frac{dv}{dx}$

7. $\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\, \frac{du}{dx}$

8. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ $\quad (v(x) \neq 0)$

Higher Derivatives:

If the derivative f' of a function f itself differentiable, then the derivative of f`is denoted by f'' and is called second derivative of f. As long as we have differentiability, we can continue the process of differentiating derivative to obtain third, fourth, fifth, and even higher derivative of f.

$$f''(x), \quad y'', \quad \frac{d}{dx}f', \quad \frac{dy'}{dx}, \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

$$f'''(x), \quad y''', \quad \frac{d}{dx}f'', \quad \frac{dy''}{dx}, \quad \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right) = \frac{d^3y}{dx^3}$$

$f''''(x)$, … … ..

Derivative of Trigonometric Functions:

1. $D_x \sin x = \cos x$
2. $D_x \cos x = -\sin x$
3. $D_x \tan x = \sec^2 x$
4. $D_x \cot x = -\csc^2 x$
5. $D_x \sec x = \sec x \tan x$
6. $D_x \csc x = -\csc x \cot x$

Ex: Prove that $D_x \sin x = \cos x$.

$$D_x \sin x = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x\,(\cos h - 1) + \sin h \cos x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x\,(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\sin h \cos x}{h}$$

$$= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}$$

$$= (\sin x \lim_{h\to 0} \frac{\cos h - 1}{h} \cdot \frac{\cos h + 1}{\cos h + 1}) + \cos x$$

$$= (\sin x \lim_{h\to 0} \frac{-\sin^2 h}{h(\cos h + 1)}) + \cos x = \cos x$$

Derivative of The Inverse Trigonometric Functions:

1. $D_x \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
2. $D_x \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$
3. $D_x \tan^{-1} x = \frac{1}{1+x^2}$
4. $D_x \cot^{-1} x = \frac{-1}{1+x^2}$
5. $D_x \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$
6. $D_x \csc^{-1} x = \frac{-1}{|x|\sqrt{x^2-1}}$

Example: Prove that $D_x \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

Y = $\sin^{-1}$ x $\Longrightarrow$ x = sin y $\qquad \frac{-\pi}{2} \le y \le \frac{\pi}{2}$

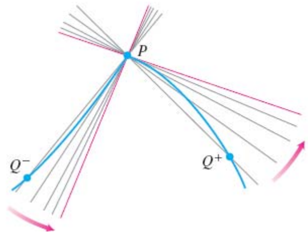
$$1 = \cos y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

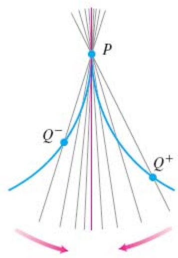
Existence of Derivative:

1. a corner, where the one – sided derivative differ.

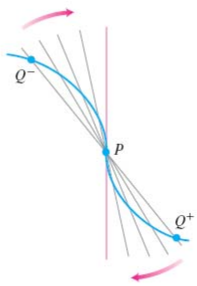
   LD ≠ RD



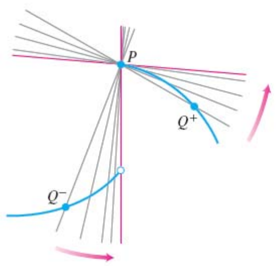
2. a cusp, where the slope of PQ approaches ∞ from one side and - ∞ from the other

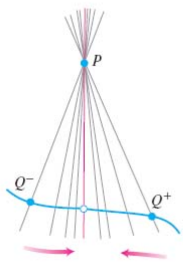


a vertical tangent, where the slope of PQ approaches ∞ from both sides or approaches -∞ from both sides (here, -∞)



3. a discontinuity .





Theorem:

If *f* has a derivative at x = c, then *f* is continuous at x = c.

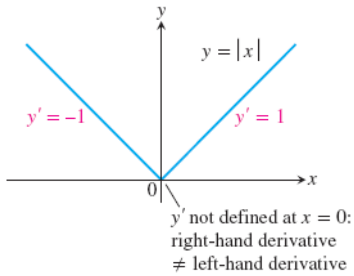
RD and LD:

Right derivative = RD = $\lim_{x \to a^+} f'(x) = \lim_{\epsilon \to 0} f'(a+\epsilon)$

Left derivative = LD = $\lim_{x \to a^-} f'(x) = \lim_{\epsilon \to 0} f'(a-\epsilon)$

If LD =RD $\Longrightarrow$ differentiable function

Ex: Show that $f(x) = |x|$ is not differentiable at x = 0.



Sol.: $f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

$$f'(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

$RD = \lim_{x \to 0^+} f'(x) = \lim_{x \to 0} 1 = 1$

$LD = \lim_{x \to 0^-} f'(x) = \lim_{x \to 0} -1 = -1$

RD ≠ LD

Ex: $y = \sqrt{x}$ is not differentiable at x=0.

$$f'(x) = \frac{1}{2\sqrt{x}} \quad \Longrightarrow f'(0) = \frac{1}{0} = \infty$$

The graph has a vertical tangent at the origin.

The Chain Rule:

If *f(u)*is differentiable at the point *u = g(x)* and *g(x)* is differentiable at x, then the composite function *(f o g))(x)=f(g(x))* is differentiable at x, and

(f o g)'(x)=f'(g(x))g'(x).

If *y = f(u)* and *u = g(x),* then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example: y=sin 2y find y'

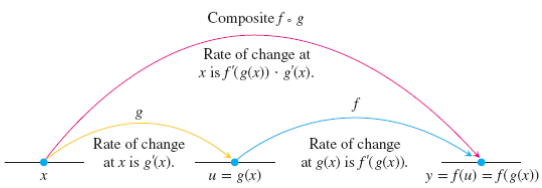
Solution.

y'=cos 2y (2) = 2cos 2y

And if *y=f(u)* , *u=g(v)* , *v=h(x)*

Y=f(g(h(x)))

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$



Example: $y = \sec^2 \sqrt{x}$ find y'.

$$y' = 2\sec\sqrt{x} \cdot \sec\sqrt{x} \cdot \tan\sqrt{x} \left(\frac{1}{2\sqrt{x}}\right)$$

Example: Find $\frac{dw}{dt}$ $if\ w = \tan x\ \ and\ \ a = 4t^3 + t$

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}$$

$$= \frac{d}{dx}(\tan x\,) \cdot \frac{d}{dt}\,(4t^3 + \ t) \quad = \sec^2 x\,(12\,t^2 + \ 1) = \sec^2(4t^3 + \ t) \cdot (12\,t^2 + \ 1)$$

Example: Find $\frac{dy}{dt}$ $if\ y = t^2 \sec\sqrt{wt}\,.$

$$\frac{dy}{dt} = t^2\ sec\sqrt{wt}\ \ tan\sqrt{wt}\ \ \frac{w}{2\sqrt{wt}} + \sec\sqrt{wt}\ \ (2t)$$

Example: Find y''' if $y = 2x\sin^{-1} 2x + \ \sqrt{1-4x^2}\,.$

$$y' = 2x \cdot \frac{2}{\sqrt{1-4x^2}} + \sin^{-1}(2x)(2) + \frac{-8x}{2\sqrt{1-4x^2}}$$

$$= \frac{4x}{\sqrt{1-4x^2}} + 2\sin^{-1}2x - \frac{4x}{\sqrt{1-4x^2}} = 2\sin^{-1} 2x$$

$$y'' = 2 \cdot \frac{2}{\sqrt{1-4x^2}} = \frac{4}{\sqrt{1-4x^2}}$$

$$y''' = 4\left(\frac{-1}{2}\right)(-8x)(1-4x^2)^{\frac{-3}{2}} = \frac{16x}{(1-4x^2)^{\frac{3}{2}}}$$

H.W.: Prove that $\frac{dy}{dx} = \sqrt{a^2 - \ x^2}\ \ if\ y = \left(\frac{a}{2}^2\right)\sin^{-1}\left(\frac{x}{a}\right) + \ \left(\frac{x}{2}\right)\sqrt{a^2 - \ x^2}$

$$\frac{dy}{dx} = \frac{a^2}{2} * \frac{\frac{1}{a}}{\sqrt{1-\frac{x^2}{a^2}}} + \frac{x}{2}\ \frac{-2x}{2\sqrt{a^2 - \ x^2}} + \frac{1}{2}\sqrt{a^2 - \ x^2}$$

Parametric Equations:

If x and y are given as functions $x = f(t),\ y = g(t)$ over an interval of t-values, then the set of points $(x, y) = \big(f(t), g(t)\big)$ defined by these equations is a **parametric curve**. The equations are parametric equations for the curve.

Slopes of Parameterized Curves:

A Parameterized curve $x = f(t)$ and $y = g(t)$ is differentiable at t if f and g differentiable at t.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

Example: Find $\frac{d^2y}{dx^2}$ from the parametric equations $y = \sec^{-1}\frac{1}{\sqrt{1-t^2}}$, $x = \frac{t}{\sqrt{1-t^2}}$

Solution: $y = \cos^{-1}\sqrt{1-t^2}$

$$\frac{dy}{dt} = \frac{-1}{\sqrt{1-(1-t^2)}} \cdot \frac{-2t}{2\sqrt{1-t^2}} = \frac{2t}{2t\sqrt{1-t^2}} = \frac{1}{\sqrt{1-t^2}}$$

$$\frac{dx}{dt} = \frac{\sqrt{1-t^2} - t \cdot \frac{-2t}{2\sqrt{1-t^2}}}{1-t^2} = \frac{\sqrt{1-t^2} + \frac{t^2}{\sqrt{1-t^2}}}{1-t^2} = \frac{1}{(1-t^2)^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{\sqrt{1-t^2}}}{\frac{1}{(1-t^2)^{\frac{3}{2}}}} = \frac{1}{\sqrt{1-t^2}} \cdot (1-t^2)^{\frac{3}{2}} = (1-t^2)$$

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{-2t}{\frac{1}{(1-t^2)^{\frac{3}{2}}}} = -2t(1-t^2)^{\frac{3}{2}}$$

H.W.: For the following parametric equation $= \cot^{-1}\sqrt{t}$, $y = \frac{1}{t+1}$, find $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ at t = 2.

Solution: $\frac{dy}{dt} = \frac{-1}{(t+1)^2}$, $\frac{dx}{dt} = \frac{-1}{1+t}(\frac{1}{2\sqrt{t}})$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{-1}{(t+1)^2}}{\frac{-1}{1+t}(\frac{1}{2\sqrt{t}})} = \frac{1}{(1+t)^2}(1+t)(2\sqrt{t}) = \frac{2\sqrt{2}}{3}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{(1+t)\frac{1}{\sqrt{t}} - 2\sqrt{t}}{(1-t)^2}}{\frac{-1}{(1+t)(2\sqrt{t})}} = \frac{2}{3}$$

Implicit Differentiation:

1. $y = \sqrt{x^2+1}$ $\quad y\ fct.\ Of\ \ x\ (explicit/y)$
2. $y^2 + xy = 2$ $\quad y\ fct.\ Of\ \ x\ (implicit/y)$

Example: Find $\frac{dy}{dx}$ if $5y^2 + \sin y = x^2$.

Solution: $10yy' + \cos y\,(y') = 2x \implies y' = \frac{2x}{10y+\cos y}$

H.W.: Find $\frac{dy}{dx}$ if $\frac{1}{x+y} - \frac{1}{x-y} = 2$

$$\frac{x-y-(x+y)}{(x+y)(x-y)} = 2 \Rightarrow x-y-x-y = 2(x+y)(x-y)$$

$$-2y = 2(x+y)(x-y)$$

$$-y = x^2 + xy - xy - y^2$$

$$y^2 - y - x^2 = 0 \Rightarrow 2yy' - y' - 2x = 0 \Rightarrow y'(2y-1) = 2x \Rightarrow y' = \frac{2x}{2y-1}$$

Example: Find the points of function $x^2 - xy + y^2 = 1$ at which the tangent is horizontal line.

Solution.

$$2x - (xy' + y) + 2yy' = 0 \Rightarrow 2x - xy' - y + 2yy' = 0$$

$$y'(2y - x) = y - 2x \Rightarrow y' = \frac{y-2x}{2y-x}$$

The tangent is horizontal $\Rightarrow \frac{dy}{dx} = 0 \Rightarrow \frac{y-2x}{2y-x} = 0 \Rightarrow y = 2x$

H.W. : Find $\frac{dy}{dx}$ if $\tan\left(\frac{sinx}{cosx}\right) + ycscx = x^2y.$

Sol.: $\sec^2(tanx)\sec^2 x + y(-\cot x \csc x) + \csc x\,(y') = x^2y' + 2xy$

$$y'(\csc x - x^2) = 2xy - \sec^2(\tan x)\sec^2 x + y\cot x \csc x$$

$$y' = \frac{2xy - \sec^2(\tan x)\sec^2 x + y\cot x\csc x}{\csc x - x^2}$$

H.w. Find y' for $\cot^{-1}\frac{1}{y} = xy.$

Sol.:

$$\frac{-1}{1+\left(\frac{1}{y}\right)^2}(-y^{-2}\,y') = xy' + y$$

$$\frac{y^{-2} * y^2}{y^2+1}y' - xy' = y \Rightarrow y'\left(\frac{1}{y^2+1} - x\right) = y$$

H.W.: Find the slope of the tangent line at (4 , 0) to the graph of $7y^4 + x^3y + x = 4.$

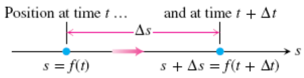
Sol.: $28y^3y' + x^3y' + y(3x^2) + 1 = 0$

$$y'(28y^3y' + x^3) = -3yx^2 - 1$$

$$y' = \frac{-3yx^2-1}{28y^3+x^3} \Rightarrow \frac{dy}{dx}@(4,0) = \frac{-1}{64}$$

Displacement, Velocity, and Acceleration:

Suppose that an object is moving along a coordinate line (say an s-axis)so that we know its position s on that line as a function of time t:



$$s = f(t)$$

The displacement of the object over the time interval from t to t + Δt is:

$$\Delta s = f(t + \Delta t) - f(t),$$

and the average velocity of the objective over that time interval is:

$$v_{av} = \frac{displacement}{travel\ time} = \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

To find the body's velocity at the exact instant t, we take the limit of the average velocity over the interval from t to $t + \Delta t$ as Δt shrinks to zero. This limit is the derivative of f with respect to t.

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

Acceleration is the derivative of velocity with respect to time. If a body's position at time t is at time t is $s = f(t)$, then the body's acceleration at time t is:

$$a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

Related Rates

In this section we look at problems that ask for the rate at which some variable change. In each case the rate is a derivative that has to compute from the rate is a derivative that has to be computed from the rate at which some other variable (or perhaps several variables) is known to change.

Related Rates Equations

Suppose we are pumping air into a spherical balloon. Both the volume and radius of the balloon are increasing over time. If V is the volume and r is the radius of the balloon at an instant of time then

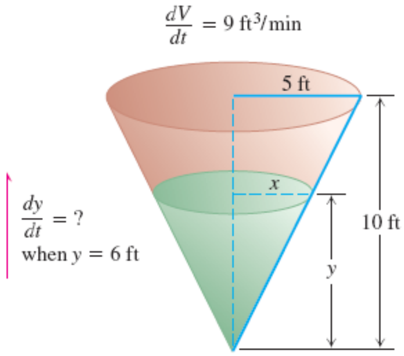
$$V = \frac{4}{3}\pi r^3$$

Using the chain rule, we differentiate to find the related rates equation

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

So if we know the radius r of the balloon and the rate dV/dt at which the volume is increasing at a given instant of time, then we can solve this last equation for dr/dt to find how fast the radius is increasing at that instant.

Example: Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?



Solution:

The figure shows a partially filled conical tank. The variables in the problem are

V=volume (ft^3) , x=radius(ft) , y=depth(ft)

$$y = 6 \quad and \quad \frac{dV}{dt} = 9\, ft^3/min$$

$$V = \frac{1}{3}\,\pi\, x^2\, y$$

$$\frac{x}{y} = \frac{5}{10} \quad or \quad x = \frac{y}{2}$$

Therefore, $V = \frac{1}{3}\,\pi\,\left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}\,y^3$

$$\frac{dV}{dt} = \frac{\pi}{12}\cdot 3y^2\frac{dy}{dt} = \frac{\pi}{4}\,y^2\,\frac{dy}{dt}$$

$$9 = \frac{\pi}{4}\,(6)^2\,\frac{dy}{dt} \quad \Longrightarrow \quad \frac{dy}{dt} = \frac{1}{\pi} \approx 0.32$$

Linearization

As we can see the tangent to the curve $y = x^2$ lies close to the curve near the point of tangency. For a brief interval to either side, the y-values along the tangent line give good approximation to the y-values on the curve.

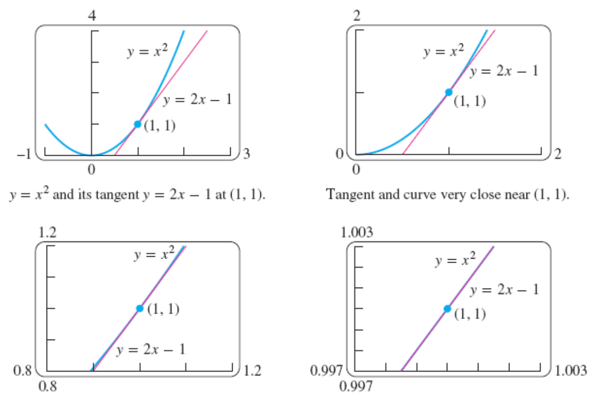
If f is differentiable at x = a, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the linearization of f at a . The approximation

$$f(x) \approx L(x)$$

of f by L is the standard linear approximation of f at a. The point $x = a$ is the center of the approximation.

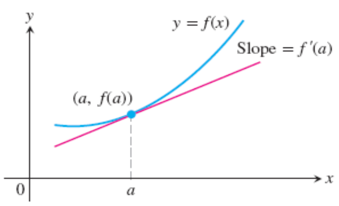


$y = x^2$ and its tangent $y = 2x - 1$ at (1, 1).

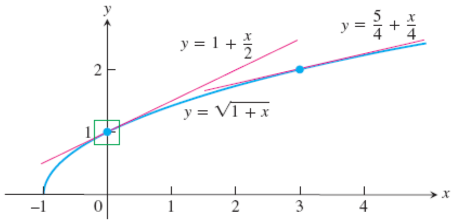
Tangent and curve very close near (1, 1).

Tangent and curve very close throughout entire x-interval shown.

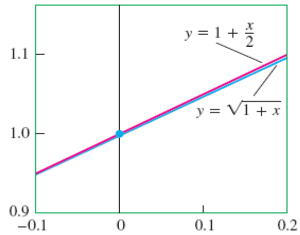
Tangent and curve closer still. Computer screen cannot distinguish tangent from curve on this x-interval.



Example1: Find the linearization of $f(x) = \sqrt{1+x}$ $at\ x = 0$



Solution:



$$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$$

We have $f(0) = 1\ and\ f'(0) = \frac{1}{2}$, giving the linearization

$$L(x) = f(a) + f'(a)(x-a) = 1 + \frac{1}{2}(x-0) = 1 + \frac{x}{2}$$

Look at how accurate the approximation $\sqrt{1+x} \approx 1 + \frac{x}{2}$ for values of x near 0.

As we move away from zero, we lose accuracy. For example, for x=2, the linearization gives 2 as the approximation for $\sqrt{3}$, which is not even accurate to one decimal place.

Approximation	True value	|True value − approximation|
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$<10^{-2}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$<10^{-3}$
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$<10^{-5}$

Example2:Find the linearization of $f(x) = \sqrt{1+x}$ $at\ \ x = 3.$

Solution: $f(3) = 2$, $\quad f'(3) = \frac{1}{2}(1+x)^{-\frac{1}{2}} = \frac{1}{4}$

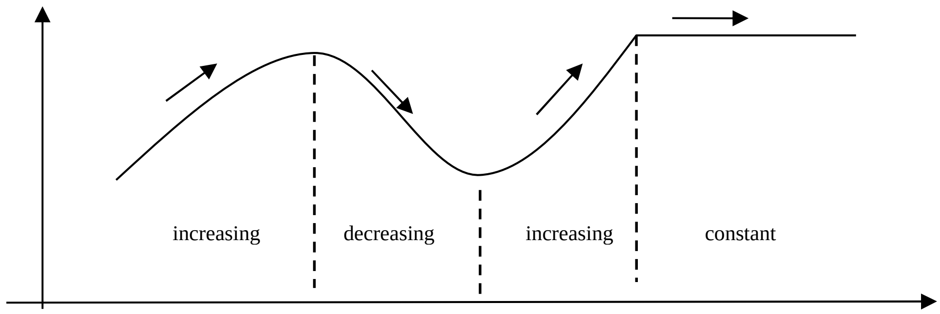
$$L(x) = 2 + \frac{1}{4}(x-3) = \frac{5}{4} + \frac{x}{4}$$

At x=3.2, the linearization in example 2 gives

$$\sqrt{1+x} = \sqrt{1+3.2} \approx 1 + \frac{3.2}{2} = 2.6$$

Which differs from the true value $\sqrt{4.2} \approx 2.04939$ by less than 10^{-3}

Intervals of Increasing and Decreasing:

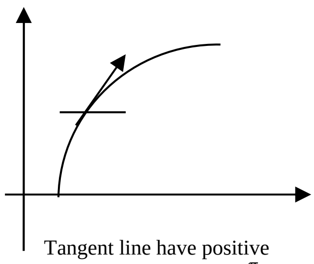


Theorem: Let f be a function that is continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b).

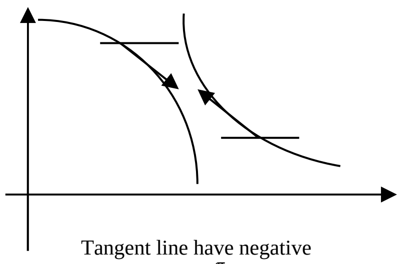
a) If $f'(x) > 0$ for every value of x in (a,b) ,then f is increasing on $[a,b]$.

b) If $f'(x) < 0$ for every value of x in (a,b) ,then f is decreasing on $[a,b]$.

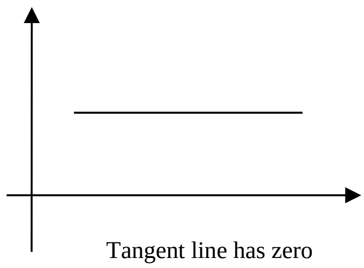
c) If $f'(x) = 0$ for every value of x in (a,b) ,then f is constant on $[a,b]$.



Tangent line have positive slope $\tan\theta \ \ 0 < \theta < \frac{\pi}{2}$
$f'(x) < 0$



Tangent line have negative slope $tan\theta \ \ \frac{\pi}{2} < \theta < \pi$
$f'(x) < 0$



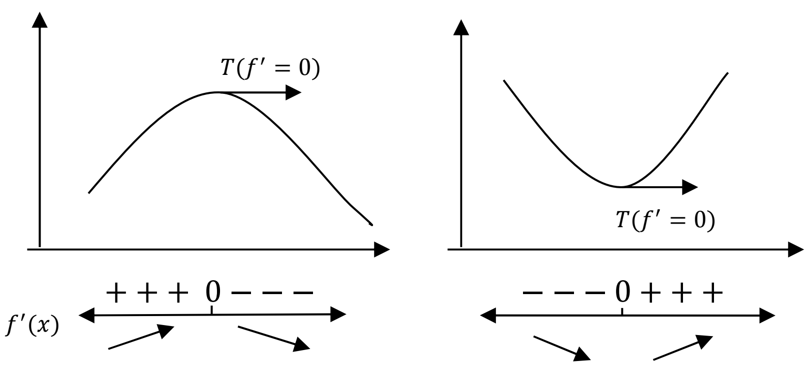
Tangent line has zero slopes $f'(x) = 0$

Concavity:

The graph of a differentiable $= f(x)$. (T=transition point)

a) Concave up on an interval (a,b) if f' is increasing on (a,b)

b) Concave down on an interval (a,b) if f' is decreasing on (a,b)



The second derivative test for concavity:

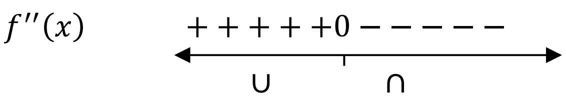
The graph of $y = f(x)$ is:

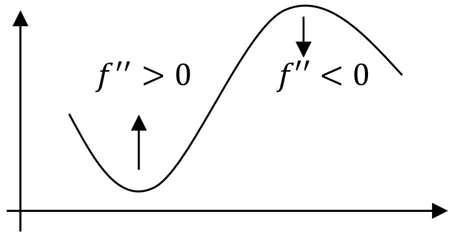
Concave down on any interval where $y'' < 0$.

Concave up on any interval where $y'' > 0$.

Points of inflection:

A point on the graph of a differentiable function where concavity changes is called a point of inflection.

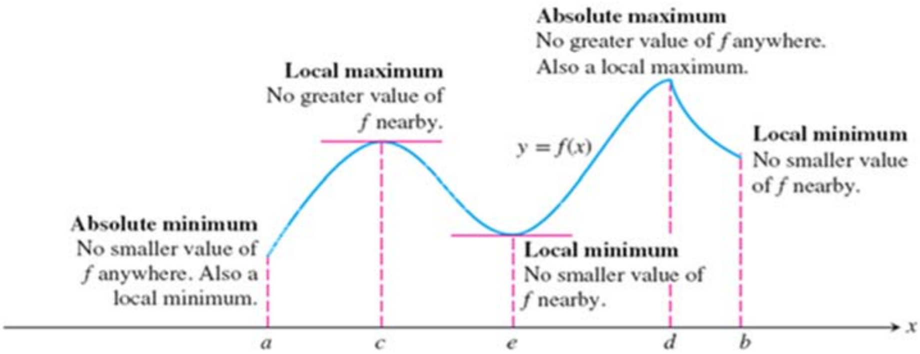




Maxima and Minima-Relative vs. Absolute:

A function f has a local maximum value at an interior point c of its domain if $f(x) \leq f(c)$ for all x in the same open interval about c. The function has an absolute maximum value at c if $f(x) \leq f(c)$ for all x in the domain.

Similarly, f has a local minimum value at an interior point c of its domain if $f(x) \geq f(c)$ for all x in an open interval about c. The function has an absolute minimum value at c if $f(x) \geq f(c)$ for all x in the domain.



Example: Find the max. and min. values of function $y = x^2 - \frac{1}{|x|}$, $[-3, -1]$.

Solution:

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

$y = x^2 + \frac{1}{x}$

$y' = 2x - \frac{1}{x^2} = 0 \Longrightarrow \frac{2x^3 - 1}{x^2} = 0 \Longrightarrow 2x^3 - 1 = 0 \Longrightarrow x^3 = \frac{1}{2} \Longrightarrow x = \frac{1}{\sqrt[3]{2}}$

$x = \frac{1}{\sqrt[3]{2}} \notin [-3, -1]$

$f(-3) = 9 + \frac{1}{-3} = 9 - \frac{1}{3} = 8.67$ Absolute max.

$f(-1) = 1 + (-1) = 0$ Absolute min.

Example: Find the max. and min. values to the function $y = \sin x - \cos x$,to the interval $[0, \pi]$.

Solution:

$y' = \cos x + \sin x = 0$

$\cos x = -\sin x \Longrightarrow \tan x = -1 \Longrightarrow x = -\frac{\pi}{4}, \frac{3\pi}{4} \in [0, \pi]$

1) $x = 0 \Longrightarrow f(0) = \sin 0 - \cos 0 = -1$ Absolute min.

2) $x = \frac{3\pi}{4} \Longrightarrow f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4} = \sqrt{2}$ Absolute max.

3) $x = \pi \Longrightarrow f(\pi) = \sin\pi - \cos\pi = 1$

H.W:$y = \sec x + \tan x$, $\left[-\frac{\pi}{4}, \frac{\pi}{3}\right]$.

Sol .: $y' = \sec x \tan x + \sec^2 x = 0 \Longrightarrow \sec x\,(\tan x + \sec x) = 0$

$\sec x \neq 0$,$\tan x + sec\ x = 0 \Longrightarrow \tan x = -\sec x \Longrightarrow \frac{\sin x}{\cos x} = \frac{-1}{\cos x} \Longrightarrow \sin x = -1$

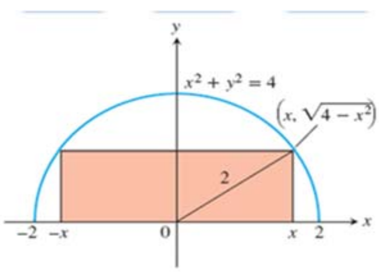
$x = -\frac{\pi}{2}, \frac{3\pi}{2} \notin \left[-\frac{\pi}{4}, \frac{\pi}{3}\right]$

1) $x = -\frac{\pi}{4} \Longrightarrow f\left(-\frac{\pi}{4}\right) = 0.414$ Absolute min.

2) $x = \frac{\pi}{3} \Longrightarrow f\left(\frac{\pi}{3}\right) = 3.732$ Absolute max.

Applied Optimization Problems:

Example: A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?



Solution: Let length=2x, height =$\sqrt{4 - x^2}$,

area=$2x \cdot \sqrt{4 - x^2}$

$A(x) = 2x \cdot \sqrt{4 - x^2}$ on the domain[0,2]

$$\frac{dA}{dx} = \frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2}$$

$$\frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2} = 0$$

$-2x^2 + 2(4 - x^2) = 0 \Rightarrow 8 - 4x^2 = 0 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}.$

$x = -\sqrt{2} \notin Area\ domain$

Critical – point value: $A(\sqrt{2}) = 2\sqrt{2}\sqrt{4 - 2} = 4$

End point values: $A(0) = 0, \quad A(2) = 0.$

The max. area = 4 when h=$2\sqrt{2}$

Graphs of Polynomials and Rational Functions:

Strategy for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find y' and y''.
3. Find the critical points of f, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve.

Example: Draw the function $y = \frac{x^2 - 1}{x^3}$

Solution:

$D_f: (-\infty, \infty)/x = 0.$

$$y' = \frac{x^3(2x) - (x^2 - 1)(3x^2)}{x^6} = \frac{2x^4 - 3x^4 + 3x^2}{x^6} = \frac{-x^4 + 3x^2}{x^6} = 0 \quad \Rightarrow -x^4 + 3x^2 = 0$$

$x^2(-x^2 + 3) = 0 \Rightarrow x = 0\ or\ x = \pm\sqrt{3}$

$- - - - 0 + + + + \infty + + + + 0 - - - -$

$f'(x)$ ↘ $-\sqrt{3}$ ↗ 0 ↗ $\sqrt{3}$ ↘

$$y'' = \frac{x^6(-4x^3+6x)-(-x^4+3x^2)6x^5}{x^{12}} = \frac{-4x^9+6x^7-18x^7}{x^{12}} = \frac{2x^9-12x^7}{x^{12}} = 0$$

$2x^9 - 12x^7 = 0 \implies x^7(2x^2-12) = 0 \implies x = 0 \;\; or \;\; x = \pm\sqrt{6}$

$- - - - 0 + + + + \infty - - - - 0 + + + +$

$f''(x) \quad \cap \quad -\sqrt{6} \quad \cup \quad 0 \quad \cap \quad \sqrt{6} \quad \cup$

$$\lim_{x\to\infty} \frac{x^2-1}{x^3} = \lim_{x\to\infty} \frac{\frac{1}{x}-\frac{1}{x^3}}{1} = \frac{0}{1} = 0$$

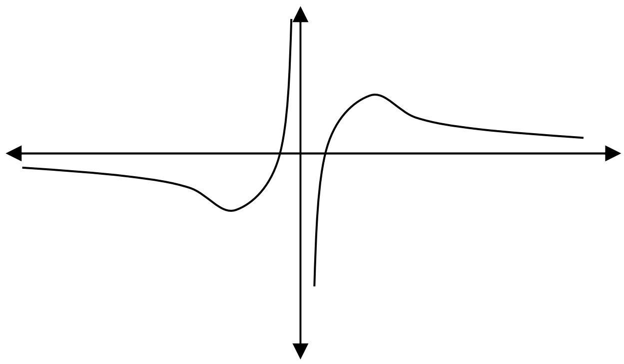
$$\lim_{x\to-\infty} \frac{x^2-1}{x^3} = \lim_{x\to\infty} \frac{\frac{1}{x}-\frac{1}{x^3}}{1} = \frac{0}{1} = 0$$

$$\lim_{x\to0^+} \frac{x^2-1}{x^3} = \lim_{\epsilon\to0} \frac{(0+\epsilon)^2-1}{(0+\epsilon)^3} = \frac{-1}{0} = -\infty$$

$$\lim_{x\to0^-} \frac{x^2-1}{x^3} = \lim_{\epsilon\to0} \frac{(0-\epsilon)^2-1}{(0-\epsilon)^3} = \frac{(-\epsilon)^2-1}{(-\epsilon)^3} = \frac{1}{0} = \infty$$

$y = 0 \implies \frac{x^2-1}{x^\wedge 3} = 0 \implies x = \mp 1$

x	y
0.5	-6
$-o.5$	6
1	0
-1	0



H.W.: Draw the function . $y = \frac{x^2}{x^2-1}$.

L'Hopital's Rule:

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g(a)$ exist, and that $g'(a) \neq 0$.

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)} = \frac{0}{0}$$

Example: Evaluate the following limits:

$$1.\ \lim_{x\to 0}\frac{1-\cos x}{3x^2}=\frac{0}{0}=\lim_{x\to 0}\frac{\sin x}{6x}=\lim_{x\to 0}\frac{\cos x}{6}=\frac{1}{6}$$

$$2.\ \lim_{x\to 1}\frac{\cot^{-1}\sqrt{x}-\frac{\pi}{4}}{x^2-1}=\lim_{x\to 1}\frac{\frac{-1}{1+x}\left(\frac{1}{2\sqrt{x}}\right)-0}{2x}=\lim_{x\to 1}\frac{-1}{4x^{\frac{3}{2}}(1+x)}=\frac{-1}{8}$$

$$3.\ \lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{x-\sin x}{x\sin x}=\lim_{x\to 0}\frac{1-\cos x}{x\cos x+\sin x}=\lim_{x\to 0}\frac{\sin x}{-x\sin x+\cos x+\cos x}=\frac{0}{2}$$
$$=0$$

$$4.\ \lim_{t\to 0}\frac{\cot 2t}{\cot 3t}=\frac{\infty}{\infty}=\lim_{t\to 0}\frac{-2\csc^2 2t}{-3csc^2 3t}=\frac{2}{3}\lim_{t\to 0}\frac{\sin^2 3t}{\sin^2 2t}=\frac{2}{3}\lim_{t\to 0}\frac{6\sin 3t\cos 3t}{4\sin 2t\cos 2t}=\frac{2}{3}\lim_{t\to 0}\frac{3\sin 6t}{2\sin 4t}$$
$$=\frac{2}{3}\lim_{t\to 0}\frac{18\cos 6t}{8\cos 4t}=\frac{2}{3}\cdot\frac{18}{8}=\frac{3}{2}$$

$$5.\ \lim_{x\to\frac{\pi}{2}}\left(x-\frac{\pi}{2}\right)\tan x=\lim_{x\to\frac{\pi}{2}}\frac{x-\frac{\pi}{2}}{\cot x}=\lim_{x\to\frac{\pi}{2}}\frac{1}{-\csc^2 x}=\lim_{x\to\frac{\pi}{2}}-\sin^2 x=-1$$

$$or\ \lim_{x\to\frac{\pi}{2}}\frac{\left(x-\frac{\pi}{2}\right)\sin x}{\cos x}$$

H.W.:

$$1.\ \lim_{t\to 0}\frac{\sqrt{1+t^2}-1}{t^2}$$

$$2.\ \lim_{x\to 0}\frac{x-\cot^{-1}\frac{1}{x}}{x}$$

$$3.\ \lim_{h\to 2}\frac{\csc\frac{\pi}{h}-1}{h-2}$$

$$4.\ \lim_{x\to 0}(x-\sin^{-1}x)\csc^3 x$$

9 INTEGRATION

Integration by Parts:

Integration by parts is a technique used mainly for simplifying integrals of the form,

$$\int f(x)\, g(x)\, dx$$

The formula for integration by parts comes from the product rule,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

In its differential form, the rule becomes:

$$d(uv) = u\, dv + v\, du$$
$$u\, dv = d(uv) - v\, du$$

And integrated to give the following formula,

$$\int u\, dv = uv - \int v\, du$$

The equivalent formula for definite integrals is:

$$\int_{v_1}^{v_2} u\, dv = u_2 v_2 - u_1 v_1 - \int_{u_1}^{u_2} v\, du$$

Example 1: Evaluate the integral

$$\int x \ln x^2\, dx$$

Solution

$$\int x \ln x^2\, dx = 2\int x \ln x\, dx$$

$$u = \ln x, \qquad du = \frac{1}{x}dx$$

$$dv = xdx, \qquad v = \frac{x^2}{2}$$

$$= 2\left(\frac{x^2}{2}\ln x\right) - 2\int \frac{x^2}{2}\cdot\frac{1}{x}dx$$

$$= x^2 \ln x - \int x\,dx$$

$$= x^2 \ln x - \frac{x^2}{2} + c$$

Example 2: Evaluate the integral

$$\int \sin^{-1}\frac{x}{2}\,dx$$

Solution

$$\int \sin^{-1}\frac{x}{2}\,dx$$

$$u = \sin^{-1}\frac{x}{2}, \qquad du = \frac{\frac{1}{2}dx}{\sqrt{1-\left(\frac{x}{2}\right)^2}} = \frac{dx}{\sqrt{4-x^2}}$$

$$dv = dx, \qquad v = x$$

$$= x\sin^{-1}\frac{x}{2} - \int \frac{x}{\sqrt{4-x^2}}dx$$

$$= x\sin^{-1}\frac{x}{2} + \frac{\frac{1}{2}\sqrt{4-x^2}}{\frac{1}{2}} + c$$

Example 3: Evaluate the integral

$$\int x\tan^{-1}2x\,dx$$

Solution

$$\int x\tan^{-1}2x\,dx$$

$$u = \tan^{-1}2x, \qquad du = \frac{2}{1+4\,x^2}dx$$

$$dv = x\,dx, \qquad v = \frac{x^2}{2}$$

$$\frac{1}{4}$$
$$4x^2+1 \;\big)\; x^2$$
$$\mp x^2 \mp \frac{1}{4}$$
$$-\frac{1}{4}$$

$$= \frac{x^2}{2}\tan^{-1}2x - \int \frac{x^2}{1+4\,x^2}dx$$

$$= \frac{x^2}{2}\tan^{-1}2x - \int\left(\frac{1}{4} - \frac{\frac{1}{4}}{4x^2+1}\right)dx$$

$$= \frac{x^2}{2} \tan^{-1} 2x - \frac{1}{4} x + \frac{1}{8} \tan^{-1} 2x + c$$

Example 4: Evaluate the integral

$$\int (1 + x) \sin 2x \ dx$$

Solution

$$\int (1 + x) \sin 2x \ dx$$

$u = 1 + x, \qquad du = dx$

$dv = \sin 2x, \qquad v = -\frac{1}{2} \cos 2x$

$$= -\frac{1}{2}(1 + x) \cos 2x + \int \frac{1}{2} \cos 2x \ dx$$

$$= -\frac{1}{2} \cos 2x + \frac{1}{4} \sin 2x$$

Example 5: Evaluate the integral

$$\int \cos \sqrt{x + 1} \ dx$$

Solution

$$\int \cos \sqrt{x + 1} \ dx$$

Let $y = \sqrt{x + 1}$

$y^2 = x + 1$

$2y \, dy = dx$

$$= 2 \int y \cos y \ dy$$

$u = y, \qquad du = dy$

$dv = \cos y \ dy, \qquad v = -\sin y$

$$= -y \sin y + \int \sin y \ dy \ = -y \sin y - \cos y + c$$

Example 6: Evaluate the integral

$$\int x^2 e^{-x} \ dx$$

Solution

$$\int x^2 e^{-x} \ dx$$

$u = x^2, \qquad du = 2x \ dx$

$dv = e^{-x} dx, \qquad v = -e^{-x}$

$$= -x^2 e^{-x} + 2 \int x \ e^{-x} dx$$

$u = x, \qquad du = dx$

$dv = e^{-x}dx, \qquad v = -e^{-x}$

$= -x^2e^{-x} + 2\left[-xe^{-x} + \int e^{-x}\,dx\right] = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + c$

Example 7: Evaluate the integral

$$\int \sec^3 x\,dx$$

Solution

$\int \sec^3 x\,dx$

$u = \sec x, \qquad du = \sec x \tan x\,dx$

$dv = \sec^2 x\,dx \qquad v = \tan x$

$= \sec x \tan x - \int \sec x \tan^2 x\,dx$

$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x\,(\sec^2 x - 1)\,dx$

$\int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx - \int \sec x\,dx$

$2\int \sec^3 x\,dx = \sec x \tan x - \int \sec x\,dx$

$\int \sec^3 x\,dx = \frac{1}{2}[\sec x \tan x - \ln|\sec x + \tan x|] + c$

Example 8: Evaluate the integral

$$\int \sec x \tan^2 x\,dx$$

Solution

$\int \sec x \tan^2 x\,dx$

$u = \tan x, \qquad du = \sec^2 x\,dx$

$dv = \sec x \tan x\,dx \qquad v = \sec x$

$= \tan x \sec x - \int \sec^3 x\,dx$

$= \tan x \sec x - \int \sec x\,(\tan^2 x + 1)\,dx$

$= \tan x \sec x - \int \sec x \tan^2 x\,dx + \int \sec x\,dx$

$2\int \sec x \tan^2 x\,dx = \tan x \sec x + \int \sec x\,dx$

$$\int \sec x \tan^2 x \, dx = \frac{1}{2}[\tan x \sec x + \ln|\sec x + \tan x|] + c$$

Example 9: Evaluate the integral

$$\int e^{-x} \cos x \, dx$$

Solution

$$\int e^{-x} \cos x \, dx$$

$u = e^{-x}$, $\quad du = -e^{-x} dx$

$dv = \cos x \, dx$, $\quad v = \sin x$

$$= e^{-x} \sin x + \int e^{-x} \sin x \, dx$$

$u = e^{-x}$, $\quad du = -e^{-x} dx$

$dv = \sin x \, dx$, $\quad v = -\cos x$

$$= e^{-x} \sin x - e^{-x} \cos x - \int e^{-x} \cos x \, dx$$

$$2\int e^{-x} \cos x \, dx = e^{-x} \sin x - e^{-x} \cos x$$

$$\int e^{-x} \cos x \, dx = \frac{1}{2}[e^{-x} \sin x - e^{-x} \cos x]$$

Example 10: Evaluate the integral

$$\int \sin \ln x \, dx$$

Solution

$$\int \sin \ln x \, dx$$

$u = \sin \ln x$, $\quad du = \dfrac{\cos \ln x}{x}$

$dv = dx$, $\quad v = x$

$$= x \sin \ln x - \int \cos \ln x \, dx$$

$u = \cos \ln x$, $\quad du = \dfrac{-\sin \ln x}{x}$

$dv = dx$, $\quad v = x$

$$= x \sin \ln x - x \cos \ln x - \int \sin \ln x \, dx$$

$$\int \sin \ln x \, dx = \frac{x}{2} [\sin \ln x - \cos \ln x] + c$$

Example 11: Evaluate the integral

$$\int e^{x+2\ln x} \, dx$$

Solution

$$\int e^{x+2\ln x} \, dx = \int e^{x} \, e^{\ln x^2} dx$$

$$= \int x^2 e^x \, dx$$

$u = x^2,$ $\qquad du = 2x \, dx$

$dv = e^x dx,$ $\qquad v = e^x$

$$= x^2 e^x - 2 \int x \, e^x \, dx$$

$u = x \,,$ $\qquad du = dx$

$dv = e^x dx,$ $\qquad v = e^x$

$$= x^2 e^x - 2 \left[x \, e^x - \int e^x \, dx \right]$$

$$= x^2 e^x - 2x \, e^x + 2 \, e^x + c$$

Example 12: Evaluate the integral

$$\int_0^{1/2} 2x \tanh^{-1} x \;\, dx$$

Solution

$$\int_0^{1/2} 2x \tanh^{-1} x \;\, dx$$

$u = \tanh^{-1} x \,,$ $\qquad du = \frac{1}{1-x^2} dx$

$dv = 2 \, x \, dx \,,$ $\qquad v = x^2$

$$\begin{array}{r|l} & \; 1 \\ \hline x^2 - 1 & \; x^2 \\ & \mp x^2 \pm 1 \\ \hline & \quad 1 \end{array}$$

$$= x^2 \tanh^{-1} x - \int_0^{1/2} \frac{x^2}{1-x^2} dx = x^2 \tanh^{-1} x + \int_0^{1/2} \frac{x^2}{x^2-1} dx$$

$$= x^2 \tanh^{-1} x + \int_0^{1/2} 1 + \frac{1}{x^2-1} dx$$

$$= [x^2 \tanh^{-1} x + x - \tanh^{-1} x] \begin{matrix} 1/2 \\ 0 \end{matrix} = \frac{1}{4} \tanh^{-1} 0.5 + 0.5 - \tanh^{-1} 0.5 = 0.088$$

Tabular Integration

We have seen that integrals of the form $\int f(x)g(x)\,dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts. However, if many repetitions are required, the calculations can be cumbersome. In situations like this, there is a way to organize the calculations that saves a great deal of work. It is called tabular integration and is illustrated in the following illustrative examples

Example 13: Evaluate the following integral by using **Tabular Integration**

$$\int x^2 e^x dx$$

Solution

With $f(x) = x^2$ and $g(x) = e^x$ one can list the following derivatives and integrals

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + c \qquad Ans.$$

Example 14: Evaluate the following integral by using **Tabular Integration**

$$\int x^3 \sin x \, dx$$

Solution

With $f(x) = x^3$ and $g(x) = \sin x$ one can list the following derivatives and integrals

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + c$$

Integration of Rational Functions by Partial Functions

$$\frac{2}{x-4} + \frac{3}{x+1} = \frac{2(x+1) + 3(x-4)}{(x-4)(x+1)} = \frac{5x-10}{x^2-3x-4}$$

$$\int \frac{5x-10}{x^2-3x-4} dx = \int \frac{2}{x-4} dx + \int \frac{3}{x+1} dx$$

$$= 2\ln|x-4| + 3\ln|x+1| + c$$

$$\frac{5x-10}{x^2-3x-4} = \frac{A}{(x-4)} + \frac{b}{(x+1)}$$

$A(x+1) + B(x-4) = 5x - 10$

$Ax + A + Bx - 4B = 5x - 10 \Rightarrow x(A+B) + A - 4B = 5x - 10$

Coefficient of x^0: $A - 4B = -10$ (1)

Coefficient of x^1: $(A + B) = 5$ (2)

From $Eq.(2)$, $A = B - 5$, substitute into $Eq.(1)$ yields

$B - 5 - 4B = -10 \Rightarrow B = 3$

Then $A = 3 - 5 = 2$

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:

- The degree of $f(x)$ must be less than the degree of $g(x)$. That is, the fraction must be proper. If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term.
- We must know the factors of $g(x)$. In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.

Here is how we find the partial fractions of a proper fraction $f(x)/g(x)$ when the factors of g are known

Method of Partial Fraction ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.
4. Equate the coefficients of corresponding powers of x and solve the resulting

Example 15 : Evaluate the integral

$$\int \frac{(x+2)}{x(x-1)(x+1)}\,dx$$

Solution

$$\frac{(x+2)}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x+1)}$$

$A(x-1)(x+1) + Bx(x+1) + Cx(x-1) = x+2$

$Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx = x+2$

Coefficient of x^0: $-A = 2$ (1)

Coefficient of x^1: $B - C = 1$ (2)

Coefficient of x^2: $A + B + C = 0$ (3)

From $Eq. (1)$, $A = -2$,

From $Eq. (2)$, $B = 1 + C$, substitute into $Eq. (3)$

$-2 + 1 + C + C = 0 \Rightarrow C = \frac{1}{2}$ $\quad \Longrightarrow B = 1 + \frac{1}{2} = \frac{3}{2}$

$$\int \frac{(x+2)}{x(x-1)(x+1)} dx = \int \frac{-2}{x} dx + \int \frac{3/2}{(x-1)} dx + \int \frac{1/2}{(x+1)} dx$$

$$= -2\ln|x| + \frac{3}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + c$$

Example 16 : Evaluate the integral

$$\int \frac{x}{(x-1)^2(x+2)} dx$$

Solution

Let

$$\frac{x}{(x-1)^2(x+2)} = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{(x+2)}$$

$A(x+2) + B(x-1)(x+2) + C(x-1)^2 = x$

$Ax + 2A + Bx^2 + Bx - 2B + Cx^2 - 2Cx + C = x + 2$

Coefficient of x^0: $2A - 2B + C = 0$ (1)

Coefficient of x^1: $A + B - 2C = 1$ (2)

Coefficient of x^2: $B + C = 0$ (3)

From $Eq. (3)$, $B = -C$, substitute into $Eqs. (1) and$ (2)

$2A + 2C + C = 2 \Rightarrow 2A + 3C = 0$ (4)

$A - C - 2C = 1 \Rightarrow A - 3C = 1$ (5)

Then solving $Eqs. (4) and$ (5)

$A = \frac{1}{3},\ \ C = -\frac{2}{9}\ \ \Rightarrow B = \frac{2}{9}$

$$\int \frac{x}{(x-1)^2(x+2)} = \frac{1}{3}\int \frac{dx}{(x-1)^2} + \frac{2}{9}\int \frac{dx}{(x-1)} - \frac{2}{9}\int \frac{dx}{(x+2)}$$

$$= -\frac{1}{3(x-1)} + \frac{2}{9}\ln|x-1| - \frac{2}{9}\ln|x+2|$$

Example 17 : Evaluate the integral

$$\int \frac{dx}{x^2(x^2+4)}$$

Solution

$$\frac{1}{x^2(x^2+4)} = \frac{A}{x^2} + \frac{B}{x} + \frac{Cx+D}{(x^2+4)}$$

$A(x^2+4) + Bx(x^2+4) + (Cx+D)x^2 = 1$

$Ax^2 + 4A + Bx^3\ +\ 4Bx + Cx^3 + Dx^2 = 1$

Coefficient of x^0: $4A = 1$ (1)

Coefficient of x^1: $4B = 0$ (2)

Coefficient of x^2: $A + D = 0$ (3)

Coefficient of x^3: $B + C = 0$ (4)

From $Eqs.(1)\ and\ (2)$

$A = \frac{1}{4}\ \ and\ \ B = 0$

Substituting into $Eqs.(3)\ and\ (4)$

$D = -\frac{1}{4}\ and\ C = 0$

Then

$$\int \frac{dx}{x^2(x^2+4)} = \frac{1}{4}\int \frac{dx}{x^2} - \frac{1}{4}\int \frac{dx}{(x^2+4)} = -\frac{1}{4x} - \frac{1}{4}\cdot\frac{1}{2}\tan^{-1}\frac{x}{2} + c$$

Example 18 : Evaluate the integral

$$\int \frac{dx}{x^3-8}$$

Solution

$$\int \frac{dx}{x^3-8} = \int \frac{dx}{(x-2)(x^2+2x+4)}$$

Let

$$\frac{1}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{(x^2+2x+4)}$$

$A(x^2+2x+4) + (Bx+C)(x-2) = 1$

$Ax^2 + 2Ax + 4A + Bx^2 - 2Bx + Cx - 2C = 1$

Coefficient of x^0: $4A - 2C = 1$ (1)

Coefficient of x^1: $2A - 2B + C = 0$ (2)

Coefficient of x^2: $A + B = 0$ (3)

From $Eq.(3)$, $B = -A$, substitute into $Eq.(2)$

$2A + 2A + C = 0 \Rightarrow \quad 4A + C = 0$ (4)

Then solving $Eqs.(1) and\ (4)$

$3C = -1 \Rightarrow C = -\frac{1}{3} \quad and\ A = \frac{1}{12}$

Then

$B = -\frac{1}{12}$

$$\int \frac{dx}{x^3-8} = \frac{1}{12}\int \frac{dx}{x-2} + \int \frac{-\frac{1}{12}x - \frac{1}{3}}{(x^2+2x+4)}dx = \frac{1}{12}\int \frac{dx}{x-2} - \frac{1}{12}\int \frac{x+4}{(x^2+2x+4)}dx$$

or

$$\int \frac{dx}{x^3-8} = \frac{1}{12}\int \frac{dx}{x-2} - \frac{1}{24}\int \frac{2x+8}{(x^2+2x+4)}dx$$

$$= \frac{1}{12}\int \frac{dx}{x-2} - \frac{1}{24}\left[\int \frac{2x+4}{(x^2+2x+4)}dx + \int \frac{6}{(x^2+2x+4)}dx\right]$$

$$= \frac{1}{12}\int \frac{dx}{x-2} - \frac{1}{24}\left[\int \frac{2x+4}{(x^2+2x+4)}dx + \int \frac{6}{(x+1)^2+3}dx\right]$$

$$= \frac{1}{12}\ln|x-2| - \frac{1}{24}\ln|x^2+2x+4| - \frac{1}{4\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + c$$

Example 19: Evaluate the integral

$$\int \frac{x^3 - x^2 + x + 2}{x^3 - 2x^2 - x + 2} dx$$

Solution

For the integral

$$\int \frac{x^3 - x^2 + x + 2}{x^3 - 2x^2 - x + 2} dx \Leftrightarrow \int \frac{f(x)}{g(x)} dx$$

The degree of $f(x)$ must be less than the degree of $g(x)$. Then one has to divide $f(x)$ by $g(x)$ and work with the remainder term

$$\begin{array}{r|l} & 1 \\ x^3 - 2x^2 - x + 2 & x^3 - x^2 + x + 2 \\ & \underline{x^3 - 2x^2 - x + 2} \\ & x^2 + 2x \end{array}$$

$$\int \frac{x^3 - x^2 + x + 2}{x^3 - 2x^2 - x + 2} dx = \int 1 + \frac{x^2 + 2x}{x^3 - 2x^2 - x + 2} dx$$

Also one can simplify the following expression

$$x^3 - 2x^2 - x + 2 = x^3 - 2x^2 - (x - 2) = x^2(x - 2) - (x - 2) = (x - 2)(x^2 - 1)$$
$$= (x - 2)(x - 1)(x + 1)$$

Then

$$\frac{x^3 - x^2 + x + 2}{x^3 - 2x^2 - x + 2} = 1 + \frac{x^2 + 2x}{(x - 2)(x - 1)(x + 1)}$$

Let

$$\frac{x^2 + 2x}{(x - 2)(x - 1)(x + 1)} = \frac{A}{(x - 2)} + \frac{B}{(x - 1)} + \frac{C}{(x + 1)}$$

$$A(x - 1)(x + 1) + B(x - 2)(x + 1) + C(x - 2)(x - 1) = x^2 + 2x$$

$$A(x^2 - 1) + B(x^2 - x - 2) + C(x^2 - 3x + 2) = x^2 + 2x$$

$$Ax^2 - A + Bx^2 - Bx - 2B + Cx^2 - 3Cx + 2C = x^2 + 2x$$

Coefficient of x^0: $-A - 2B + 2C = 0$ (1)

Coefficient of x^1: $-B - 3C = 2$ (2)

Coefficient of x^2: $A + B + C = 1$ (3)

Solving the above simultaneous equations yields

$$A = \frac{8}{3}, \qquad B = -\frac{3}{2} \qquad and \qquad C = -\frac{1}{6}$$

Then

$$\int \frac{x^3 - x^2 + x + 2}{x^3 - 2x^2 - x + 2} dx = \int dx + \frac{8}{3}\int \frac{dx}{(x-2)} - \frac{3}{2}\int \frac{dx}{(x-1)} - \frac{1}{6}\int \frac{dx}{(x+1)}$$

or

$$\int \frac{x^3 - x^2 + x + 2}{x^3 - 2x^2 - x + 2} dx = x + \frac{8}{3}\ln|x-2| - \frac{3}{2}\ln|x-1| - \frac{1}{6}\ln|x+1| + c$$

HOMEWORKS

HW 1: Evaluate the integral

$$\int \frac{e^x dx}{e^x(e^{2x} + 3e^x + 2)}$$

Hint: Let $y = e^x \Rightarrow dy = e^x dx$ the above integral will be transformed into the following form

$$\int \frac{dy}{y(y^2 + 3y + 2)} = \int \frac{dy}{y(y+2)(y+1)}$$

HW 2: Evaluate the integral

$$\int \frac{dx}{\cos x\,(\sin^2 x - 4)}$$

Hint: Multiply by $\cos x$ / $\cos x$ yields

$$\int \frac{\cos x\, dx}{\cos^2 x\,(\sin^2 x - 4)} = \int \frac{\cos x\, dx}{(1 - \sin^2 x)(\sin^2 x - 4)}$$

Let $y = \sin x \Rightarrow dy = \cos x\, dx$ the above integral will be transformed into the following form

$$\int \frac{dy}{(1 - y^2)(y^2 - 4)} = \int \frac{dy}{(1-y)(1+y)(y-2)(y+2)}$$

HW 3: Evaluate the integral

$$\int \frac{dx}{\tan^2 x + 2\tan x}$$

Hint: Multiply by $\sec^2 x$ / $\sec^2 x$ yields

$$\int \frac{\sec^2 x\, dx}{\sec^2 x\,(\tan^2 x + 2\tan x)} = \int \frac{\sec^2 x\, dx}{(1 + \tan^2 x)\,(\tan^2 x + 2\tan x)}$$

Let $y = \tan x \Rightarrow dy = \sec^2 x\, dx$ *the above integral will be transformed into the following form*

$$\int \frac{dy}{(1+y^2)(y^2+2y)}$$

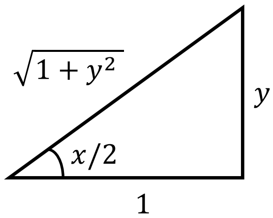
Trigonometric Substitutions of $y = \tan(x/2)$

One can use the following substitution to transform the trigonometric function to algebraic functions

Let $y = \tan(x/2)$

then $\tan^{-1} y = x/2 \Rightarrow x = 2\tan^{-1} y$

$$dx = \frac{2dy}{1+y^2}$$



Using the following identity and the figure shown one can find the following expressions

$\sin x = 2\sin(x/2)\cdot\cos(x/2)$

$$= 2\frac{y}{\sqrt{1+y^2}}\frac{1}{\sqrt{1+y^2}} = \frac{2y}{1+y^2}$$

Also

$\cos x = \cos^2(x/2) - \sin^2(x/2)$

$$= \frac{1}{1+y^2} - \frac{y^2}{1+y^2} = \frac{1-y^2}{1+y^2}$$

then

$$\sin x = \frac{2y}{1+y^2}$$

$$\cos x = \frac{1-y^2}{1+y^2}$$

Example 20 : Evaluate the integral

$$\int \frac{dx}{2+\sin x}$$

Solution

By using the trigonometric substitution of $y = \tan x/2$

$$\cos x = \frac{1-y^2}{1+y^2}, \qquad dx = \frac{2dy}{1+y^2}$$

$$\int \frac{\frac{2dy}{1+y^2}}{2+\frac{1-y^2}{1+y^2}} = \int \frac{2dy}{2(1+y^2)+(1-y^2)} = 2\int \frac{dy}{y^2+3} = \frac{2}{\sqrt{3}}\tan^{-1}\frac{y}{\sqrt{3}} + c'$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{\tan(x/2)}{\sqrt{3}} + c$$

Example 21 : Evaluate the integral

$$\int \frac{\sin x}{\sin x + 2} dx$$

Solution

$$\int \frac{\sin x}{\sin x+2}dx = \int \frac{\sin x + 2 - 2}{\sin x + 2}dx = \int dx - 2\int \frac{dx}{\sin x+2} = x - 2\int \frac{dx}{\sin x + 2}$$

By using the trigonometric substitution of $y = \tan x/2$

$$\sin x = \frac{2y}{1+y^2}, \qquad dx = \frac{2dy}{1+y^2}$$

$$\int \frac{dx}{\sin x+2} = \int \frac{\frac{2dy}{1+y^2}}{\frac{2y}{1+y^2}+2} = \int \frac{dy}{y+(1+y^2)} = \int \frac{dy}{y^2+y+1}$$

$$y^2+y+1 = y^2+y+\frac{1}{4}+\frac{3}{4} = \left(y+\frac{1}{2}\right)^2+\frac{3}{4}$$

$$\int \frac{dy}{y^2+y+1} = \int \frac{dy}{\left(y+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{y+\frac{1}{2}}{\sqrt{3}/2}\right) + c'$$

$$\int \frac{\sin x}{\sin x+2}dx = x - 2\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan\left(\frac{x}{2}\right) + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$$

$$= x - \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan(x/2) + 1}{\sqrt{3}}\right) + c$$

Example 22 : Evaluate the integral

$$\int \frac{dx}{2\sec x + 3}$$

Solution

$$\int \frac{dx}{2\sec x + 3} \times \frac{\cos x}{\cos x} = \int \frac{\cos x\, dx}{2 + 3\cos x} = \frac{1}{3}\int \frac{3\cos x + 2 - 2}{3\cos x + 2} dx$$

$$= \frac{1}{3}\left[\int dx - 2\int \frac{dx}{3\cos x + 2}\right] = \frac{1}{3}\left[x - 2\int \frac{dx}{3\cos x + 2}\right]$$

By using the trigonometric substitution of $y = \tan x/2$

$$\cos x = \frac{1 - y^2}{1 + y^2}, \qquad dx = \frac{2dy}{1 + y^2}$$

$$\int \frac{dx}{3\cos x + 2} = \int \frac{\frac{2dy}{1 + y^2}}{3\left(\frac{1 - y^2}{1 + y^2}\right) + 2} = 2\int \frac{dy}{3(1 - y^2) + 2(1 + y^2)} = 2\int \frac{dy}{5 - y^2}$$

$$= \frac{2}{\sqrt{5}} \tan^{-1}\left(\frac{y}{\sqrt{5}}\right) + c'$$

$$\int \frac{dx}{2\sec x + 3} = \frac{1}{3}\left[x - 2\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{\tan(x/2)}{\sqrt{5}}\right)\right] + c = \frac{x}{3} - \frac{4}{3\sqrt{5}}\tan^{-1}\left(\frac{\tan(x/2)}{\sqrt{5}}\right) + c$$

Products of Powers of Sines and Cosines

The integrals of the form

$$\int \sin^m x \cos^n x\, dx$$

Where m and n are nonnegative integers (positive or zero). One can divide the work into three cases:

Case 1: If m is odd, one can write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x$$

Then we combine the single $\sin x$ with dx in the integral and set xdx equal to – $d(\cos x)$.

Case 2: If m is even and n is odd, one can write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^m x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x$$

Then we combine the single $\cos x$ with dx in the integral and set xdx equal to $d(\sin x)$.

Case 3: If m and n are both even, one can substitute

$$\sin^2 x = \frac{1-\cos 2x}{2}, \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

To reduce the integral to one in lower powers of $\cos 2x$.

Also, one can use the following general **reduction formulas**

$$\int \sin^n x \cos^m x\,dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{m+n} + \frac{n-1}{m+n}\int \sin^{n-2} x \cos^m x\,dx \qquad (n \neq -m)$$

Or

$$\int \sin^n x \cos^m x\,dx = -\frac{\sin^{n+1} x \cos^{m-1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{n} x \cos^{m-2} x\,dx$$

Example 23 : Evaluate the integral

$$\int \sin^3 x \cos^2 x\,dx$$

Solution

The integral having the form

$$\int \sin^m x \cos^n x\,dx$$

With $m = 3$ odd, one can write $m = 3 = 2k + 1 \Rightarrow k = 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^3 x = \sin^{2\times 1+1} x = (\sin^2 x)^1 \sin x = (1-\cos^2 x)^1 \sin x$$

$$\int \sin^3 x \cos^2 x\,dx = \int (1-\cos^2 x)\sin x \cos^2 x\,dx$$

Let $u = \cos x \Rightarrow du = -\sin x\,dx$

$$\int (1-\cos^2 x)\sin x \cos^2 x\,dx = -\int (1-u^2)\;u^2\,du = \int (u^4-u^2)du = \frac{u^5}{5} - \frac{u^3}{3} + c'$$

$$\therefore \int \sin^3 x \cos^2 x\,dx = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + c$$

Example 24 : Evaluate the integral

$$\int \cos^5 x\,dx$$

Solution

The integral having the form

$$\int \sin^m x \cos^n x \, dx$$

With $n = 5$ odd and $m = 0$, one can write $n = 5 = 2k + 1 \Rightarrow k = 2$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^5 x = \cos^{2\times2+1} x = (\cos^2 x)^2 \cos x = (1 - \sin^2 x)^2 \cos x$$

$$\int \cos^5 x \, dx = \int (1 - \sin^2 x)^2 \cos x \, dx$$

Let $u = \sin x \Rightarrow du = \cos x \, dx$

$$\int (1 - \sin^2 x)^2 \cos x \, dx = \int (1 - u^2)^2 \, du = \int (1 - 2u^2 + u^4) \, du = u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + c'$$

$$\therefore \int \cos^5 x \, dx = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c$$

HOMEWORKS

1. $\int (2x+1)^{10} x^2 \, dx$
2. $\int \frac{x^3}{(x+2)^{\frac{3}{2}}} dx$
3. $\int \frac{x}{4 + 9x^4} dx$
4. $\int \frac{dx}{\sqrt{-x^2 + 6x}}$
5. $\int \frac{dx}{x^2 + 4x + 5}$
6. $\int \frac{dx}{1 - \sin 2x}$
7. $\int \frac{\sin^3 x}{\sqrt[3]{\cos x}} dx$
8. $\int \csc^6 x \, dx$
9. $\int \sin^2 x \cos^2 x \, dx$
10. $\int \sec^4 x \tan^{\frac{1}{2}} dx$
11. $\int \sqrt{1 - \cos 2x} \, dx$
12. $\int \frac{\sin x}{(1 + \cos x)^2} dx$
13. $\int \frac{\sqrt{\tan x}}{1 - \sin^2 x} dx$
14. $\int \cot^4 2x \, dx$
15. $\int \sqrt{\sin x} \cos^3 x \, dx$
16. $\int \frac{dx}{1 - \sec x}$
17. $\int \frac{dx}{\sqrt{4x - x^2 + 5}}$
18. $\int \frac{dx}{4x^2 + 2x + 3}$
19. $\int \frac{dx}{\sqrt{x}\sqrt{1 - x}}$

20 $\int \frac{\tan x \sec^2 x}{\sqrt{1 + \tan x}} dx$

21. $\int \frac{x^5 - x + 1}{x^2 + 1} dx$

Definite Integral

Sigma notation:

$$\sum_{k=1}^{4} K^2 = 1^2 + 2^2 + 3^2 + 4^2$$

$$\sum_{k=-3}^{1} K^3 = (-3)^3 + (-2)^3 + (-1)^3 + (0)^3 + (1)^3$$

$$\sum_{j=1}^{n} a_j x^j = a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

$$\sum_{i=1}^{4} 2 = 2 + 2 + 2 + 2$$

$$\sum_{i=1}^{n} \frac{i+1}{i+3} = \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \cdots$$

$$\sum_{k=1}^{3} k \sin\left(\frac{k\pi}{5}\right) = \sin\left(\frac{\pi}{5}\right) + 2\sin\left(\frac{2\pi}{5}\right) + 3\sin\left(\frac{3\pi}{5}\right)$$

Properties:

1. $\sum_{j=1}^{n} k a_j = k \sum_{j=1}^{n} a_j$

2. $\sum_{k=1}^{n} (a_k \mp b_k) = \sum_{k=1}^{n} a_k \mp \sum_{k=1}^{n} b_k$

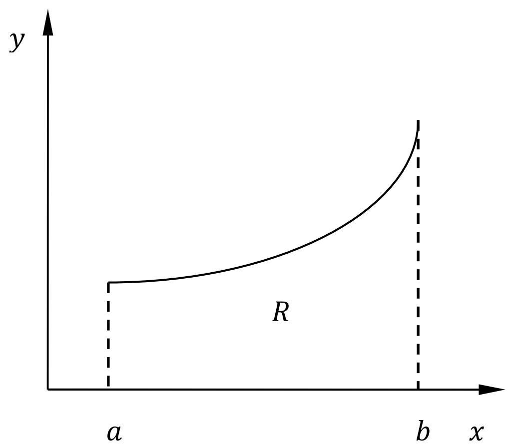
3. $\sum_{k=1}^{n} a_k = a_1 + a_2 + \sum_{k=3}^{n} a_k = \sum_{k=1}^{n-1} a_k + a_n = \sum_{k=1}^{4} a_k + \sum_{k=5}^{n} a_k$

4. $\sum_{k=1}^{n} a_k = \sum_{j=1}^{n} a_j = \sum_{i=1}^{n} a_i$

Summation Formulas:

1. $$\sum_{k=1}^{n} k = 1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}$$
2. $$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
3. $$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Area under a curve:



Problem: Find the area of a region R bounded below by the x-axis, on the sides by the lines x=a ,and x=b, and above by a curve $y = f(x)$, where f is continuous on $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$, A_a^b ?

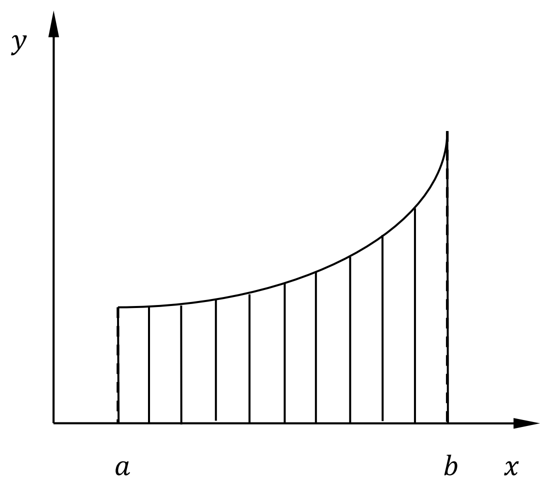
Solution

1. Choose an arbitrary positive integer n and divided the interval $[a, b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$ by points

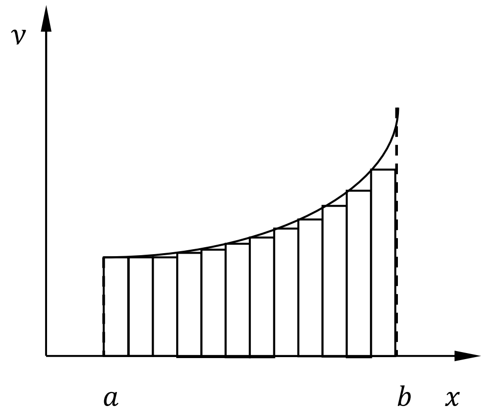
$a = x_0, x_1, x_2, x_3, \dots, x_k, \dots, x_{n-1}, x_n = b$

2. Draw vertical lines through the points $x_0, x_1, x_2, \dots, x_n$ to divide the region R into n strips of uniform width, the lines

$y_0 = f(a), y_1, y_2, y_3, \dots, y_k, \dots, y_{n-1}, y_n = f(b)$



Inscribed rectangle:

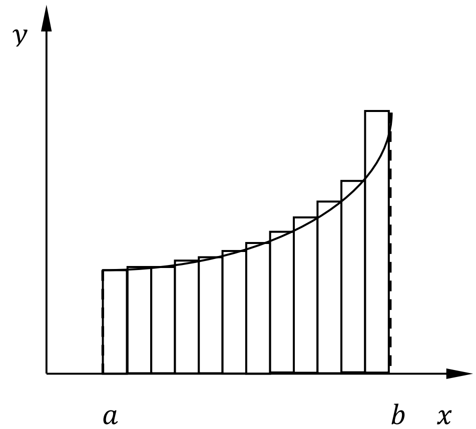


$\underline{S_{\boldsymbol{n}}} = \Delta x f(x_0) + \Delta x f(x_1) + \cdots + \Delta x f(x_k) + \cdots + \Delta x f(x_{n-1})$

$$= \sum_{k=1}^{n} \Delta x f(x_k)$$

or Circumscribed rectangles:

$\overline{S_n} = \Delta x f(x_1) + \Delta x f(x_2) + \cdots + \Delta x f(x_k) + \cdots + \Delta x f(x_n)$



$$= \sum_{k=1}^{n} \Delta x f(x_k)$$

$$\underline{S_{\boldsymbol{n}}} < A_{\boldsymbol{a}}^{\boldsymbol{b}} < \overline{S_n}$$

If we allow n to increase, the widths of the rectangles will get smaller, so that the approximation of R will get better as the smaller rectangles fill in more of the gaps under the curve. Thus we can consider the exact areas of R to be the limit of the areas of the approximately regions as n goes to plus infinity.

$$A_{\boldsymbol{a}}^{\boldsymbol{b}} = \lim_{n\to\infty} \overline{S_n} = \lim_{n\to\infty} \underline{S_{\boldsymbol{n}}} \Longrightarrow A_{\boldsymbol{a}}^{\boldsymbol{b}} = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta x f(x_k) = \lim_{\Delta x\to 0} \sum_{k=1}^{n} \Delta x f(x_k)$$

Properties:

1. $A_a^a = 0 \qquad (\Delta x = 0)$
2. $A_b^a = -A_a^b \qquad (\Delta x < 0)$
3. $A_a^b = A_a^c + A_c^b$

Definite Integrals:

The limit in the definition above is so important that there is some special notation for it. We write

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{k=1}^{n} \Delta x f(x_k)$$

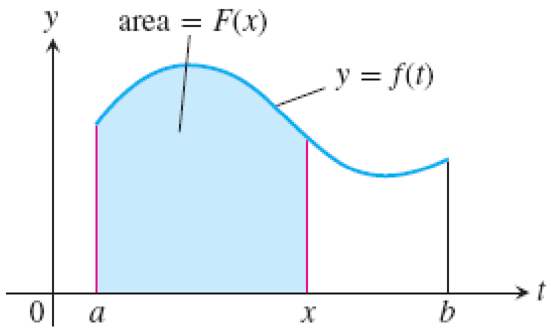
or

$$A_a^b = \begin{bmatrix} area\ under\ y = f(x) \\ over\ [a, b] \end{bmatrix} = \int_a^b f(x)dx$$

Theorem: The Fundamental Theorem of Calculus, Part 1

If f is continuous on [a , b] and differentiable on (a , b) and its derivative is $f(x)$;

$$F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x).$$



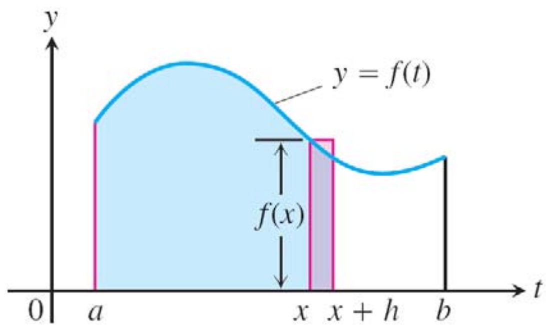
Proof of Theorem:

$$F(x) = \int_a^x f(t)\,dt$$

$$\frac{d}{dx}F(x) = \lim_{h \to 0}\frac{F(x+h) - F(x)}{h}$$

$$F(x+h) - F(x) = h\,f(x)$$

$$f(x)$$



Ex: $\frac{d}{dx}\int_1^x \frac{dt}{t+1} = \frac{1}{x+1}$

Ex: $\frac{d}{du}\int_0^u \frac{ds}{s^2-1} = \frac{1}{u^2-1}$

Ex: $\frac{d}{dx}\int_x^1 \frac{dt}{\sqrt{t^2+1}} = -\frac{d}{dx}\int_1^x \frac{dt}{\sqrt{t^2+1}} = \frac{-1}{\sqrt{x^2+1}}$

If $F(x) = \int_a^{u=g(x)} f(t)dt$

$$F'(x) = \frac{d}{du}\int_a^u f(t)dt \cdot \frac{du}{dx}$$

$$= f(u) \cdot \frac{du}{dx}$$

Ex: find F', F'' to the function $F(x) = \int_0^{\sin x} \sqrt{1-t^2}\, dt$.

Sol.: $F'(x) = \sqrt{1-\sin^2 x} \cdot \cos x = \cos^2 x$

$$F''(x) = -2\cos x \sin x = -\sin 2x$$

Example: Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ to the function $y = \int_{\sin x}^{\cos x} \frac{t}{\sqrt{1-t^2}} dt$.

Solution:

$$y = \int_{\sin x}^{c} \frac{t}{\sqrt{1-t^2}} dt + \int_{c}^{\cos x} \frac{t}{\sqrt{1-t^2}} dt = -\int_{c}^{\sin x} \frac{t}{\sqrt{1-t^2}} dt + \int_{c}^{\cos x} \frac{t}{\sqrt{1-t^2}} dt$$

$$\frac{dy}{dx} = \frac{-\sin x}{\sqrt{1-\sin^2 x}} \cdot \cos x + \frac{\cos x}{\sqrt{1-\cos^2 x}} \cdot (-\sin x)$$

$$= -\sin x - \cos x$$

$$\frac{d^2y}{dx^2} = -\cos x + \sin x$$

Theorem: The Fundamental Theorem of Calculus, Part 2

If f is continuous at every point of [a , b] and F is any antiderivative of f on [a , b], then

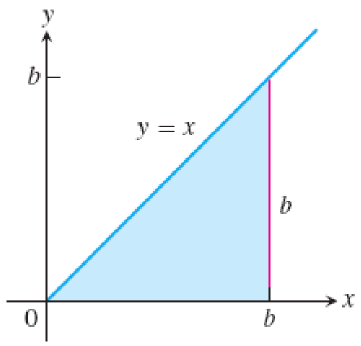
$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof:

Find the area under the line y = x to the interval [0,b].

Solution: We compute the area in two ways.

(a) To compute the definite integral as the limit of Riemann sums,



$$\Delta x = \frac{b-0}{n} = \frac{b}{n}$$

$$P = \left\{0, \frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}, \dots, \frac{nb}{n}\right\} \; and \; x_k = \frac{kb}{n}.$$

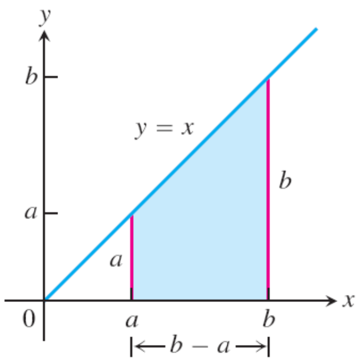
So

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x =$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{kb}{n} \cdot \frac{b}{n} =$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{kb^2}{n^2} =$$

$$\frac{b^2}{n^2} \lim_{n\to\infty} \sum_{k=1}^{n} k =$$



$$\lim_{n\to\infty} \frac{b^2}{n^2} \cdot \frac{n(n+1)}{2} = \lim_{n\to\infty} \frac{b^2}{2}\left(1 + \frac{1}{n}\right) = \frac{b^2}{2}$$

(b) $A = \left(\frac{1}{2}\right) \cdot b \cdot b = \frac{b^2}{2}$. Therefore,

$$\int_0^b x\,dx = \frac{b^2}{2}.$$

To find the area to the interval [a , b].

$$\int_a^b x\,dx = \int_a^0 x\,dx + \int_0^b x\,dx$$

$$= -\int_0^a x\,dx + \int_0^b x\,dx$$

$$= -\frac{a^2}{2} + \frac{b^2}{2}$$

$$\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}$$

Rules satisfied by definite integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A Definition

2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — Also a Definition

3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any Number k

   $\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$ — $k = -1$

4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

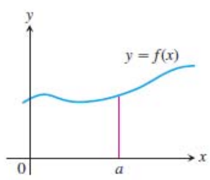
5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. *Max-Min Inequality:* If f has maximum value $\max f$ and minimum value $\min f$ on $[a, b]$, then

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a).$$

7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

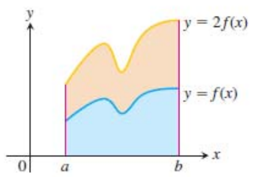
   $f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ (Special Case)



(a) *Zero Width Interval:*

$\int_a^a f(x)\,dx = 0.$

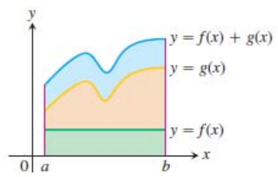
(The area over a point is 0.)



(b) *Constant Multiple:*

$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$

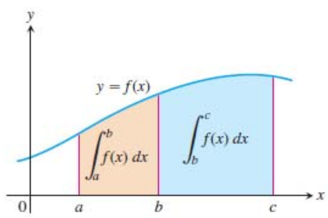
(Shown for $k = 2$.)



(c) *Sum:*

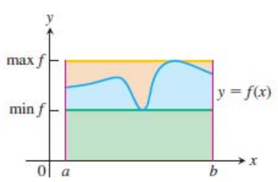
$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(Areas add)



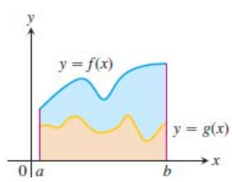
(d) *Additivity for definite integrals:*

$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$



(e) *Max-Min Inequality:*

$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a)$



(f) *Domination:*

$f(x) \ge g(x)$ on $[a, b]$

$\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

Example: Find the area under the curve $y = x^3$ from $x = -1$ to $x = 1$.

Solution:

$y = o \quad when \quad x = 0$

$$A_{-1}^{1} = \int_{-1}^{1} |f(x)| dx = \left| \int_{-1}^{0} x^3 \, dx \right| + \left| \int_{0}^{1} x^3 \, dx \right|$$

$$= \left| \frac{x^4}{4} \right]_{-1}^{0} \left| + \right| \frac{x^4}{4} \Big]_{0}^{1} \Big| = \left| 0 - \frac{1}{4} \right| + \left| \frac{1}{4} - 0 \right| = \frac{1}{2}.$$

Example: Find the area under the curve $y = \sin x \cos^2 2x$ from $x = 0 \; to \; x = \frac{\pi}{2}$.

Solution:

$\sin x \cos^2 2x = 0 \quad \Longrightarrow \quad \sin x = 0 \quad when \quad x = 0$

$or \quad \cos^2 2x = 0 \quad when \; 2x = \frac{\pi}{2} \Longrightarrow x = \frac{\pi}{4}$

$$A_0^{\frac{\pi}{2}} = \left| \int_0^{\frac{\pi}{4}} \sin x \cos^2 2x \, dx \right| + \left| \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x \cos^2 2x \, dx \right|$$

$$\int \sin x \cos^2 2x \, dx = \int \sin x \left(\frac{1 + \cos 4x}{2}\right) dx = \frac{1}{2} \int \sin x \, dx + \frac{1}{2} \int \sin x \cos 4x \, dx$$

$$= -\frac{1}{2} \cos x + \frac{1}{4} \int (\sin(1-4)\, x + \sin(1+4)x) \, dx$$

$$= -\frac{1}{2} \cos x + \frac{1}{4} \int (\sin(-3x) + \sin 5x) \, dx$$

$$= -\frac{1}{2} \cos x + \frac{1}{4} \left(\frac{1}{3} \cos 3x - \frac{1}{5} \cos 5x \right) + c$$

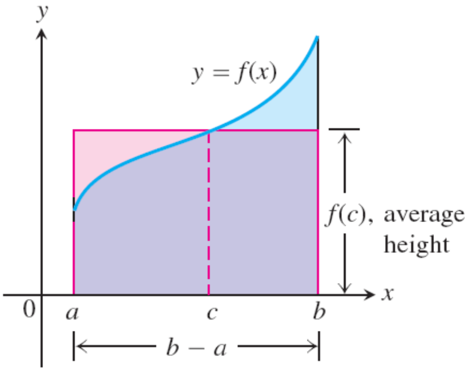
$$A_0^{\frac{\pi}{2}} = \left| -\frac{1}{2}\cos x + \frac{1}{4}\left(\frac{1}{3}\cos 3x - \frac{1}{5}\cos 5x\right)\right]_0^{\frac{\pi}{4}} \Bigg| + \left| -\frac{1}{2}\cos x + \frac{1}{4}\left(\frac{1}{3}\cos 3x - \frac{1}{5}\cos 5x\right)\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \Bigg|$$

$$= 7/15$$

Theorem: The Mean Value Theorem For Definite Integrals

If f is continuous on [a , b], then at some point c in [a , b],

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$



Example: Find the average value of $f(x) = \sin \pi x$ on the interval [-1,1].

Solution:

$$f(c) = \frac{1}{1-(-1)}\int_{-1}^{1} \sin\ \pi x\ dx = \frac{1}{2}(0) = 0$$

Example: Find the average value of $f(x) = \frac{x+2}{\sqrt{9-4x^2}}$ to the interval [0 , 1].

Solution:

$$f(c) = \frac{1}{1-0}\int_0^1 \frac{x+2}{\sqrt{9-4x^2}}dx = \int_0^1 \frac{x}{\sqrt{9-4x^2}}dx + 2\int_0^1 \frac{dx}{\sqrt{9-4x^2}}$$

$$= -\frac{1}{8}\frac{(9-4x^2)^{\frac{1}{2}}}{\frac{1}{2}}\Big]_0^1 + \sin^{-1}\frac{2x}{3}\Big]_0^1 = -\frac{\sqrt{5}}{4} + \frac{3}{4} + \sin^{-1}\frac{2}{3}$$

Example: Find the average value of $y = x + |x|$ to the interval [-1,2].

Solution:

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

$$f(c) = \frac{1}{2-(-1)} \int_{-1}^{2} (x + |x|)\, dx = \frac{1}{3}\left[\int_{-1}^{0} (x - x)dx + \int_{0}^{2} (x + x)dx\right]$$

$$= \frac{1}{3}\int_{0}^{2} 2xdx = \frac{1}{3}x^2\Big]_{0}^{1} = \frac{1}{3}(4 - 0) = \frac{4}{3}$$

Example: Evaluate the integral $\int_{-1}^{2} f(x)\, dx$ if,

$$f(x) = \begin{cases} \cos \pi x & -1 \leq x \leq 1 \\ \sqrt{x+1} & 1 < x < 2 \end{cases}$$

Solution:

$$\int_{-1}^{2} f(x)\, dx = \int_{-1}^{1} \cos \pi x + \int_{1}^{2} \sqrt{x+1}\; dx$$

$$= 2\int_{0}^{1} \cos \pi x\, dx + \int_{1}^{2} (x+1)^{\frac{1}{2}}\, dx$$

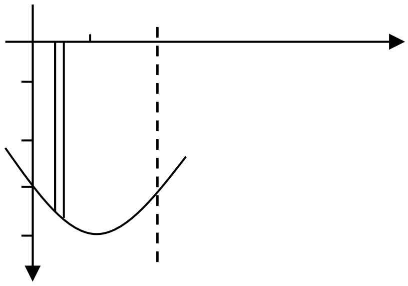
$$= \frac{2}{\pi}(\sin \pi x)\,\Big|_{0}^{1} + \frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}}\Big]_{1}^{2} = 2\sqrt{3} - \frac{4\sqrt{2}}{3}$$

7 APPLICATIONS OF DEFINITE INTEGRALS

Example: Find the area bounded by the curve $y = x^2 - 2x - 3$ and lines $y = 0, x = 0, x = 2$.

Solution:

First we sketch the area. The limits of integration are found by found the points of intersection of the area.



$y' = 2x - 2 = 0 \Longrightarrow x = 1, y = -4 \Longrightarrow (1, -4)$ Transition point.

$x = 0 \Longrightarrow y = -3$

$x = 2 \Longrightarrow y = -3$

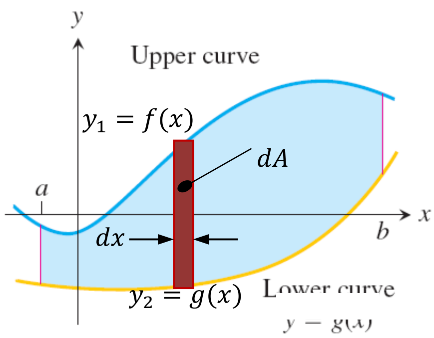
$dA = |f(x)|dx = |\ y = x^2 - 2x - 3|dx$

$$A_0^2 = \int_0^2 |x^2 - 2x - 3|\, dx = \left|\int_0^2 x^2 - 2x - 3\, dx\right| = \left|\left[\frac{x^3}{3} - x^2 - 3x\right]_0^2\right| = \left|\frac{8}{3} - 4 - 6\right| = \frac{22}{3}\ unit^2.$$

AREAS BETWEEN CURVES

Area under the curve can be found by using two approaches of integrations:

1. Integration with respect to x.



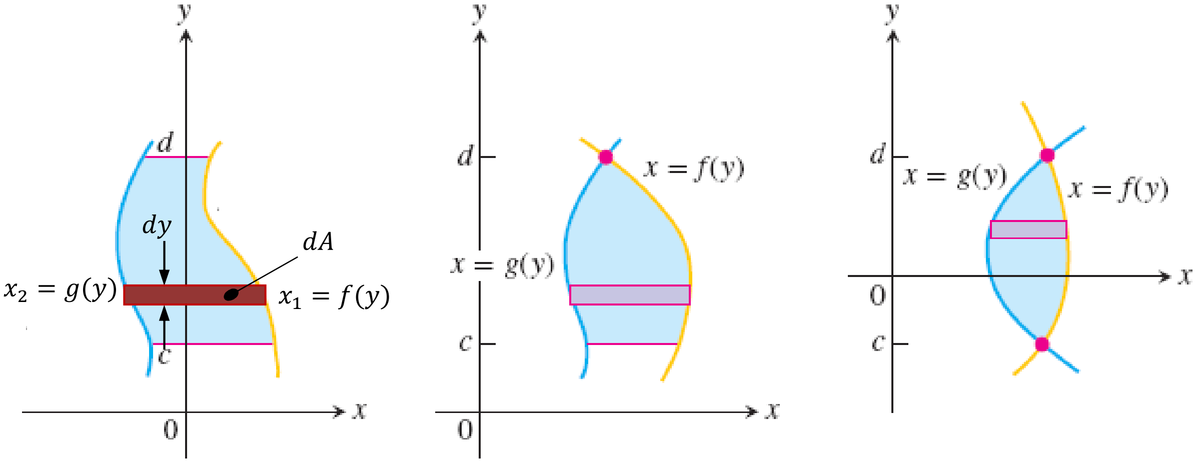
$dA = |y_1 - y_2|dx = |f(x) - g(x)|dx$

$$A = \int_a^b dA = \int_a^b |f(x) - g(x)|dx$$

2. Integration with respect to y.

$dA = |x_1 - x_2|dy = |f(y) - g(y)|dy$

$$A = \int_c^d dA = \int_c^d |f(y) - g(y)|dy$$



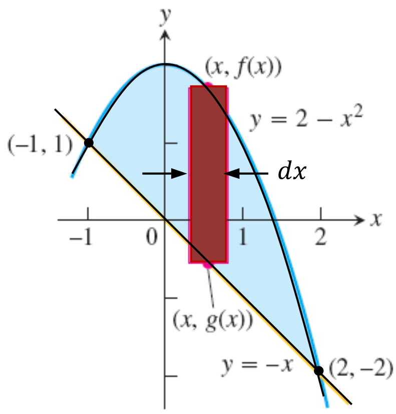
Example 1: Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution

First one should sketch the two curves.

Let $y_1 = 2 - x^2$ and $y_2 = -x$

The limits of integration can be found by solving the equation $y_1 = y_2$



$2 - x^2 = -x \Rightarrow x^2 - x - 2 = 0$

$(x + 1)(x - 2) = 0$

$x = -1$ and $x = 2$

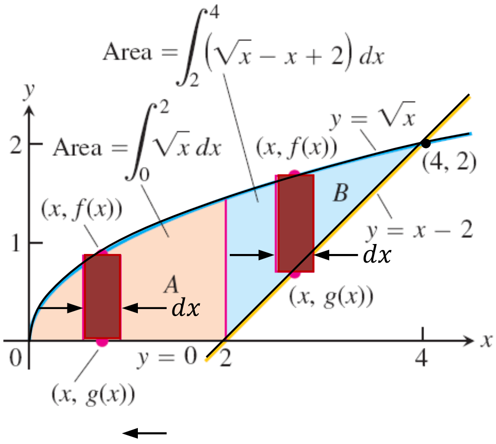
Then $a = -1$ and $b = 2$

$dA = |y_1 - y_2|dx = |2 - x^2 + x|dx$

$$A = \int_{-1}^{2} |2 - x^2 + x|dx = \left|\int_{-1}^{2} (2 - x^2 + x)dx\right| = \left|\left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}\right|$$

$$= \left|\left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right)\right| = \frac{9}{2} \quad Ans.$$

Example 2: Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.



Solution

First one should sketch the two curves.

From the sketch there are two regions:

Region A

$y_1 = \sqrt{x}$ and $y_2 = 0$

Region B

$y_1 = \sqrt{x}$ and $y_2 = x - 2$

Find the limits of each region

The limits of region A

The lower limit is $a_A = 0$ (satisfy the first quadrant). To find the upper limit let $x - 2 = 0 \Rightarrow x = 2$ then $b_A = 2$.

The limits of region B

The lower limit is $a_B = 2$ to find the upper limit let $y_1 = y_2$

$\sqrt{x} = x - 2 \Rightarrow x = (x - 2)^2 \Rightarrow x^2 - 5x + 4 = 0 \Rightarrow (x - 1)(x - 4) = 0$

Either $x = 1 < a_B$ lower limit, *neglected*

Or $x = 4 > a_B$ O.K., then $b_B = 4$

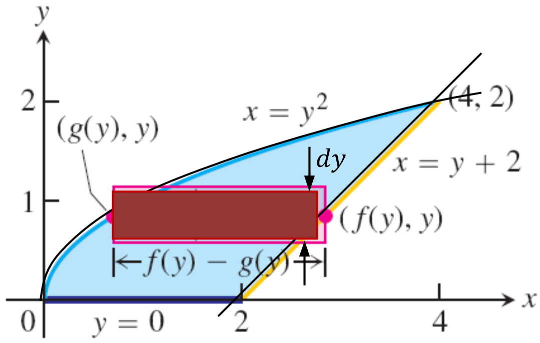
Then

	Region A	Region B
Limits	$a = 0$ and $b = 2$	$a = 2$ and $b = 4$
Functions	$y_1 = \sqrt{x}$ and $y_2 = 0$	$y_1 = \sqrt{x}$ and $y_2 = x - 2$
dA	$\lvert y_1 - y_2 \rvert dx = \sqrt{x}dx$	$\lvert y_1 - y_2 \rvert dx = \lvert \sqrt{x} - (x - 2) \rvert dx$

$$A = A_A + A_B = \int_0^2 \left|\sqrt{x}\right| dx + \int_2^4 \left|\sqrt{x} - (x-2)\right| dx = \left|\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^2\right| + \left|\left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{2} + 2x\right]_2^4\right|$$

$$= \left|\frac{2}{3}(2)^{\frac{2}{3}} - 0\right| + \left|\left(\frac{2}{3}(4)^{\frac{2}{3}} - 8 + 8\right) - \left(\frac{2}{3}(2)^{\frac{2}{3}} - 2 + 4\right)\right| = \frac{10}{3} \quad Ans.$$

Example 3: Find the area of the region in Example 2 by integrating with respect to y.



Solution

From the sketch of the graph there are one region

Find x as a function of y, $x = f(y)$

$y = x - 2 \Rightarrow x = y + 2 \Rightarrow f(y) = y + 2$

$y = \sqrt{x} \Rightarrow y^2 = x$ or $g(y) = y^2$

Then the functions are

$f(y) = x_1 = y + 2$

$g(y) = x_2 = y^2$

Find the limits of integration

The lower limit is $c = 0$ (satisfy the first quadrant).

To find the upper limit let $x_1 = x_2$ then

$y + 2 = y^2 \Rightarrow y^2 - y - 2 = 0 \Rightarrow (y+1)(y-2) = 0$

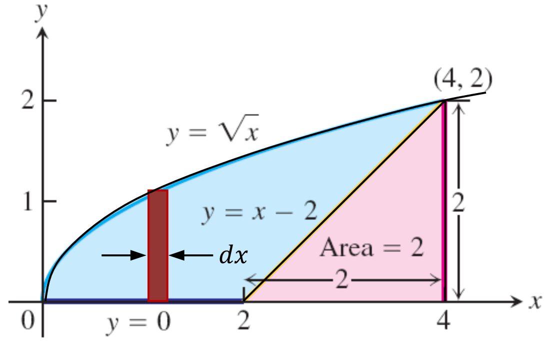
Either $y = -1 < c$, this point gives a point of intersection below the x-axis, *neglected*

Or $y = 2 > c$, O.K. then $d = 2$

$dA = |x_1 - x_2| dy$

$$A = \int_0^2 |x_1 - x_2| dy = \left|\int_0^2 (x_1 - x_2) dy\right| = \left|\int_0^2 (y + 2 - y^2) dy\right| = \left|\left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2\right| = \frac{10}{3} \quad Ans.$$

Example 4: Find the area of the region in Example 2 by using a combined method of calculus and geometry



Solution

The area will be the area between the curves $y_1 = \sqrt{x}$, and $y_2 = 0$ (x-axis) within the interval $a = 0$ and $b = 4$, minus the area of a triangle with base 2 and height 2.

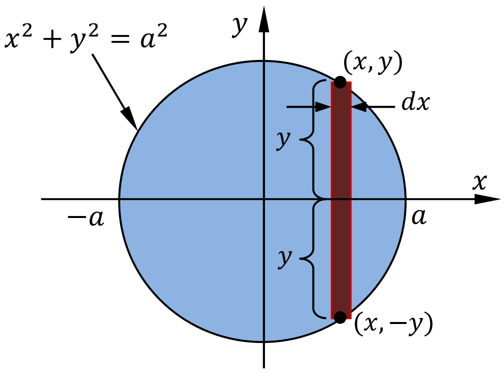
$$A = A_{y=0}^{y=\sqrt{x}} - A_{Triangle} = \int_0^4 dA - \frac{1}{2}BH$$

$$dA = |y_1 - y_2|dx = \left|\sqrt{x} - 0\right|dx = \left|\sqrt{x}\right|dx$$

$$A = \int_0^4 \left|\sqrt{x}\right|dx - \frac{1}{2}(2)(2) = \left|\int_0^4 \sqrt{x}dx\right| - 2 = \left|\frac{2}{3}x^{\frac{3}{2}}\right]_0^4\Bigg| - 2 = \frac{10}{3} \quad Ans.$$

Example 5: Prove that the area of a circle with radius a is πa^2.

Solution I



The equation of a circle is

$$x^2 + y^2 = a^2$$

Or in piece-wise form and as function of x

$$y = \begin{cases} +\sqrt{a^2 - x^2} \\ -\sqrt{a^2 - x^2} \end{cases}$$

If $y_1 = \sqrt{a^2 - x^2}$

Then $y_2 = -y_1$

$$dA = (y_1 - y_2)dx = (y_1 - (-y_1))dx = 2y_1dx = 2\sqrt{a^2 - x^2}\,dx$$

$$A = \int_{-a}^{a} dA = \int_{-a}^{a} 2\sqrt{a^2 - x^2}\,dx = 2\int_0^a 2\sqrt{a^2 - x^2}\,dx = 4\int_0^a \sqrt{a^2 - x^2}\,dx \quad even\ function$$

Let $x = a\sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right)$

$$dx = a\cos\theta\ d\theta$$

Find the limits of θ

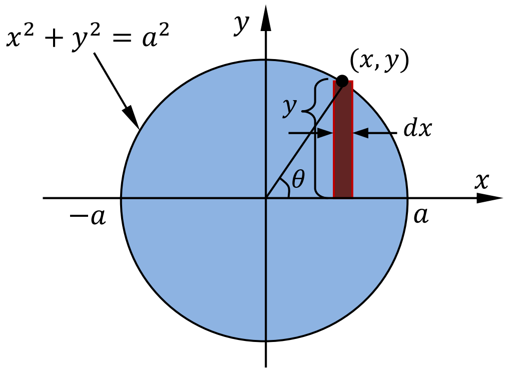
$x = 0 \Rightarrow \theta = 0 = \sin^{-1}(0) = 0$

$x = a \Rightarrow \theta = \sin^{-1}\left(\frac{a}{a}\right) = \sin^{-1}(1) = \frac{\pi}{2}$

$$A = 4\int_0^{\frac{\pi}{2}} \sqrt{a^2 - (a\sin\theta)^2}\ a\cos\theta\ d\theta = 4\int_0^{\frac{\pi}{2}} a^2\cos^2\theta\ d\theta = \frac{4a^2}{2}\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)\ d\theta$$

$$= \frac{4a^2}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = 2a^2\left[\frac{\pi}{2} + \frac{1}{2}\sin\frac{\pi}{2} - \left(0 + \frac{1}{2}\sin 0\right)\right] = \pi a^2 \qquad O.K.$$

Solution II



From the figure

$y = a\sin\theta$ and $x = a\cos\theta \Rightarrow dx = -a\sin\theta$

The limits of integration is $0 \le \theta \le 2\pi$

$dA = ydx = |a\sin\theta\,(-a\sin\theta)|d\theta = a^2\sin^2\theta\, d\theta$

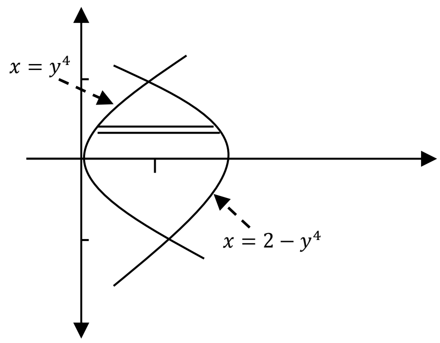
$$A = \int_0^{2\pi} dA = \int_0^{2\pi} a^2\sin^2\theta\, d\theta = \frac{a^2}{2}\int_0^{2\pi}(1-\cos 2\theta)d\theta$$

$$= \frac{a^2}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = \frac{a^2}{2}(2\pi)$$

$$= a^2\pi \qquad O.K.$$

Example: Find the area of the region enclosed by the curves $x = y^4$and the curve $x = 2 - y^4$.

Solution:



1) $\frac{dx}{dy} = 4y^3 = 0 \implies y = 0, x = 0$

2) $\frac{dx}{dy} = -4y^3 = 0 \implies y = 0, x = 2$

$y^4 = 2 - y^4 \implies 2y^4 = 2 \implies y^4 = 1 \implies y = \mp 1\,, x = 1$

$(1,1) and\ (1,-1)$ Intersection points

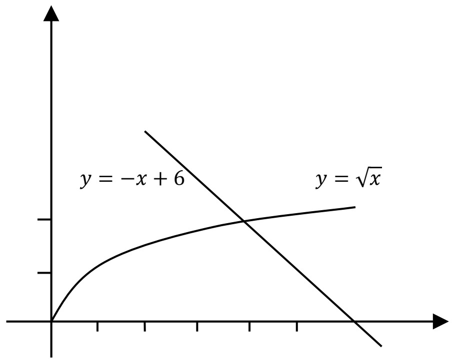
$dA = |x_1 - x_2|dy = |2 - y^4 - y^4|dy = |2 - 2y^4|dy$

$$A_{-1}^{1} = \int_{-1}^{1} |2 - 2y^4| = \left|2\int_0^1 2 - 2y^4\ dy\right| = 4\left|\int_0^1 (1 - y^4)\, dy\right| = 4\left|y - \frac{y^5}{5}\right]\begin{matrix}1\\0\end{matrix}\right|$$

$$= 4\left|1 - \frac{1}{5} - 0\right| = \frac{16}{5}$$

Example: Find the area of the region enclosed by the curve $y = \sqrt{x}$ and the lines $y = -x + 6$, $y = 0$.

Solution:



$y' = \frac{1}{2\sqrt{x}} > 0$ Increasing function

$\sqrt{x} = -x + 6$

$x + \sqrt{x} + 6 = 0$

$(\sqrt{x} - 2)(\sqrt{x} + 3) = 0$

$\sqrt{x} = 2 \Rightarrow x = 4, y = 2$ $(4,2)$ Intersection points

$\sqrt{x} \neq -3$

$dA = |(x_1 - x_2)|dy = |(6 - y) - y^2|dy$

$$A_0^2 = \int_0^2 |6 - y - y^2|\, dy = \left|\int_0^2 6 - y - y^2\, dy\right| = 6y - \frac{y^2}{2} - \frac{y^3}{3}\Big]_0^2 = \frac{22}{3}$$

Example: Find the area of the region enclosed by the curves $y = x^2 - 2, y = 2x^2 + x - 4$.

Solution:

1) $y' = 2x = 0 \Rightarrow x = 0, y = -2$ $(0,-2)$ Transition point

2) $y' = 4x + 1 = 0 \Rightarrow x = -\frac{1}{4}, y = -\frac{33}{8} = -4.125$ Transition point

$x^2 - 2 = 2x^2 + x - 4$

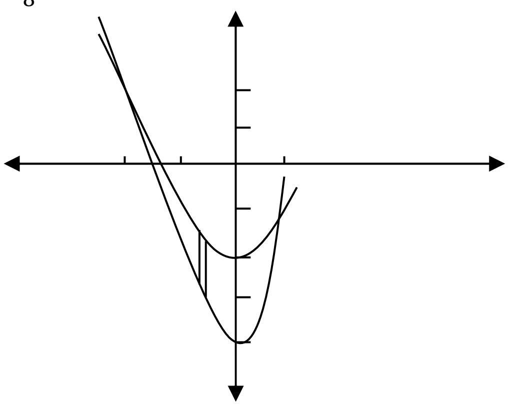
$x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0$

$x = -2 \Rightarrow y = 2$

$x = 1 \Rightarrow y = -1$

$(-2,2), (1,-1)$ Intersection points



$dA = |y_1 - y_2|dx = |x^2 - 2 - (2x^2 + x - 4)|dx = |-x^2 - x + 2|dx$

$$A_{-2}^{1} = \int_{-2}^{1} |-x^2 - x + 2|\, dx = \left|\int_{-2}^{1} -x^2 - x + 2\, dx\right| = \left|-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1}\right|$$
$$= \left|-\frac{1}{3} - \frac{1}{2} + 2 - \left(\frac{8}{3} - 2 - 4\right)\right| = \frac{9}{2}$$

Example: Find the area of the region enclosed by the curves $y = x^2 - 2x \ and, y = 4x - x^2$.

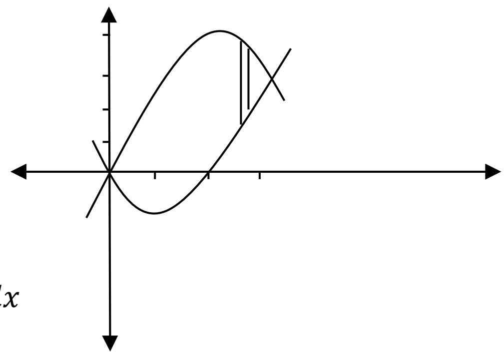
Solution:

1) $y' = 2x - 2 = 0 \Rightarrow x = 1, y = -1$ $(1,-1)$ Transition point
2) $y' = 4 - 2x = 0 \Rightarrow x = 2, y = 4$ $(2,4)$ Transition point

$x^2 - 2x = 4x - x^2$

$2x^2 - 6x = 0 \Rightarrow 2x(x - 3) = 0 \quad \Rightarrow x = 0, y = 0$

$\Rightarrow x = 3, y = 3$



(0,0),(3,3) Intersection point

$dA = |y_1 - y_2|dx = |4x - x^2 - (x^2 - 2x)|dx = |6x - 2x^2|dx$

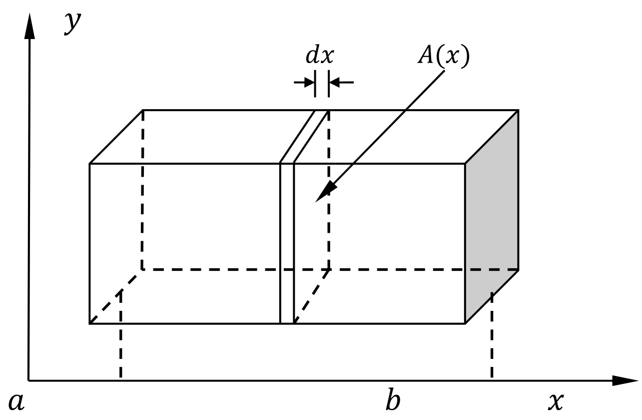
$$A_0^3 = \int_0^3 |6x - 2x^2|dx = \left|\int_0^3 6x - 2x^2\, dx\right| = \left|\frac{6x^2}{2} - \frac{2x^3}{3}\right]_0^3\right| = \left|\frac{54}{2} - \frac{54}{3}\right| = 9$$

APPLICATIONS OF DEFINITE INTEGRALS

Volumes by Slicing:

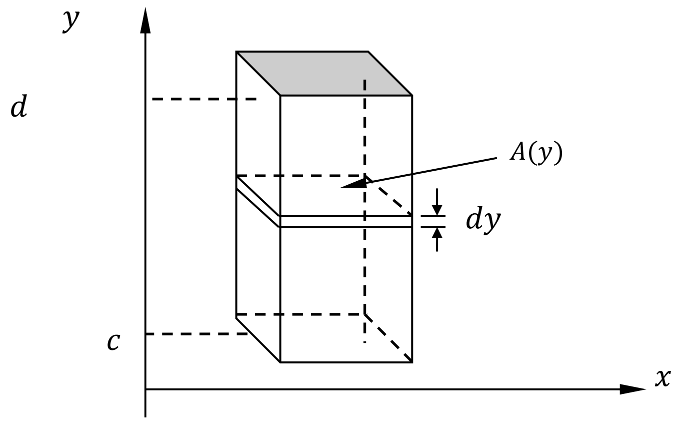
1) Volumes by cross sections Perpendicular to the x-axis.

$$V(x) = \int_a^b A(x)\, dx$$



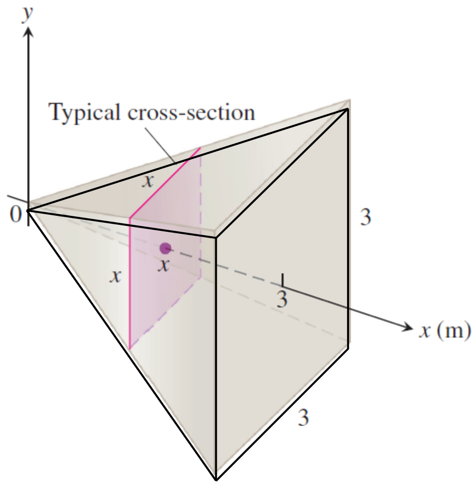
2) Volumes by cross sections

Perpendicular to the y-axis.



$$V(y) = \int_c^d A(y)\, dy$$

Example: A pyramid 3m high has a square base that is 3m on a side. Find the volume of the pyramid.



Solution:

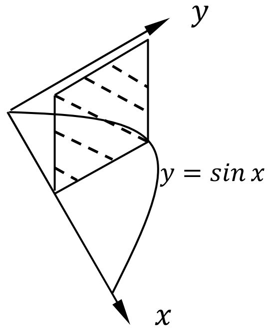
1. A sketch.
2. A formula for $A(x)$.
3. The limit of integration
4. Integrate to find the volume

$$V = \int_0^3 A(x)\, dx = \int_0^3 x^2\, dx = \frac{x^3}{3}\Big]_0^3 = 9\ m^3$$

Example: Find the volume of the solid that the base is curve $y = \sin x$ from $x = 0$ to $x = \pi$ and $y = 0$ if the if the vertical section on the x-axis is,

a) Square b) Half circle

Solution:



a) $A(x) = y^2 = \sin^2 x$

$dV = \sin^2 x\, dx$

$$V_0^\pi = \int_0^\pi \sin^2 x\ dx = \frac{1}{2}\int_0^\pi (1 - \cos 2x) dx =$$

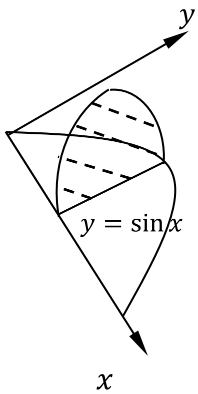
$$\frac{1}{2}\left[x - \frac{1}{2} sin\, 2x\right]_0^\pi = \frac{1}{2}(\pi - 0) = \frac{\pi}{2}$$

b) $A(x) = \frac{\pi}{2}\left(\frac{y}{2}\right)^2 = \frac{\pi}{8}\sin^2 x$

$dV = \frac{\pi}{8}\sin^2 x$

$V\,_0^{\pi} = \int_0^{\pi} \frac{\pi}{8}\sin^2 x\,dx = \frac{\pi}{8}\cdot\frac{1}{2}\int_0^{\pi}(1-\cos 2x)\,dx =$

$\frac{\pi}{16}\left[x - \frac{1}{2}sin\,2x\right]_0^{\pi} = \frac{\pi^2}{16}$



Example: Find the volume of the solid that the base is the area enclosed be the curves $y = \sin x$ and $y = \cos x$ and the line $x = 0$ if the vertical section on the x-axis is square.

Solution:

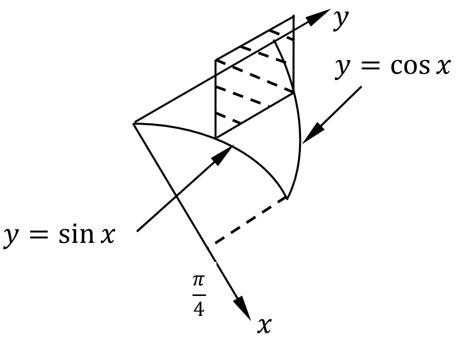
$\sin x = \cos x \implies \tan x = 1 \implies x = \frac{\pi}{4}$

$A(x) = y^2 dx = (\cos x - \sin x)^2 = \cos^2 x - 2\cos x\,\sin x + \sin^2 x$

$= 1 - \sin 2x$

$dV = (1 - \sin 2x)dx$

$V\,_0^{\pi/4} = \int_0^{\pi/4}(1 - \sin 2x)\,dx = \left[x + \frac{1}{2}\cos 2x\right]_0^{\pi/4} = \frac{\pi}{4} - \frac{1}{2}$



Solid of Revolution:

1. The Disk Method

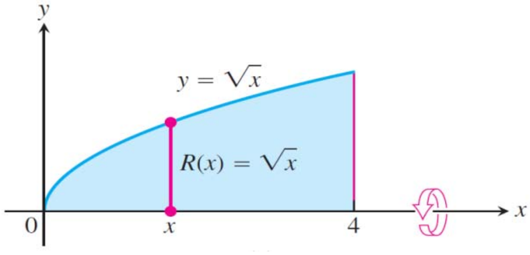
Example: (Rotation About the x-Axis)

The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$,and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

Solution:

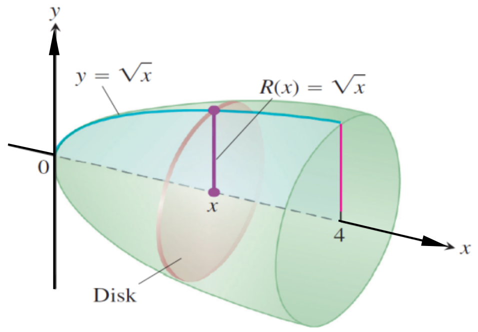
$A(x) = \pi\,[R(x)]^2$

$V = \int_a^b \pi\,[R(x)]^2\,dx$



$= \int_0^4 \pi\, [\sqrt{x}\,]^2\, dx$

$= \pi \int_0^4 x\, dx = \pi \frac{x^2}{4}\Big]_0^4 = \pi \frac{4^2}{2} = 8\pi$



Example: (Volume of a Sphere)
The circle $x^2 + y^2 = a^2$is rotated about the x-axis to generate a sphere. Find its volume.

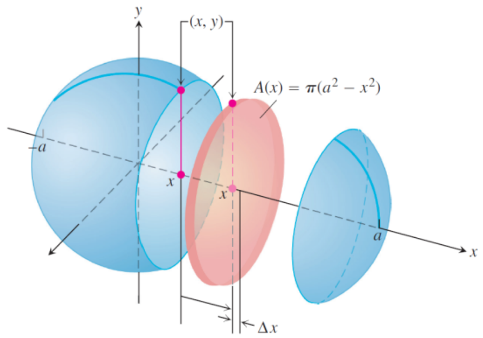
Solution:

$R(x) = y = \sqrt{a^2 - x^2}$

$A(x) = \pi\, y^2 = \pi(a^2 - x^2)$

$$V = \int_{-a}^{a} A(x)\, dx = \int_{-a}^{a} \pi(a^2 - x^2)\, dx$$

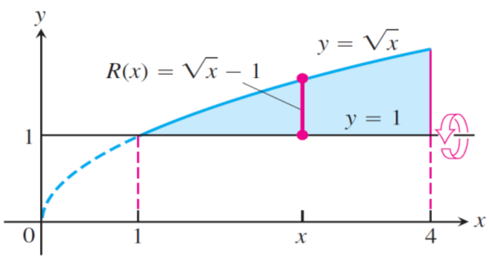
$$= \pi \left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4}{3}\pi a^3$$



Example: (Rotation About the Line $y = 1$)
Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.
Solution:

$R(x) = \sqrt{x} - 1$

$A(x) = \pi[R(x)]^2$

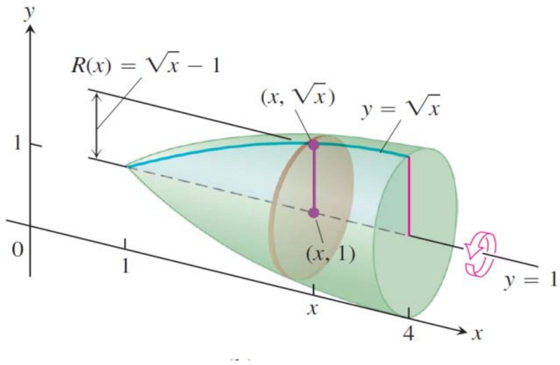


$V = \int_1^4 \pi[R(x)]^2\, dx$

$$= \int_1^4 \pi\, [\sqrt{x} - 1]^2\, dx$$

$$= \pi \int_1^4 [x - 2\sqrt{x} + 1]\, dx$$

$$= \pi \left[\frac{x^2}{2} - 2 \cdot \frac{2}{3} x^{\frac{3}{2}} + x\right]_1^4 = \frac{7\pi}{4}$$



Example: (Rotation About the y-Axis)

Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = \frac{2}{y}$, $1 \le y \le 4$, about the y-axis.

Solution:

$x' = -\frac{2}{y^2} < 0 \Longrightarrow$ Decreasing fct.

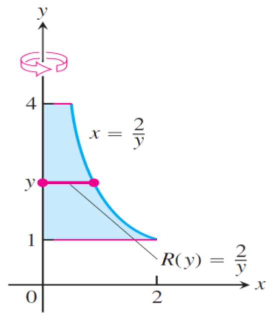
$x'' = \frac{4y}{y^4} > 0 \Longrightarrow$ Concave up

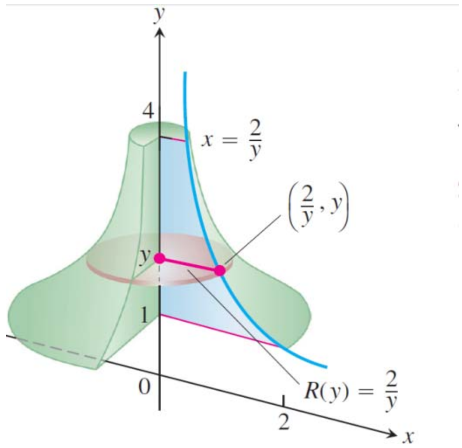
$V = \int_1^4 \pi[R(y)]^2\, dy$

$$= \int_1^4 \pi \left(\frac{2}{y}\right)^2\, dy$$

$$= \int_1^4 \frac{4}{y^2}\, dy = 4\pi \left[-\frac{1}{y}\right]_1^4$$

$$= 4\pi \frac{3}{4} = 3\pi$$





Example:(Rotation About a vertical Axis)

Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

Solution:

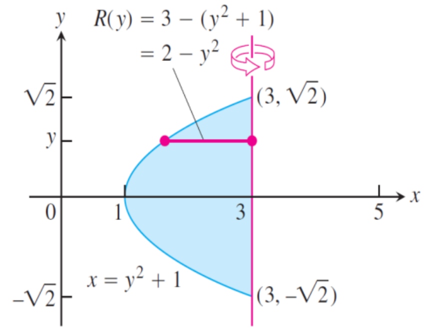
$x' = 2y = 0 \Rightarrow y = 0, x = 1$

(1,0) T point

$3 = y^2 + 1 \;\Rightarrow y = \pm\sqrt{2}$

$(3, \sqrt{2}), (3, -\sqrt{2})$ Intersection points

$R(y) = 3 - x = 3 - (y^2 + 1) = 2 - y^2$

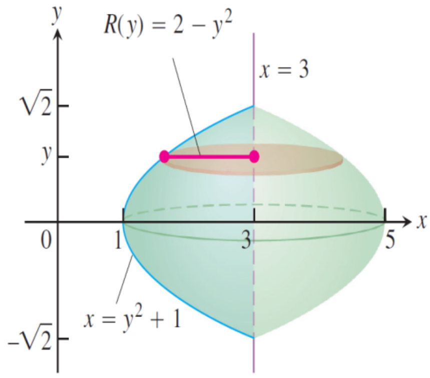


$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi [R(y)]^2 dy$

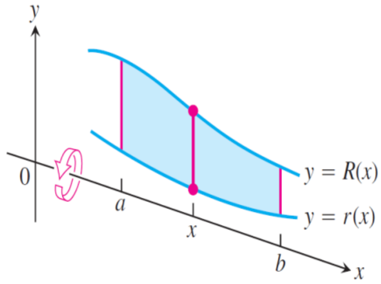
$$= \int_{-\sqrt{2}}^{\sqrt{2}} \pi [2 - y^2]^2 \, dy$$

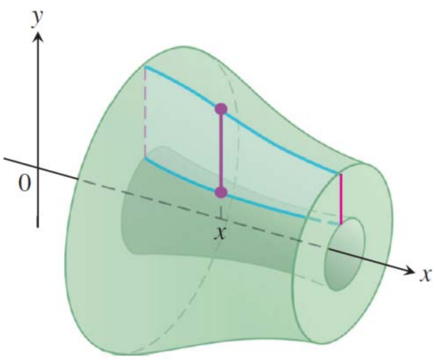
$$= \pi \int_{-\sqrt{2}}^{\sqrt{2}} (4 - 4y^2 + y^4) \, dy$$

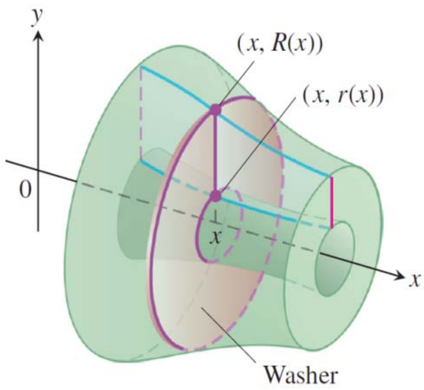
$$= \pi \left[4y - \frac{4}{3} y^3 + \frac{y^5}{5} \right]_{-\sqrt{2}}^{\sqrt{2}} = \frac{64\pi\sqrt{2}}{15}$$



2. The Washer Method







Outer radius: $R(x)$

Inner radius: $r(x)$

$A(x) = \pi([R(x)]^2 - [r(x)]^2)$

$$V = \int_a^b A(x) \, dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2) \, dx$$

Example: (Rotation About the x-Axis)

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution:

$y' = 2x = 0 \Longrightarrow x = 0, y = 1$

$(0,1)$T point

$x^2 + 1 = -x + 3 \Longrightarrow x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0 \Longrightarrow x = -2\,, y = 5$

$or\ x = 1\,, y = 2$

$(-2,5), (1,2)$Intersection points

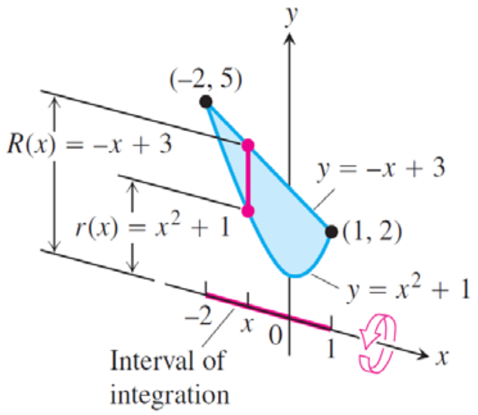
$R(x) = -x + 3$

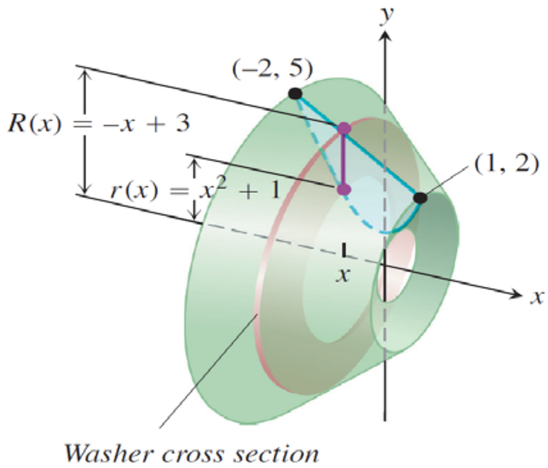
$r(x) = x^2 + 1$

$A(x) = \pi([R(x)]^2 - r(x)^2)$

$= \pi([-x + 3]^2 - [x^2 + 1]^2) = \pi(8 - 6x - x^2 - x^4)$

$$V = \int_{-2}^{1} A(x)\, dx = \int_{-2}^{1} \pi(8 - 6x - x^2 - x^4)\, dx = \pi \left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \frac{117\pi}{5}$$





Example: (Rotation About the y-Axis)

The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution:

$$x^2 = 2x \Longrightarrow x^2 - 2x = 0 \Longrightarrow x(x-2) = 0$$

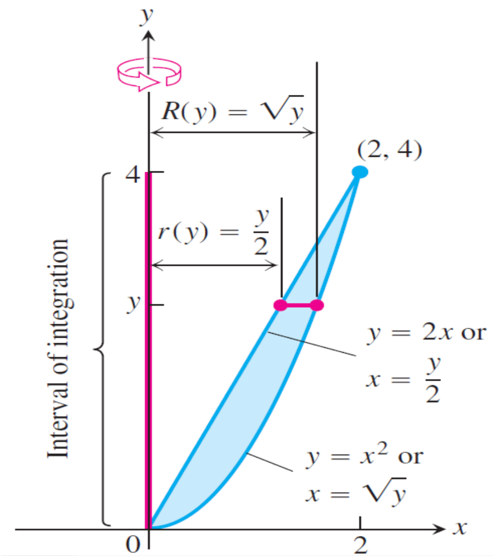
$$x = 0, y = 0 \quad or \quad x = 2, y = 4$$

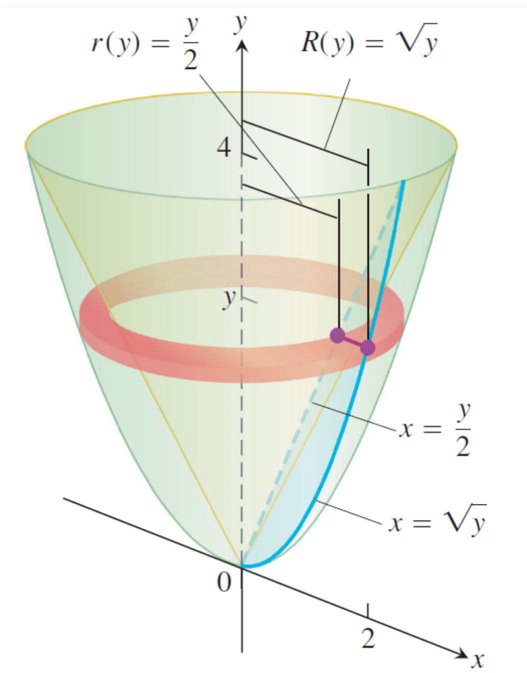
$$R(y) = \sqrt{y} \ , \ r(y) = \frac{y}{2}$$

$$A(x) = \pi([R(y)]^2 - r(y)^2)$$

$$= \pi\left(\left[\sqrt{y}\right]^2 - \left[\frac{y}{2}\right]^2\right) = \pi(y - \frac{y^2}{4})$$

$$V = \pi\int_0^4 \left(y - \frac{y^2}{4}\right) dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8}{3}\pi$$



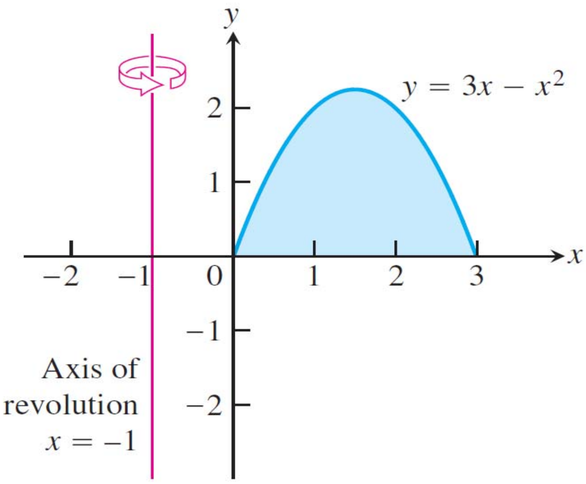


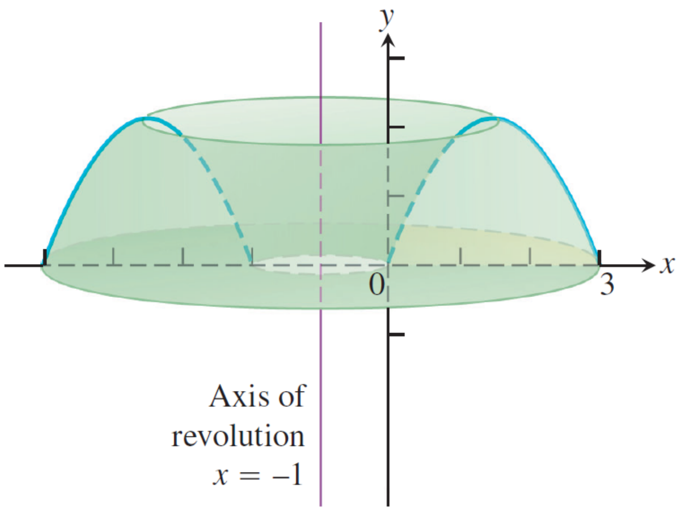
3. The Cylindrical Shells Method:

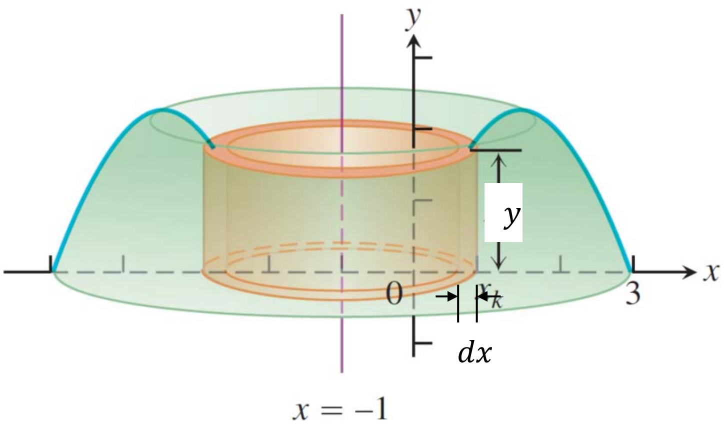
Example:(Finding a Volume Using Shells)

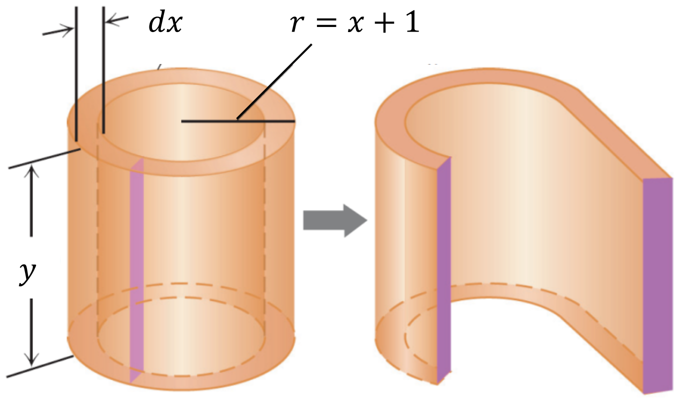
The region enclosed by the x-axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate the shape of a solid. Find the volume of the solid.

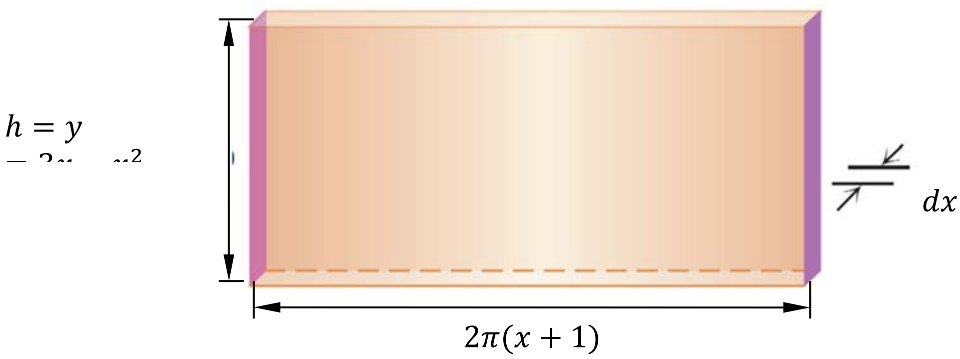
Solution:











$y' = 3 - 2x = 0 \Longrightarrow x = \frac{3}{2}, y = 2.25 \Longrightarrow (1.5, 2.25)$T point

$y = 3x - x^2 = 0 \Longrightarrow x(3 - x) = 0 \Longrightarrow x = 0 \quad or \quad x = 3 \quad \Longrightarrow (0,0), (3,0)$ Intersection points.

$$dV = 2\pi\,(shell\ radius)(shell\ height)dx = 2\pi\, r\, h\, dx$$

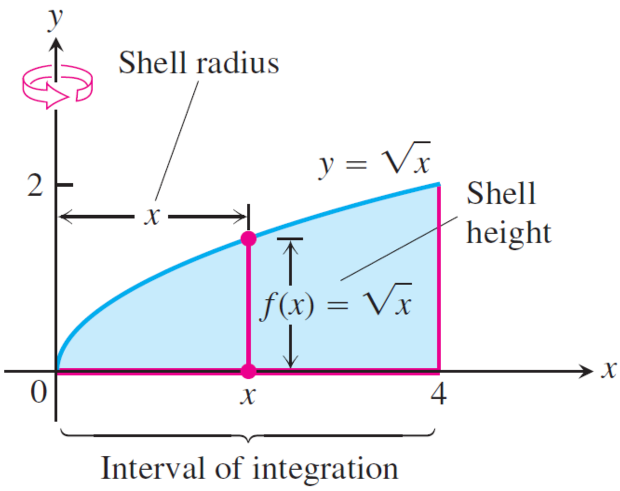
$$V = \int_0^3 dV = \int_0^3 2\pi(x+1)(3x - x^2)dx = 2\pi \int_0^3 (3x^2 - x^3 + 3x - x^2)\,dx$$

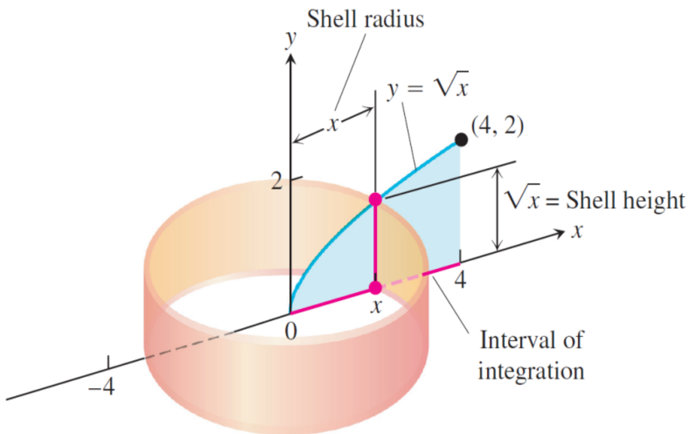
$$= 2\pi \int_0^3 2x^2 - x^3 + 3x\,dx = 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4} + \frac{3x^2}{2}\right]_0^3 = \frac{45\pi}{2}$$

Example: (Cylindrical Shells Revolving About the y-Axis)

The region bounded by the curve $y = \sqrt{x}$,the x-axis,and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution:





$$V = \int_a^b 2\pi\, r\, h\, dx$$

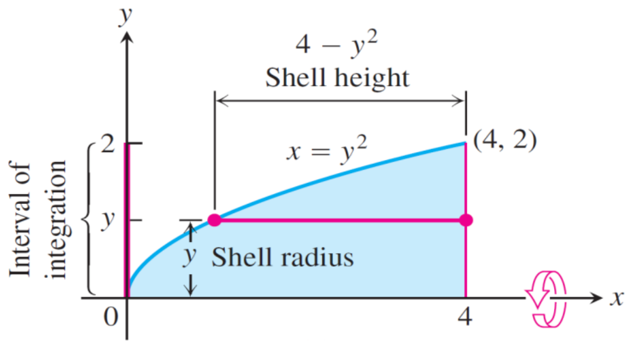
$$r = x, \qquad h = y = \sqrt{x}$$

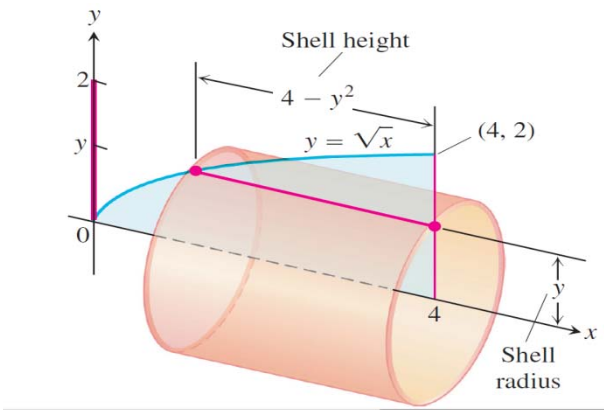
$$V = \int_0^4 2\pi\,(x)(\sqrt{x})\,dx = 2\pi \int_0^4 x^{\frac{3}{2}}\,dx = 2\pi \left[\frac{2}{5} x^{\frac{5}{2}}\right]_0^4 = \frac{128\pi}{5}$$

Example: (Cylindrical Shells Revolving About the x-Axis)

The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution:





$dV = 2\pi\, r\, h\, dy$

$r = y\ ,\qquad h = 4 - x = 4 - y^2$

$$V = \int_0^2 2\,\pi\,(y)(4 - y^2)\,dy = \int_0^2 2\pi(4y - y^3)dy = 2\pi\left[2y^2 - \frac{y^4}{4}\right]_0^2 = 8\pi$$

HOMEWORKS

H.W.1: Derive the formula for the volume of a sphere of radius a.

a) by using disk method   b) by using cylindrical shells

H.W.2: Find the volume of the solid generated by revolving the region bounded by $y = x^2$, $y = 0$ and $x = 2$,

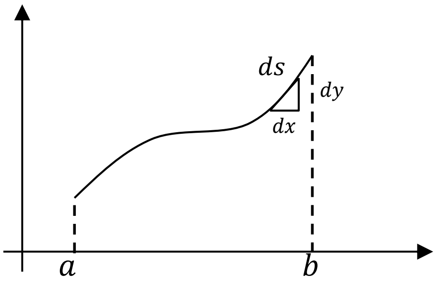
a)about $y = 0$ b) about $y = 4$ c) about $x = 0$ d) about $x = 2$

H.W.3: Find the volume of the solid generated by revolving $x^2 + y^2 = 4$ about $x = 3$.

H.W.4: Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{4 - x^2}$ and the line $y = 1$ about the x-axis.

Length of Curve in the Plane:

If the function f has a continuous first derivative throughout the interval [a,b], the length of the curve $y = f(x)$ from a to b is:



$$ds^2 = (dx)^2 + (dy)^2$$

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

1. If $y = f(x)$ $\qquad a \le x \le b$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$L_a^b = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

2. If $x = g(y)$ $\qquad c \le y \le d$

$$ds = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\ dy$$

$$L_c^d = \int_c^d \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\ dy$$

3. If $x = f(t), y = f(t)$ $\qquad t_1 \le t \le t_2$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$L_{t_1}^{t_2} = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Example: Find the length of the circle of radius r defined parametrically by
$x = r\cos t \quad and \quad y = r\sin t\,, \qquad 0 \le t \le 2\pi.$

Solution:

$$L_{t_1}^{t_2} = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = -r\sin t \quad , \quad \frac{dy}{dt} = r\cos t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = r^2(\sin^2 t + \cos^2 t\,) = r^2$$

$$L = \int_0^{2\pi} \sqrt{r^2}\, dt = r[t]\,_0^{2\pi} = 2\pi r$$

Example: Find the length of the curve $y = \int_0^x \sqrt{\cos t}\, dt$ from $t = 0$ to $t = \frac{\pi}{2}$.

Solution:

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; dx$$

$$\frac{dy}{dx} = \sqrt{\cos x}$$

$$ds = \sqrt{1 + \cos x}\; dx$$

$$L = \int_0^{\pi/2} \sqrt{1 + \cos x}\; dx = \int_0^{\pi/2} \sqrt{2\cos^2\left(\frac{x}{2}\right)}\, dx = \sqrt{2}\int_0^{\pi/2} \cos\left(\frac{x}{2}\right) dx = \sqrt{2}\,\left[2\sin\left(\frac{x}{2}\right)\right]_0^{\pi/2} = 2.$$

Areas of Surfaces of Revolution:

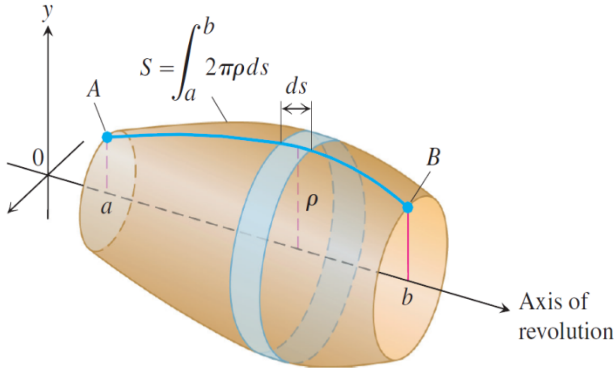
If the function $f(x) \ge 0$ is continuously differentiable on [a, b], the area of the surface generated by revolving the curve $y = f(x)$ about the x-axis is

$$dS = 2\pi\, y\, ds$$

$$S = \int_a^b 2\pi y ds = \int_a^b 2\pi\, f(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; dx$$

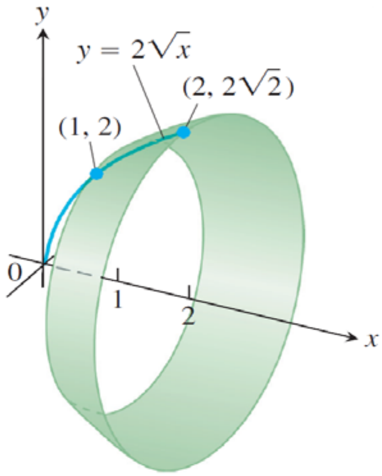
Or $S = \int_a^b 2\pi\, \rho\, ds$

Where ρ =distance between the curve and the line of revolution



Example: Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}\,, 1 \le x \le 2\,,$ about the x-axis .

Solution:



$$S = \int_a^b 2\pi\,\rho\,\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$a = 1, \qquad b = 2, \qquad y = 2\sqrt{x}\;, \qquad \frac{dy}{dx} = \frac{1}{\sqrt{x}}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\frac{1}{\sqrt{x}}\right)^2}$$

$$= \sqrt{1+\frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}$$

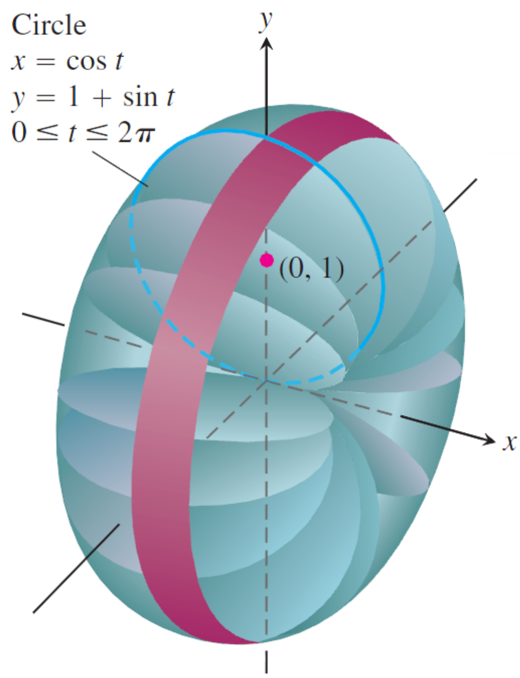
$$S = \int_1^2 2\pi\cdot 2\sqrt{x}\,\frac{\sqrt{x+1}}{\sqrt{x}}\,dx = 4\pi\int_1^2 \sqrt{x+1}\,dx = 4\pi\cdot\frac{2}{3}(x+1)^{3/2}\Big]_1^2 = \frac{8\pi}{3}\left(3\sqrt{3}-2\sqrt{2}\,\right).$$

Example: The standard parameterization of the circle of radius 1 centered at the point (0, 1) in the xy-plane is

$$x = \cos t\,, \qquad y = 1+\sin t\,, \qquad 0 \le t \le 2\pi\,.$$

Use this parameterization to find the area of the surface generated by revolving the circle about the x-axis.

Solution:



$$S = \int_a^b 2\pi\,y\,\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\;dt$$

$$= \int_0^{2\pi} 2\pi\,(1+\sin t)\,\sqrt{(-\sin t)^2+(\cos t)^2}\;dt$$

$$= \int_0^{2\pi} (1+\sin t)\,dt$$

$$= 2\pi[t-\cos t]_0^{2\pi} = 4\pi^2$$

8 TRANSCENDENTAL FUNCTIONS

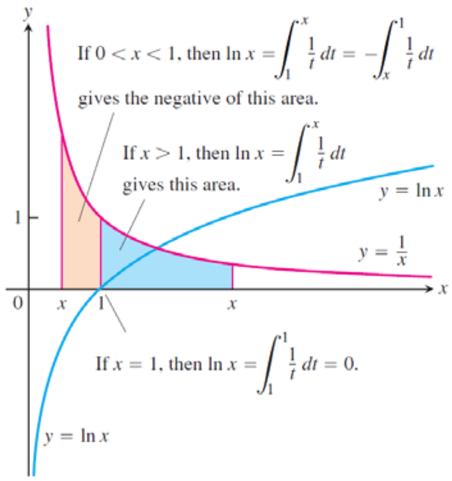
Functions can be classified into two broad groups. Polynomial functions are called *algebraic*, as are functions obtained from them by addition, multiplication, division, or taking powers and roots. Functions that are not algebraic are called *transcendental*. The trigonometric, exponential, logarithmic, and hyperbolic functions are transcendental, as are their inverses.

Natural Logarithms

$$\ln x = \int_1^x \frac{1}{t}\, dt\,, \qquad x > o$$

$$\ln 1 = \int_1^1 \frac{1}{t}\, dt = 0$$

$\ln x > 0$ *when* $x > 1$
$\ln x = 0$ *when* $x = 1$
$\ln x < 0$ *when* $0 < x < 1$



Definition: (The Number e)
The number e is that number in the domain of the natural logarithm satisfying

$$\ln(e) = 1$$

(Geometrically, the number e corresponds to the point on the x-axis for which the area under the graph of $y = \frac{1}{t}$ and above the interval $[1\,, e]$ is the exact area of the unit square.

The Derivative of $y = \ln x$

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\, dt = \frac{1}{x}$$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}\,, \qquad u > 0$$

Example: Derivative of Natural Logarithms

(a) $\frac{d}{dx}\ln 2x = \frac{1}{2x}\,(2) = \frac{1}{x}$

(b) $\frac{d}{dx}\ln(3x^2+4) = \frac{1}{3x^2+4}\,(6x) = \frac{6x}{3x^2+4}$

Properties of logarithms:

1. $\ln ax = \ln a + \ln x$
2. $\ln\frac{a}{x} = \ln a - \ln x$
3. $\ln\frac{1}{x} = -\ln x \qquad (\ln 1 = 0)$
4. $\ln x^n = n\ln x$

Example: (Applying the Properties to Function Formulas)

(a) $\ln\sqrt{\cos x} = \frac{1}{2}\ln\cos x$

(b) $\ln 4 + \ln\sin x = \ln(4\sin x)$

(c) $\ln\frac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$

(d) $\ln\sec x = \ln\frac{1}{\cos x} = -\ln\cos x$

The Integral $\int\frac{1}{u}du$

$$\int\frac{1}{u}\,du = \ln u + c$$

$$\int\frac{1}{u}\,du = \int\frac{1}{(-u)}\,d(-u) = \ln(-u) + c$$

$$\boxed{\int\frac{1}{u}du = \ln|u| + c}$$

Logarithmic Differentiation

Example: (Using Logarithmic Differentiation**)**

Find $\frac{dy}{dx}$ if $y = \sqrt[5]{\frac{x^2\,(x+3)^3\,(x^2-1)}{(x+2)(x+1)}}$

Solution:

$$\ln y = \ln \sqrt[5]{\frac{x^2\,(x+3)^3\,(x^2-1)}{(x+2)(x+1)}}$$

$$\ln y = \frac{1}{5}[2\ln x + 3\ln(x+3) + \ln(x^2-1) - \ (\ln(x+2) + \ln(x+1))]$$

$$\frac{y'}{y} = \frac{1}{5}\left[\frac{2}{x} + \frac{3}{x+3} + \frac{2x}{x^2-1} - \frac{1}{x+2} - \frac{1}{x+1}\right]$$

$$y' = \frac{y}{5}\left[\frac{2}{x} + \frac{3}{x+3} + \frac{2x}{x^2-1} - \frac{1}{x+2} - \frac{1}{x+1}\right]$$

$$\frac{dy}{dx} = \frac{1}{5}\sqrt[5]{\frac{x^2\,(x+3)^3\,(x^2-1)}{(x+2)(x+1)}}\ \left[\frac{2}{x} + \frac{3}{x+3} + \frac{2x}{x^2-1} - \frac{1}{x+2} - \frac{1}{x+1}\right]$$

The Graph of $\ln x$ **:**

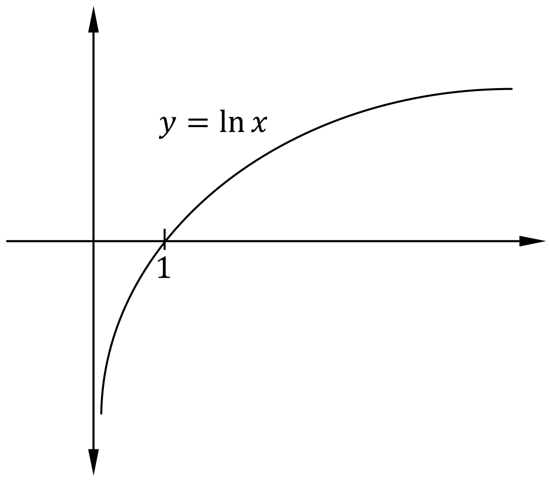
$D_f: x > 0$

$\frac{d}{dx}\ln x = \frac{1}{x} > 0$ *increasing function*

$\frac{d^2y}{dx^2} = -\frac{1}{x^2} < 0$ *concave dowen*

$\lim_{x\to\infty} \ln x = \infty$

$\lim_{x\to 0} \ln x = -\infty$

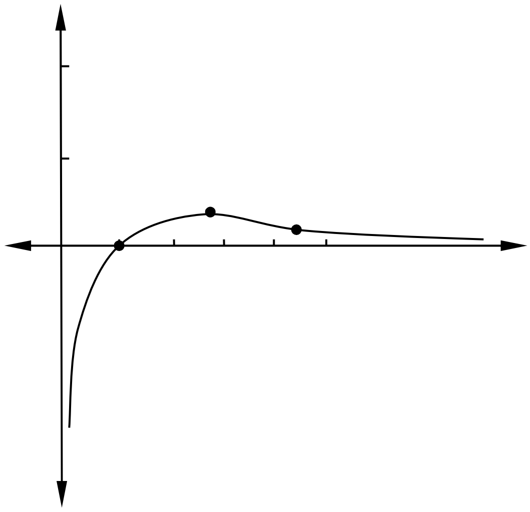


Example: Draw the function $= \frac{\ln x}{x}$.

Solution:

$D_f: x > 0$

$y' = \frac{x\left(\frac{1}{x}\right)-\ln x}{x^2} = \frac{1-\ln x}{x^2} = 0$
$\Rightarrow \ln x = 1 \;\Rightarrow x = e^1 = 2.7\,, y = 0.36$
$(2.7\,, 0.36) T\ point$



$$y'' = \frac{x^2\left(-\frac{1}{x}\right)-(1-\ln x)(2x)}{x^4} = \frac{-3x+2x\ln x}{x^4}$$
$$= \frac{2\ln x - 3}{x^4} = 0$$
$\Rightarrow \ln x = \frac{3}{2} \Rightarrow x = e^{\frac{3}{2}} \Rightarrow x = 4.48\,, y = 0.33$
$(4.48\,, 0.33) Inflection\ point$

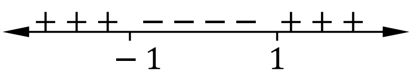
$$\lim_{x\to 0}\frac{\ln x}{x} = -\frac{\infty}{0} = \lim_{x\to 0}\frac{1}{x}\cdot\ln x = \infty(-\infty) = -\infty$$
$$\lim_{x\to\infty}\frac{\ln x}{x} = \frac{\infty}{\infty} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 0$$

$\frac{\ln x}{x} = 0 \Rightarrow \ln x = 0 \;\Rightarrow x = 1$
$(1,0) Intersection\ point\ with\ x - axis$

Example: Draw the function $y = \ln\frac{x-1}{x+1}$.

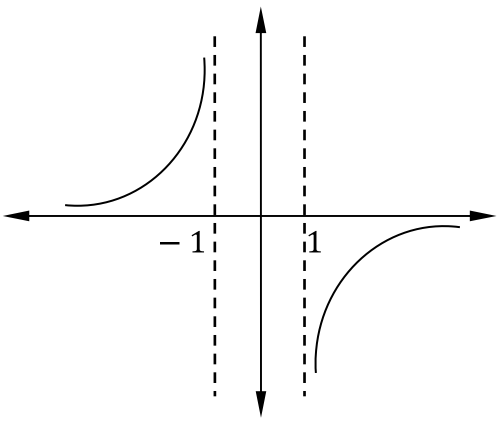
Solution:

$\frac{x-1}{x+1} > 0$
$x = 1$
$x = -1$
$D_f: (-\infty\,, -1) \cup (1\,, \infty)$



$y = \ln(x-1) - \ln(x+1)$
$$y' = \frac{1}{x-1} - \frac{1}{x+1} = \frac{x+1-(x-1)}{(x-1)(x+1)}$$
$$= \frac{2}{x^2-1} > 0\ increasing\ function$$

$$y'' = \frac{2(-1)(2x)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)} = 0 \Rightarrow x \neq 0$$
$< 0 \;\Rightarrow x > 1$
$> 0 \;\Rightarrow x < -1$



$$\lim_{x\to 1}\ln\frac{x-1}{x+1} = -\infty$$

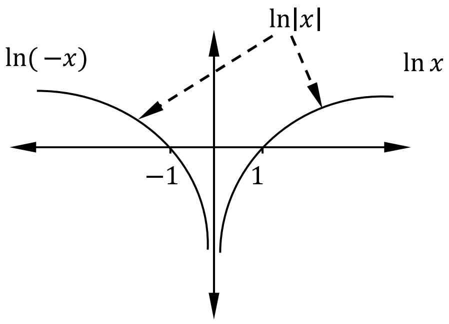
$$\lim_{x\to -1} \ln\frac{x-1}{x+1} = \infty$$

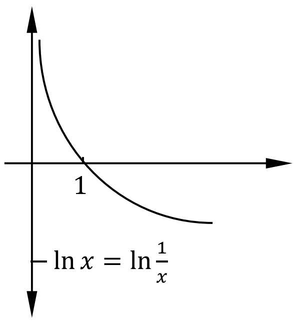
$$\lim_{x\to\infty} \ln\frac{x-1}{x+1} = \lim_{x\to\infty} \frac{\frac{x}{x}-\frac{1}{x}}{\frac{x}{x}+\frac{1}{x}} = \lim_{x\to\infty} \ln 1 = 0$$

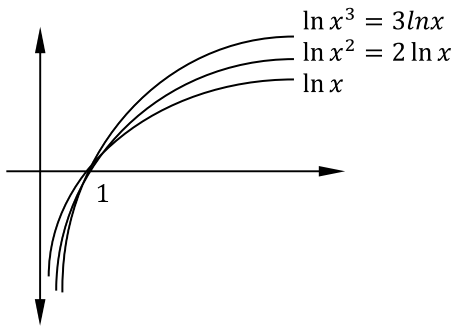
$$\lim_{x\to -\infty} \ln\frac{x-1}{x+1} = \lim_{x\to -\infty} \frac{\frac{x}{x}-\frac{1}{x}}{\frac{x}{x}+\frac{1}{x}} = \lim_{x\to -\infty} \ln 1 = 0$$

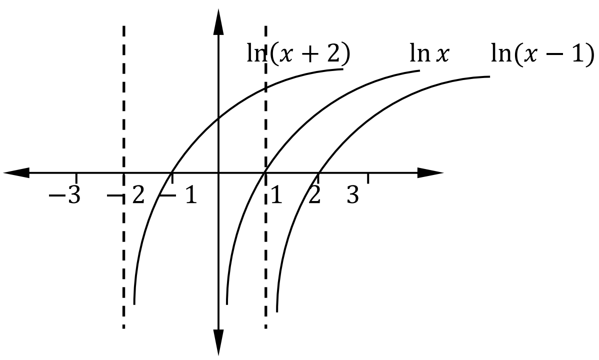
Sketch to Some Functions:

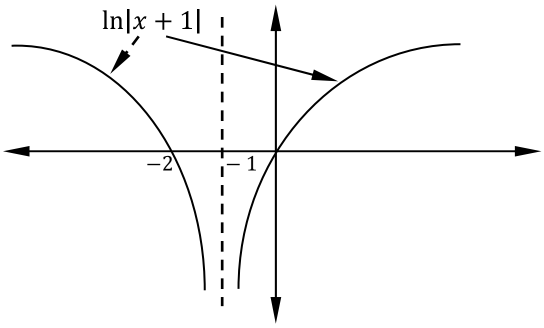
$y = \ln -x\,, -\ln x \;\;, \ln|x| \;\;, \ln x^2\,,\;\; \ln x^3\,,\;\; \ln(x-1), \ln(x+2), \ln|x+1|$











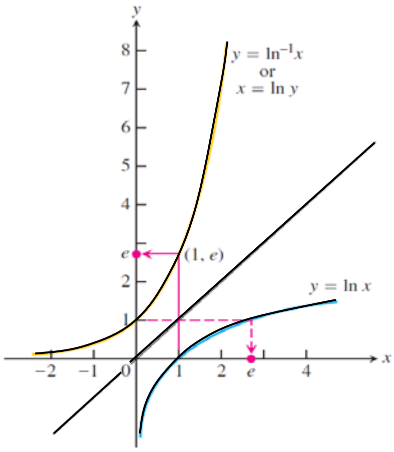
Note: $\lim_{x\to 0} \ln\frac{\sin x}{x} = 0$

The Exponential Function $\exp(x) = e^x$

The natural logarithm function ($\ln x$), is one-to-one function with domain $(0, \infty)$ and range $(-\infty, \infty)$, has an inverse whose domain is $(-\infty, \infty)$ and the range is $(0, \infty)$. We call the inverse $\exp(x)$, the exponential function of x.

$$e^x = \exp(x) = \ln^{-1} x$$

$$e^{\ln x} = x \qquad x > 0$$
$$\ln(e^x) = x \qquad all\ x\ (-\infty, \infty)$$



Example: (Using the Inverse Equations)

a) $\ln e^2 = 2$
b) $\ln \sqrt{e} = \frac{1}{2}$
c) $\ln e^{\sin x} = \sin x$
d) $e^{\ln 2} = 2$
e) $e^{3\ln 2} = e^{\ln 2^3} = 2^3$

Laws of e^x:
For all numbers x, x_1,and x_2,

1. $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$
2. $e^{-x} = \frac{1}{e^x}$
3. $\frac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$
4. $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$

Example: (Applying the Exponent Laws)

a) $e^{x+\ln x} = e^x \cdot e^{\ln 2} = 2e^x$
b) $e^{-\ln x} = \frac{1}{e^{\ln x}} = \frac{1}{x}$
c) $\frac{e^{2x}}{e} = e^{2x-1}$

The Derivative and Integral of e^x

$$y = e^x \implies x = \ln y$$
$$1 = \frac{y'}{y} \implies y' = y$$
$$\therefore \frac{d}{dx} e^x = e^x$$
and $\frac{d}{dx} e^u = e^u \cdot \frac{du}{dx}$

$$\int e^u \ du = e^u + c$$

Example: Find y' if $y = \tan^{-1} e^x + \ln\sqrt{\frac{e^{2x}+1}{e^{2x}-1}}$

Solution:

$$y = \tan^{-1} e^x + \frac{1}{2}[\ln(e^{2x}+1) - \ln(e^{2x}-1)]$$

$$y' = \frac{e^x}{(e^x)^2+1} + \frac{1}{2}\left[\frac{2\,e^{2x}}{e^{2x}+1} - \frac{2e^{2x}}{e^{2x}-1}\right]$$

$$= \frac{e^x}{e^{2x}+1} + \frac{e^{2x}}{e^{2x}+1} - \frac{e^{2x}}{e^{2x}-1}$$

Example: Find y'' if $x = \frac{1}{2}(e^t + e^{-t}),\ y = \frac{1}{2}(e^t - e^t)$.

Solution:

$$\frac{dx}{dt} = \frac{1}{2}(e^t - e^{-t}) \quad , \quad \frac{dy}{dt} = \frac{1}{2}(e^t + e^{-t})$$

$$\frac{dy}{dx} = \frac{\frac{1}{2}(e^t + e^{-t})}{\frac{1}{2}(e^t - e^{-t})} = \frac{(e^t + e^{-t})}{(e^t - e^{-t})}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{(e^t - e^{-t})(e^t - e^{-t}) - (e^t + e^{-t})(e^t + e^{-t})}{(e^t - e^{-t})^2}}{\frac{1}{2}(e^t - e^{-t})}$$

$$= \frac{e^{2t} - 1 - 1 + e^{-2t} - (e^{2t} + 1 + 1 + e^{-2t})}{\frac{1}{2}(e^t - e^{-t})^3}$$

$$= \frac{-4}{\frac{1}{2}(e^t - e^{-t})^3} = \frac{-8}{(e^t - e^{-t})^3}$$

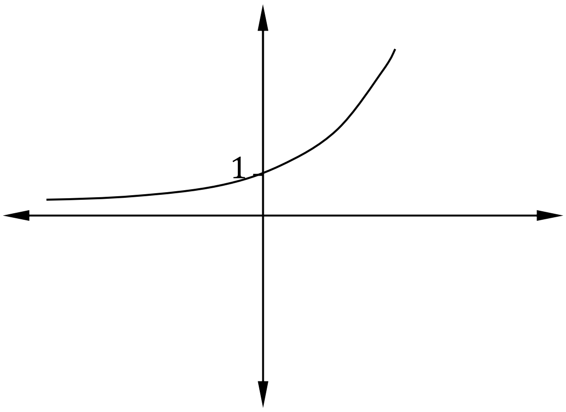
Note:$\sin^{-1} e^{-x} = \csc^{-1} e^x$

The Graph of e^x :

$D_f: (-\infty , \infty)$
$R_f: (0 , \infty)$

$x = 0 \ \Rightarrow y = e^0 = 1$

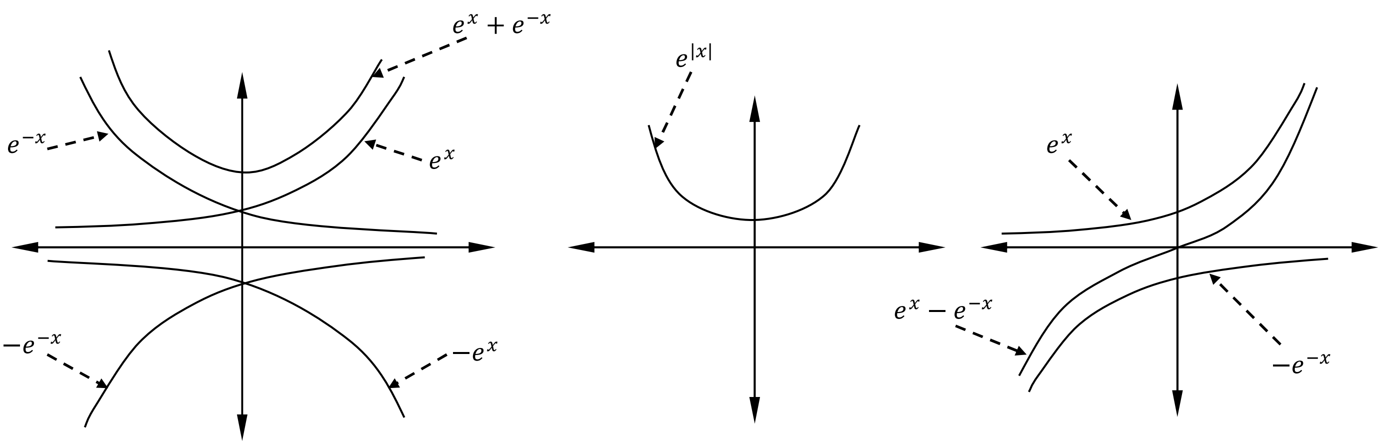
$y' = e^x > 0 \ \ increasing\ function$
$y'' = e^x > 0 \ \ concave\ up$



$\lim_{x\to\infty} e^x = e^\infty = \infty$
$\lim_{x\to-\infty} e^x = e^{-\infty} = 0$

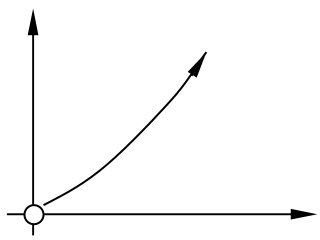
Sketch to some functions:

$y = e^x\,, \quad e^{-x}, \quad -e^x, \quad -e^{-x}, \quad e^{|x|}, \quad e^x + e^{-x}, \quad e^x - e^{-x} = (e^x + (-e^{-x}))$



Example: Draw the function $= e^{x+\ln x}$.

Solution:



$D_f: x > 0$

$y = e^{x+\ln x} = e^x \cdot e^{\ln x} = x\, e^x$
$y' = x\, e^x + e^x = e^x(x+1) > 0$ *increasing function*
$e^x > 0$ *and* $x > 0$

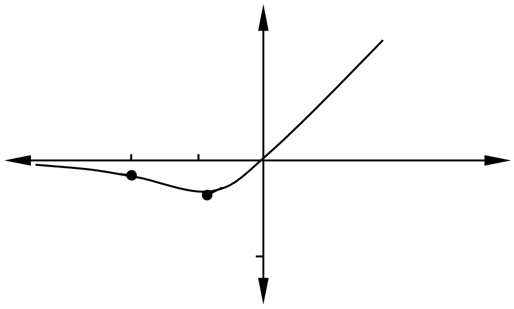
$y'' = e^x + (x+1)e^x = e^x\big(1 + (x+1)\big) = e^x(x+2) > 0$ *concave up*

$\lim_{x\to 0} x\, e^x = 0 \cdot 1 = 0$
$\lim_{x\to\infty} x\, e^x = \infty \cdot \infty = \infty$

Example: Draw the function $= x\, e^x$.

Solution:

$D_f: (-\infty, \infty)$



$y' = x\,e^x + e^x = e^x(x+1) = 0$
$e^x \neq 0 \quad or \quad x = -1, y = -0.36$
$(-1, -0.36)\ T\ point$

$y'' = e^x + e^x(x+1) = e^x(x+2) = 0$
$e^x \neq 0 \quad or \quad x = -2, y = -0.27$
$(-2, -0.27)\ Inflection\ point$

$\lim_{x\to\infty} x\,e^x = \infty\cdot\infty = \infty$

$$\lim_{x\to-\infty} x\,e^x = -\infty\cdot 0 = \lim_{x\to-\infty}\frac{x}{e^{-x}} = \frac{-\infty}{\infty} = \lim_{x\to-\infty}\frac{1}{-e^{-x}} = \frac{1}{-\infty} = 0$$

Example: Find the area enclosed by the curve $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ and the x-axis from $x = 0$ to $x = \ln 2$ (draw the function).

Solution:

$D_f: (-\infty, \infty)$

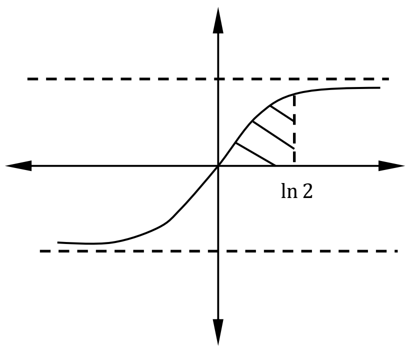
$$y' = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$
$$= \frac{e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2} > 0\ increasing\ function$$

$$y'' = \frac{4(2)(e^x + e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^4} = 0 \Longrightarrow (e^x + e^{-x})(e^x - e^{-x}) = 0$$

$e^x + e^{-x} \neq 0 \quad or \quad e^x - e^{-x} = 0 \Rightarrow e^x = e^{-x} \Rightarrow x = 0, y = 0 \Longrightarrow (0, 0) Inflection\ point$

$$\lim_{x\to\infty}\frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to\infty}\frac{e^x - e^{-x}}{e^x + e^{-x}}\cdot\frac{e^{-x}}{e^{-x}} = \lim_{x\to\infty}\frac{1 - e^{-2x}}{1 + e^{-2x}} = 1$$

$$\lim_{x\to-\infty}\frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to-\infty}\frac{e^x - e^{-x}}{e^x + e^{-x}}\cdot\frac{e^x}{e^x} = \lim_{x\to-\infty}\frac{e^{2x} - 1}{e^{2x} + 1} = -1$$



$$dA = y\,dx = \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$$

$$A = \int_0^{\ln 2}\frac{e^x - e^{-x}}{e^x + e^{-x}}dx = \ln|e^x + e^{-x}|\Big]_0^{\ln 2} = \ln\frac{5}{2} - \ln 2$$

The Function a^x

For any numbers $a > 0$ and x, the exponential function with base a is

$$a^x = e^{x \ln a}$$

Example:(Evaluating Exponential Functions)

a) $2^{\sqrt{3}} = e^{\sqrt{3}\ln 2} \approx e^{1.20} \approx 3.32$

b) $2^{\pi} = e^{\pi \ln 2} \approx e^{2.18} \approx 8.8$

Laws of Exponents:

1) $a^{x_1} \cdot a^{x_2} = a^{x_1+x_2}$
2) $a^{-x} = \frac{1}{a^x}$
3) $\frac{a^{x_1}}{a^{x_2}} = a^{x_1-x_2}$
4) $(a^{x_1})^{x_2} = (a^{x_2})^{x_1} = a^{x_1 x_2}$
5) $a^0 = 1$
6) $(a \cdot b)^x = a^x \cdot b^x$

The Derivative of a^x :

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x \ln a} = e^{x \ln a} \cdot \ln a = a^x \ln a$$

If $a > 0$ and u is a differential function of x, then a^u is a differential function of x and

$$\frac{d}{dx}a^u = a^u \cdot \ln a \cdot \frac{du}{dx}$$

These equations show why e^x is the exponential function preferred in calculus. If $a = e$, then $\ln a = 1$ and the derivative of a^x simplifies to

$$\frac{d}{dx}e^x = e^x \ln e = e^x$$

Example: (Differentiating General Exponential Functions)

a) $\frac{d}{dx}\left(\frac{1}{2}\right)^x = \frac{d}{dx}2^{-x} = 2^{-x} \cdot \ln 2 \cdot (-1) = -2^{-x} \ln 2 < 0$

b) $\frac{d}{dx} 3^{\sin x} = 3^{\sin x} \cdot \ln 3 \cdot (\cos x)$

Note: $a^{\infty} = \infty \qquad a > 1$

$a^{\infty} = 0 \qquad 0 < a < 1$

The Integral of a^x :

If $a \neq 1$ so that $\ln a \neq 0$

$$\int a^u \, du = \frac{a^u}{\ln a} + c$$

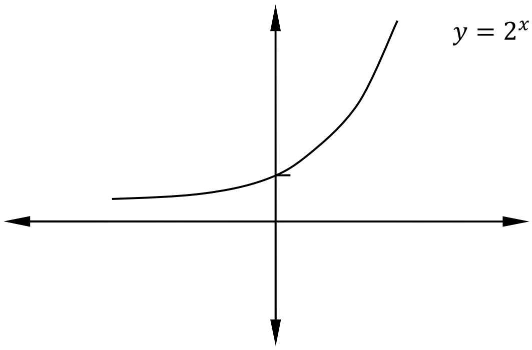
Example: Draw the functions $y = 2^x$.

Solution:

$D_f : (-\infty, \infty)$

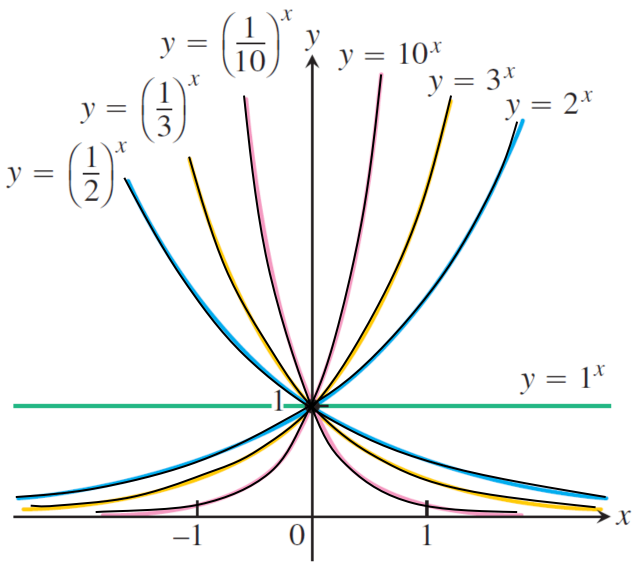
$y' = 2^x \ln 2 > 0 \quad increasing\ function$
$y'' = 2^x (\ln x)^2 > 0 \ \ concave\ up$

$\lim_{x \to \infty} 2^x = 2^{\infty} = \infty$
$\lim_{x \to -\infty} 2^x = 2^{-\infty} = 0$
$2^0 = 1$



Example: Sketch the function $y = 2^{-x}$.

$y = 2^{-x} = \left(\frac{1}{2}\right)^x$



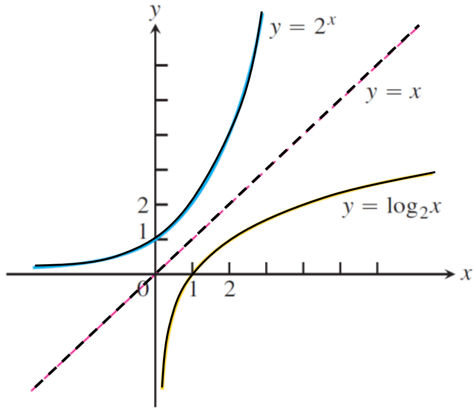
Logarithm with Basic a ($\log_a x$)

As we saw, if a is any positive number other than 1, the function a^x is one-to-one and has a nonzero derivative at every point. It therefore has a differentiable inverse. We call the inverse the **logarithm of x with base** a and denote it by **$\log_a x$.**

Definition $\log_a x$

For any positive number $a \neq 1$,
$\log_a x$ is the inverse function of a^x .

The graph of $y = \log_a x$ can be obtained by reflecting the graph of $y = a^x$ across the line $y = x$.



$\log_a x$, $D_f : (0, \infty)$
$R_f : (-\infty, \infty)$

When $a = e$, we have $\log_e x =$ inverse of $e^x = \ln x$. Since $\log_a x$ and a^x are inverses of one another, composing them in either order gives the identity function.

Inverse Equation for a^x and $\log_a x$:

$a^{\log_a x} = x \qquad x > 0$
$\log_a(a^x) = x \qquad all\ x\ (-\infty, \infty)$

Example: (Applying the Inverse Equations)

a) $\log_2(2^5) = 5$ b) $\log_{10}(10^{-7}) = -7$

c) $2^{\log_2 3} = 3$ d) $10^{\log_{10} 4} = 4$

Evaluation of $\log_a x$

$$\log_a x = \frac{\ln x}{\ln a}$$

We can derive this equation from equation $a^{\log_a x} = x$
$a^{\log_a x} = x$
$\ln a^{\log_a x} = \ln x$
$\log_a x \cdot \ln a = \ln x \implies \log_a x = \dfrac{\ln x}{\ln a}$

For example, $\log_{10} 2 = \frac{\ln 2}{\ln 10} \approx \frac{0.69315}{2.30259} \approx 0.30103$

Rules for base a logarithms

For any numbers $x > 0$ and $y > 0$,

1. $\log_a xy = \log_a x + \log_a y$
2. $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. $\log_a \frac{1}{y} = -\log_a y$
4. $\log_a x^y = y \log_a x$
5. $\log_a a = 1$
6. $\log_a 1 = 0$

Derivatives and Integrals Involving $\log_a x$

$$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}(\ln u) = \frac{1}{\ln a} \cdot \frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u}\frac{du}{dx}$$

Example:

a) $\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{\ln 10} \cdot \frac{1}{3x+10}(3) = \frac{3}{(\ln 10)(3x+1)}$

b) $\int \frac{\log_2 x}{x} dx = \frac{1}{\ln 2}\int \frac{\ln x}{x} dx = \frac{1}{\ln 2} \cdot \frac{(\ln x)^2}{2} + c$

c) $D_x\sqrt{\log_{10} x} = D_x\left(\frac{\ln x}{\ln 10}\right)^{\frac{1}{2}} = \frac{1}{2}\left(\frac{\ln x}{\ln 10}\right)^{-\frac{1}{2}} \cdot \frac{1}{x \ln 10} = \frac{1}{2}(\log_{10} x)^{-\frac{1}{2}} \cdot \frac{1}{x \ln 10}$

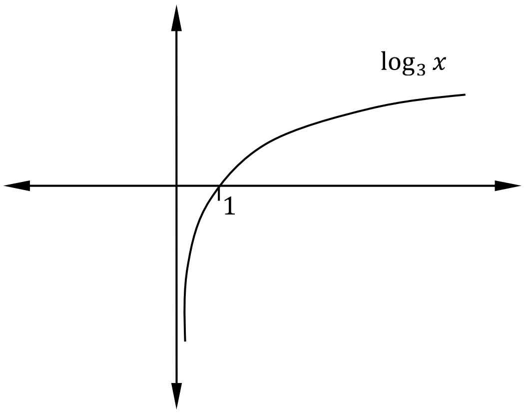
Example: Draw the function $\log_3 x$

Solution:

$D_f: (0, \infty)$, $\quad R_f: (-\infty, \infty)$

$y' = \frac{1}{x \ln 3} > 0$ *Increasing function*

$y'' = \frac{-1}{x^2 \ln 3} < 0$ *Concave down*



$$\lim_{x\to 0} \log_3 x = \lim_{x\to 0}\frac{\ln x}{\ln 3} = -\infty$$

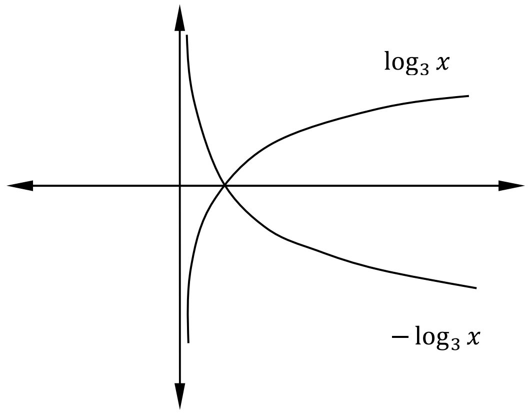
$$\lim_{x\to \infty} \log_3 x = \lim_{x\to \infty}\frac{\ln x}{\ln 3} = \infty$$

$$\log_3 x = 0 \Longrightarrow \frac{\ln x}{\ln 3} = 0 \Longrightarrow \ln x = 0 \Longrightarrow x = 1$$

Example: Sketch the function $\log_{\frac{1}{3}} x$.

Solution:

$\log_{\frac{1}{3}} x = -\log_3 x$



Example: Find the value of x to the following:

a) $\log_3 x = 5$
$$\frac{\ln x}{\ln 3} = 5 \Rightarrow \ln x = 5\ln 3 \Rightarrow \ln x = \ln 3^5 \Rightarrow x = 3^5$$

b) $\log_{0.01} 100 = x$
$$\frac{\ln 100}{\ln 0.01} = x \Rightarrow \ln 100 = x \ln 0.01 \Rightarrow \ln 100 = \ln 0.01^x$$
$$\Rightarrow 100 = 0.01^x \Rightarrow 10^2 = 10^{-2x} \Rightarrow 2 = -2x \Rightarrow x = -1$$

c) $\log_6(x+1) + \log_6 x = 1$
$$\log_6 x(x+1) = 1 \Rightarrow \frac{\ln x(x+1)}{\ln 6} = 1 \Rightarrow \ln x(x+1) = \ln 6$$
$$\Rightarrow x(x+1) = 6 \Rightarrow x^2 + x + 6 = 0 \Rightarrow (x+3)(x-2) = 0$$
$$x = -3 \notin D_f \quad or \quad x = 2$$

d) $4^{3x-1} = 5$
$$\ln 4^{3x-1} = \ln 5 \Rightarrow (3x-1)\ln 4 = \ln 5 \Rightarrow 3x\ln 4 - \ln 4 = \ln 5$$
$$3x = \frac{\ln 5 + \ln 4}{\ln 4} \Rightarrow x = \frac{\ln 5 + \ln 4}{3\ln 4}$$

e) $e^x + e^{-x} = 3$
$$(e^x + e^{-x} = 3) \quad * e^x$$
$$e^{2x} + 1 = 3\,e^x \Rightarrow e^{2x} - 3\,e^x + 1 = 0$$
$$e^x = \frac{3 \mp \sqrt{9-4}}{2} = \frac{3 \mp \sqrt{5}}{2} \Rightarrow x = \ln\left(\frac{3 \mp \sqrt{5}}{2}\right)$$

f) $\ln\left(x + \sqrt{x^2+1}\right) = 4$
$$x + \sqrt{x^2+1} = e^4 \Rightarrow \sqrt{x^2+1} = e^4 - x \Rightarrow x^2 + 1 = (e^4 - x)^2$$
$$x^2 + 1 = e^8 - 2xe^4 + x^2 \Rightarrow 2xe^4 = e^8 - 1 \Rightarrow x = \frac{e^8 - 1}{2e^4}$$

g) $\frac{2^x - 2^{-x}}{2^x + 2^{-x}} = \frac{1}{2}$

$$\frac{2^x - 2^{-x}}{2^x + 2^{-x}} \cdot \frac{2^x}{2^x} = \frac{1}{2} \implies \frac{2^{2x} - 1}{2^{2x} + 1} = \frac{1}{2} \implies 2(2^{2x} - 1) = 2^{2x} + 1$$

$$2(2^{2x}) - 2 = 2^{2x} + 1 \implies 2^{2x} = 3 \implies \ln 2^{2x} = \ln 3$$

$$\implies 2x \ln 2 = \ln 3 \implies x = \frac{\ln 3}{2 \ln 2}$$

Example: Find the inverse to the function $= \ln \frac{x-1}{x+1}$.

Solution:

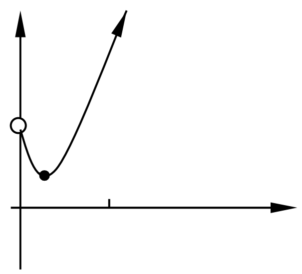
$$x = \ln \frac{x-1}{x+1} \implies e^x = \frac{y-1}{y+1} \implies e^x(y+1) = y - 1$$

$$e^x y + e^x = y - 1 \implies e^x + 1 = y - e^x y \implies y = \frac{1 + e^x}{1 - e^x}$$

The Function x^x

$$y = x^x = e^{\ln x^x} = e^{x \ln x} \qquad D_f : x > 0$$

To draw the function x^x



$$y' = e^{\ln x^x}(x \cdot \frac{1}{x} + \ln x) = e^{\ln x^x}(1 + \ln x) = 0$$

$$e^{\ln x^x} \neq 0 \quad or \quad 1 + \ln x = 0 \implies \ln x = -1 \implies x = 0.36 \, , y = 0.6$$

$$y'' = e^{\ln x^x} \cdot \frac{1}{x} + (1 + \ln x)^2 \, e^{\ln x^x} > 0 \;\; Concave \; up$$

$$\lim_{x \to \infty} e^{\ln x^x} = e^{\infty} = \infty$$

$$\lim_{x \to 0} e^{\ln x^x} =$$

$$\lim_{x \to 0} \ln x^x = \lim_{x \to 0} x \ln x = 0 \cdot -\infty = \lim_{x \to 0} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x \to 0}(-x) = 0$$

$$\implies e^0 = 1$$

Example: Find y' if $y = (\ln x^2)^{2x+3}$

Solution:

$$y = (\ln x^2)^{2x+3} = e^{\ln(\ln x^2)^{2x+3}} = e^{(2x+3)\ln(\ln x^2)}$$

$$y' = e^{(2x+3)\ln(\ln x^2)} \left[(2x+3) \cdot \frac{1}{\ln x^2} \cdot \frac{2x}{x^2} + 2 \ln(\ln x^2) \right]$$

Example: Find the tangent equation to the curve $y = (\cos x)^{\sin x}$ in (0 , 1).

Solution:

$y = (\cos x)^{\sin x} = e^{\ln(\cos x)^{\sin x}}$

$$y' = e^{\sin x \ln \cos x}\left(\sin x \cdot \frac{-\sin x}{\cos x} + \cos x \ln \cos x\right)$$

$y'@(0\,,1) = e^0(0+0) = 0$

Tangent equation $\Longrightarrow \frac{y-1}{x-0} = 0$

Note: 1^∞ , 0^0 , ∞^0 Indeterminate powers

Example: $\lim_{x\to\frac{\pi}{4}} (\sin 2x)^{\tan^2 2x} = 1^\infty$

$$= \lim_{x\to\frac{\pi}{4}} e^{\ln(\sin 2x)^{\tan^2 2x}} = \lim_{x\to\frac{\pi}{4}} e^{\tan^2(2x)\ \ln(\sin 2x)}$$

$$\lim_{x\to\frac{\pi}{4}} \tan^2(2x)\ln(\sin 2x) = \lim_{x\to\frac{\pi}{4}} \frac{\ln \sin 2x}{\cot^2 2x} = \lim_{x\to\frac{\pi}{4}} \frac{\frac{2\cos 2x}{\sin 2x}}{4\cot 2x \csc^2 2x}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{1}{2\csc^2 2x} = \frac{1}{2}$$

$$\Longrightarrow e^{\frac{1}{2}}$$

HOMEWORK

H.W (1) Find the value of x to the following:

a) $3^{\log_3 7} + 2^{\log_2 5} = 5^{\log_5 x}$

b) $x = \log_{27}\frac{1}{3} + \log_{1000} 0.01 - \log_8 2$

H.W (2) Evaluate the following integrals:

a) $\int \frac{dx}{\sqrt{e^{4x}-1}}$

b) $\int \frac{2x+3}{5-4x-x^2}\,dx$

c) $\int (\sin 3x)^4 \cos 3x\, dx$

d) $\int \frac{(\cos x)^3+1}{(\sin x)^2}\,dx$

H.W (3) Evaluate the following limits:

a) $\lim_{x\to 0} \frac{e^x+e^{-x}-x^2-2}{\sin^2 x - x^2}$

b) $\lim_{x\to 0} \frac{\ln \sin x}{\ln \tan x}$

c) $\lim_{x\to 0} x^{\ln x}$

d) $\lim_{x\to\frac{\pi}{2}} (1+\cos x)^{3\sec x}$

e) $\lim_{x\to 0} (1-x^2)^{\frac{1}{x}}$

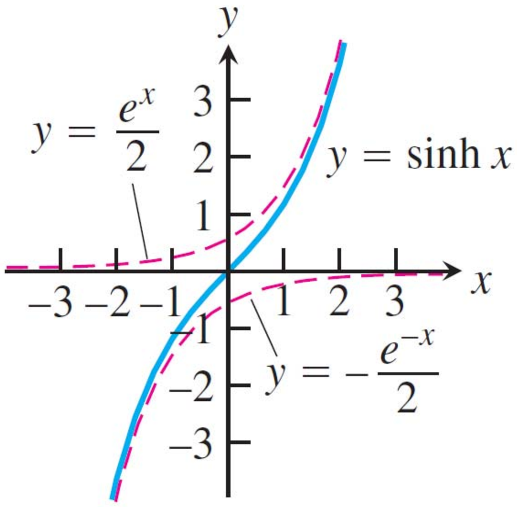
Hyperbolic Functions

The hyperbolic functions are found by taking combination of the two exponential functions e^x and e^{-x} . The hyperbolic functions simplify many mathematical expressions and they are important in application.

The six basic hyperbolic functions:

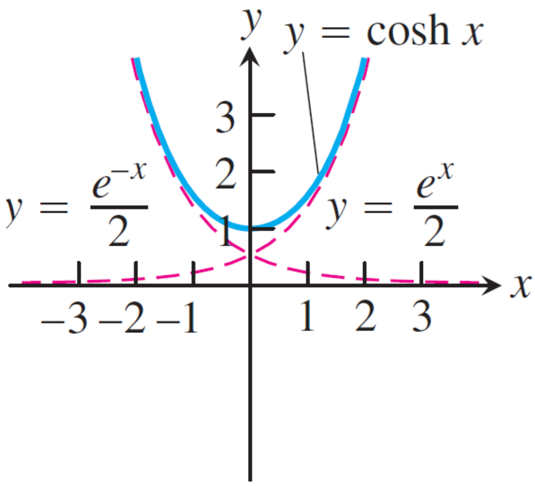
Hyperbolic sine of x:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$



Hyperbolic cosine of x:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

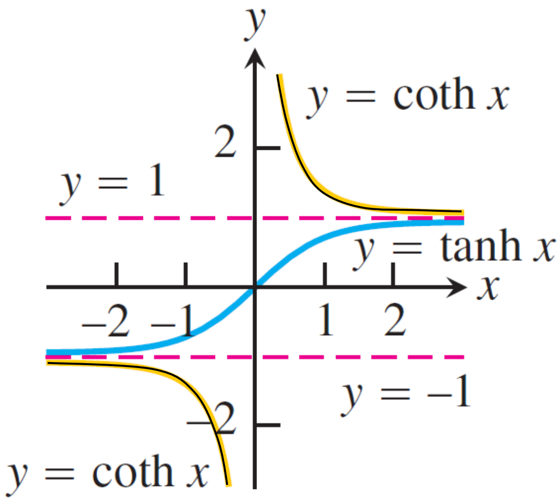


Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

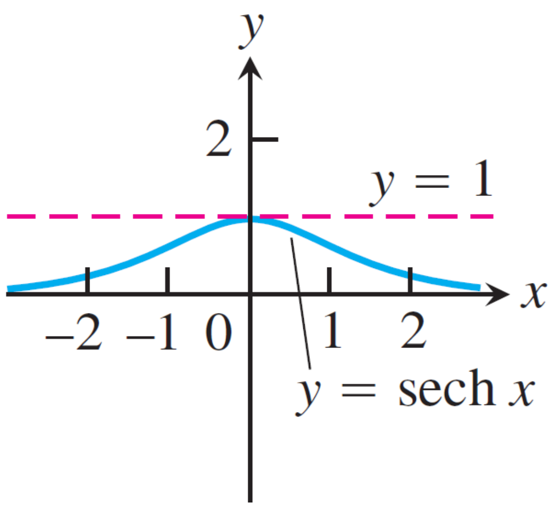
Hyperbolic cotangent

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



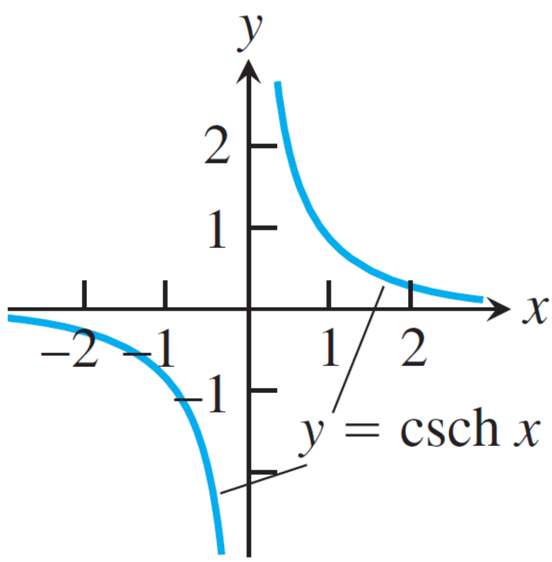
Hyperbolic secant:

$$\text{sech x} = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$



Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$



Identities for hyperbolic functions:

$\cosh^2 x - \sinh^2 x = 1$

$\cosh x + \sinh x = e^x$

$\cosh x - \sinh x = e^{-x}$

$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

$\sinh 2x = 2 \sinh x \cosh x$

$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

$\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

$\tanh^2 x = 1 - \operatorname{sech}^2 x$

$\coth^2 x = 1 + \operatorname{csch}^2 x$

The first equation is obtained as follows:

$\cosh^2 x - \sinh^2 x =$

$\left(\dfrac{e^x + e^{-x}}{2}\right)^2 - \left(\dfrac{e^x - e^{-x}}{2}\right)^2 =$

$\dfrac{1}{4}(e^{2x} + 2 + e^{-2x}) - \dfrac{1}{4}(e^{2x} - 2 + e^{-2x}) = 1$

And the fourth equation is obtained as follows:

$$\sinh(x + y) = \frac{e^{x+y} - e^{-(x+y)}}{2} = \frac{e^x \cdot e^y - e^{-x} \cdot e^{-y}}{2}$$

$= \dfrac{1}{2}[(\cosh x + \sinh x) \cdot (\cosh y + \sinh y) - (\cosh x - \sinh x) \cdot (\cosh y - \sinh y)]$

$= \dfrac{1}{2}(\cosh x \cosh y + \cosh x \sinh y + \sinh x \cosh y + \sinh x \sinh y$

$- \cosh x \cosh y + \cosh x \sinh y + \sinh x \cosh y - \sinh x \sinh y) = \sinh x \cosh y + \cosh x \sinh y$

The others are obtain similarly

Derivatives of hyperbolic functions:

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

The derivative formulas are derived from the derivative of e^u :

$$\frac{d}{dx}(\sinh u) = \frac{d}{dx}\left(\frac{e^u - e^{-u}}{2}\right) = \frac{e^u \frac{du}{dx} + e^{-u}\frac{du}{dx}}{2} = \frac{e^u + e^{-u}}{2}\frac{du}{dx} = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = \frac{d}{dx}\left(\frac{1}{\sinh u}\right) = -\frac{\cosh u}{\sinh^2 u}\frac{du}{dx} = -\frac{1}{\sinh u}\frac{\cosh u}{\sinh u}\frac{du}{dx} = -\operatorname{csch} u \coth u \frac{du}{dx}$$

The others are obtain similarly.

Integral formula for hyperbolic functions:

$$\int \sinh u \; du = \cosh u + c$$

$$\int \cosh u \; du = \sinh u + c$$

$$\int \operatorname{sech}^2 u \; du = \tanh u + c$$

$$\int \operatorname{csch}^2 u \; du = -\coth u + c$$

$$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + c$$

Example:

$$\frac{d}{dt}(\tanh\sqrt{1+t^2}\,) = \frac{t}{\sqrt{1+t^2}}\operatorname{sech}^2\sqrt{1+t^2}$$

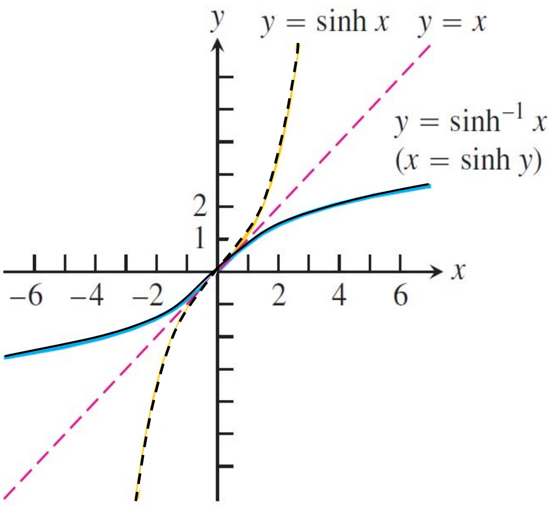
H.W. Draw the following function: (using the graphing strategy)

a) $y = \tanh x$ b) $y = \coth x$ c) $y = \operatorname{sech} x$ d) $y = \operatorname{csch} x$

Inverse Hyperbolic Functions

$y = \sinh^{-1} x \qquad -\infty \le x \le \infty$

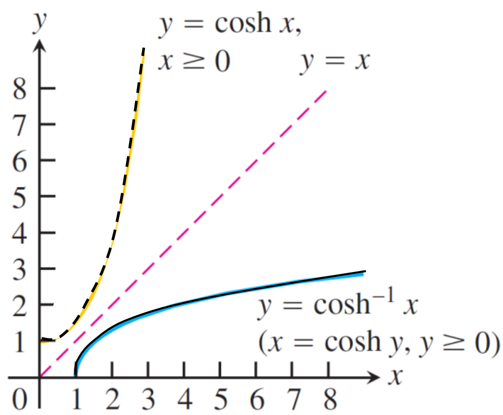
$y = \sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$



$y = \cosh^{-1} x \qquad x \ge 1$

$y = \cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$

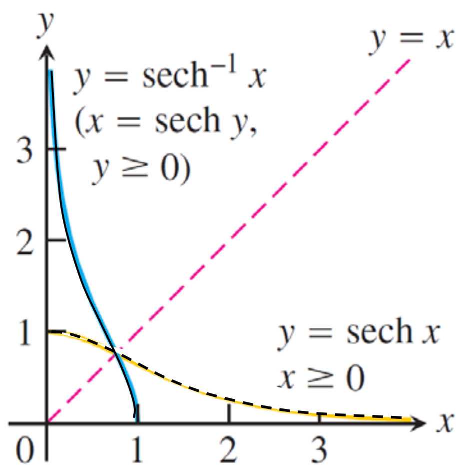
The function $y = \cosh x$ is not one-to-one.
The restricted function $y = \cosh x$, $x \ge 0$, however, is one-to-one function



$y = \text{sech}^{-1} x \qquad\qquad 0 < x \le 1$

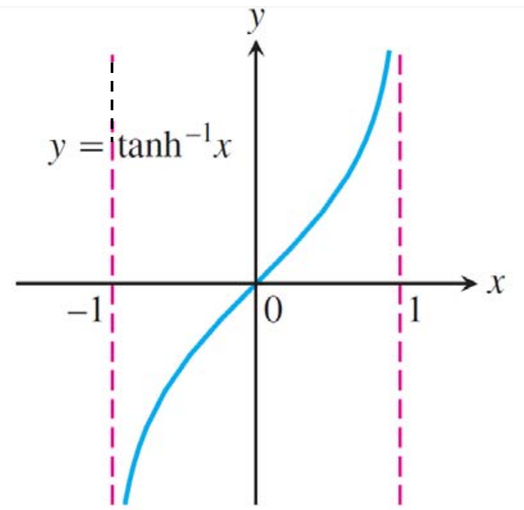
$y = \text{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$

Like $y = \cosh x$, the function $y = \text{sech}\, x = \frac{1}{\cosh x}$ fails to be one-to-one, but its restricted to nonnegative values of x does have an inverse.



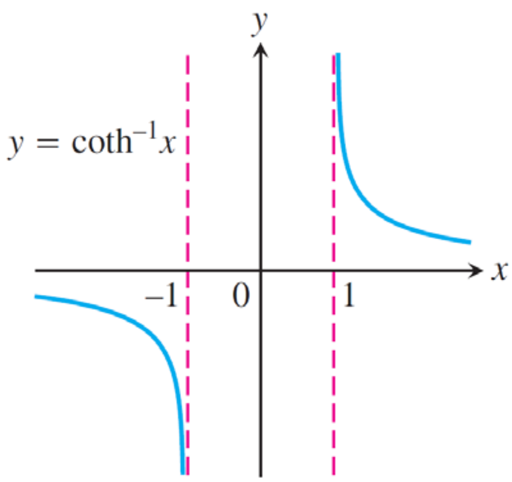
$y = \tanh^{-1} x \qquad\qquad |x| < 1$

$$y = \tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$$



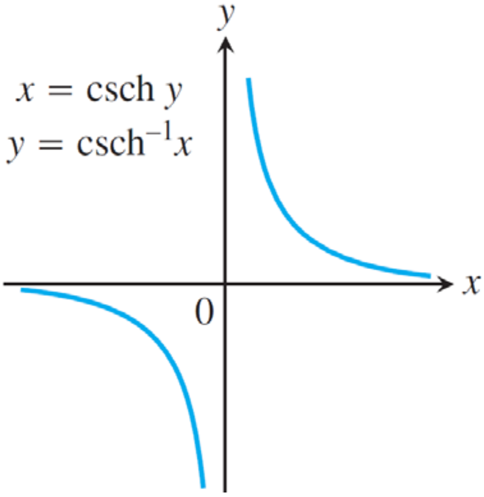
$y = \coth^{-1} x \qquad\qquad |x| > 1$

$$y = \coth^{-1} x = \frac{1}{2}\ln\frac{x+1}{x-1}$$



$y = \operatorname{csch}^{-1} x \qquad\qquad x \neq 0$

$y = \operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$



Example: Prove that $\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$.

Solution:

$$y = \tanh^{-1} x \implies x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$\frac{e^y - e^{-y}}{e^y + e^{-y}} \cdot \frac{e^y}{e^y} = \frac{e^{2y} - 1}{e^{2y} + 1} \implies x = \frac{e^{2y} - 1}{e^{2y} + 1} \implies xe^{2y} + x = e^{2y} - 1 \implies e^{2y} - xe^{2y} = x + 1$$

$$e^{2y} = \frac{1+x}{1-x} \implies 2y = \ln\frac{1+x}{1-x} \implies y = \frac{1}{2}\ln\frac{1+x}{1-x}$$

Identities for inverse hyperbolic functions:

$$\operatorname{sech}^{-1} x = \cosh^{-1}\frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1}\frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1}\frac{1}{x}$$

Derivatives:

$$\frac{d}{dx}\sinh^{-1} u = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\cosh^{-1} u = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx} \qquad u > 1$$

$$\frac{d}{dx}\tanh^{-1} u = \frac{1}{1-u^2}\frac{du}{dx} \qquad |u| < 1$$

$$\frac{d}{dx}\coth^{-1} u = \frac{1}{1-u^2}\frac{du}{dx} \qquad |u| > 1$$

$$\frac{d}{dx}\operatorname{sech}^{-1} u = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx} \qquad 0 < u < 1$$

$$\frac{d}{dx}\operatorname{csch}^{-1} u = \frac{-1}{|u|\sqrt{1+u^2}}\frac{du}{dx} \qquad u \neq 0$$

Example: Prove that $\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$.

Solution:

Let $y = \cosh^{-1} x$

$x = \cosh y$

$1 = \sinh y \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\sinh x} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2-1}}$

Example: Find $\frac{dy}{dx}$ to the function $y = \sinh^{-1}(\tan x) + \coth^{-1}\sqrt{x} + \operatorname{csch}^{-1} 2^x$.

Solution:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+\tan^2 x}} \cdot \sec^2 x + \frac{1}{1-x} \cdot \frac{1}{2\sqrt{x}} - \frac{1}{|2^x|\sqrt{1+(2^x)^2}} \cdot 2^x \cdot \ln 2$$

Integrals leading to inverse hyperbolic functions:

$$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + c\,, \qquad a > 0$$

$$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + c\,, \qquad u > a > 0$$

$$\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + c & if\ u^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + c & if\ u^2 > a^2 \end{cases}$$

$$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + c\,, \qquad 0 < u < a$$

$$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + c\,, \qquad u \neq 0\ and\ a > 0$$

6 INTEGRATION

Integration:

1. Indefinite Integrals
2. Definite Integrals

Indefinite Integrals:

If the function $f(x)$ is a derivative, then the set of all anti derivatives of f is called the indefinite integral of f ,denoted by the symbol

$$\int f(x)dx$$

As indefinite integrals , the symbol $\int$ is called an integral sign.
The function f is integrand of the integral and x is the variable of integration.

$$\int f(x)dx = F(x) + c$$

$$\frac{d}{dx}(F(x) + c) = F'(x) = f(x)$$

Summary of Basic Integration Formulas and Techniques

Basic Rules

1. $\int du = u + c$
2. $\int kdu = ku + c$
3. $\int (du + dv) = \int du + \int dv$

Power Functions

4. $\int u^n \, du = \dfrac{u^{n+1}}{n+1} + c \quad (n \neq -1)$
5. $\int \dfrac{du}{u} = \ln|u| + c$

Trigonometric Functions

6. $\int \sin u \, du = -\cos u + c$

7. $\int \cos u \, du = \sin u + c$

8. $\int \cos^2 u \, du = \tan u + c$

9. $\int \csc^2 u \, du = -\cot u + c$

10. $\int \sec u \tan u \, du = \sec u + c$

11. $\int \csc u \cot u \, du = -\csc u + c$

12. $\int \tan u \, du = \int \frac{\sin x}{\cos x} dx = -\ln|\cos u| + c \ = \ln|\sec u| + c$

13. $\int \cot u \, du = \int \frac{\cos x}{\sin x} dx = \ln|\sin u| + c = -\ln|\csc u| + c$

14. $\int \sec u \ \ du = \ln|\sec u + \tan u| + c$

$$because: \int \sec u \ \ du = \int \sec u \, . \frac{\sec u + \tan u}{\sec u + \tan u} dx = \int \frac{\sec^2 u + \sec u \ tan \ u}{\sec u + \tan u} dx = \ln|\sec u + \tan u| + c$$

15. $\int \csc u \ \ du = -\ln|\csc u + \cot u| + c$

16. $\int \sin ax \cos bx \ \ dx = \frac{1}{2} \int (\sin(a-b)x + \sin(a+b)\,x) \, dx$

17. $\int \cos ax \cos bx \ \ dx = \frac{1}{2} \int (\cos(a-b)x + \cos(a+b)\,x) \, dx$

18. $\int \sin ax \sin bx \ \ dx = \frac{1}{2} \int (cos(a-b)x - cos \ \ (a+b)\,x) \, dx$

Rational Functions

29. $\int \frac{du}{\sqrt{a^2 - \ u^2}} = sin^{-1}\left(\frac{u}{a}\right) + c$

$$because: \int \frac{du}{\sqrt{a^2 - u^2}} = \int \frac{du}{a\sqrt{1 - \left(\frac{u}{a}\right)^2}} = \int \frac{\frac{1}{a} du}{\sqrt{1 - \left(\frac{u}{a}\right)^2}} = \sin^{-1}\frac{u}{a} + c$$

30. $\int \frac{du}{a^2+u^2} = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{u}{a}\right) + c$

$$because: \int \frac{du}{a^2+u^2} = \int \frac{du}{a^2(1+\left(\frac{u}{a}\right)^2)} = \frac{1}{a}\int \frac{\frac{1}{a}du}{1+\left(\frac{u}{a}\right)^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c$$

31. $\int \frac{du}{u\sqrt{u^2-a^2}} = \left(\frac{1}{a}\right)\sec^{-1}\left|\frac{u}{a}\right| + c$

$$because: \int \frac{du}{u\sqrt{u^2-a^2}} = \int \frac{du}{ua\sqrt{\left(\frac{u}{a}\right)^2-1}} = \frac{a}{a}\int \frac{\frac{1}{a}du}{u\sqrt{\left(\frac{u}{a}\right)^2-1}} = \frac{1}{a}\int \frac{\frac{1}{a}du}{\frac{u}{a}\sqrt{\left(\frac{u}{a}\right)^2-1}}$$

$$= \frac{1}{a}\sec^{-1}|u/a| + c$$

32. $\int \frac{du}{\sqrt{a^2+u^2}} = sinh^{-1}\left(\frac{u}{a}\right) + c\,, \quad a > 0$

33. $\int \frac{du}{\sqrt{u^2-a^2}} = cosh^{-1}\left(\frac{u}{a}\right) + c\,,$

$u > a > 0$

34. $\int \frac{du}{a^2-u^2} = \left(\frac{1}{a}\right)tanh^{-1}\left(\frac{u}{a}\right) + c,$

$if\ u^2 < a^2$

35. $\int \frac{du}{u\sqrt{a^2-u^2}} = -\left(\frac{1}{a}\right)sech^{-1}\left(\frac{u}{a}\right) + c\,,$

$0 < u < a$

36. $\int \frac{du}{u\sqrt{a^2+u^2}} = -\left(\frac{1}{a}\right)csch^{-1}\left|\frac{u}{a}\right| + c\,,$

$u \neq 0\ and\ a > 0$

Integration Factors

37. $\int \frac{dx}{1-cos\,x}$

Multiply the integral by the following factor $\frac{1+\cos x}{1+\cos x}$

38. $\int \frac{dx}{1+sin\,x}$

Multiply the integral by the following factor $\frac{1-sin\,x}{1-sin\,x}$

39. $\int \frac{dx}{1-sin\,x}$

Multiply the integral by the following factor $\frac{1+\sin x}{1+\sin x}$

40. $\int \sqrt{1+sin\,x}\;dx$

Multiply the integral by the following factor $\frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}$

41. $\int \sqrt{1-cos\,x}\;dx$

Multiply the integral by the following factor $\frac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}}$

42. $\int \sqrt{1+cos\,x}\;dx$

Multiply the integral by the following factor $\frac{\sqrt{1-\cos x}}{\sqrt{1-\cos x}}$

Trigonometric Substitutions

43. $\int (a^2-u^2)^n\,du \Longrightarrow$ Let $u = a\sin\theta$

44. $\int (a^2+u^2)^n\,du \Longrightarrow$ $Let\ u = a\tan\theta$

45. $\int (u^2-a^2)^n\;du \Longrightarrow$ Let $u = a\sec\theta$

EXAMPLES

Example 1(Making a Simplifying Substitution): Evaluate the integral

$$\int \frac{2x-9}{\sqrt{x^2-9x+1}}dx$$

Solution

$$\int \frac{2x-9}{\sqrt{x^2-9x+1}}dx = \int \frac{du}{\sqrt{u}} = \int u^{-\frac{1}{2}}\,du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c = 2u^{\frac{1}{2}} + c = 2(x^2-9x+1)^{\frac{1}{2}} + c$$

$let\ u = x^2-9x+1,\ \ du = 2x-9$

Example 2: Evaluate the integral

$$\int \sqrt{2x+1}\,dx$$

Solution

$$\int \sqrt{2x+1}\,dx = \int u^{\frac{1}{2}} \cdot \frac{du}{2} = \frac{1}{2} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{1}{3}(2x+1)^{\frac{3}{2}} + c$$

$let\ u = 2x+1, \quad du = 2dx \Rightarrow dx = \frac{du}{2}$

Example 3: Evaluate the following integrals (Trigonometric Functions)

$a. \int \cos 2x\,dx = \frac{1}{2}\sin 2x + c$

$b. \int x \sin\frac{x^2}{2}\,dx = -\cos\frac{x^2}{2} + c$

$c. \int \frac{\sec^2\sqrt{x}}{\sqrt{x}}\,dx = 2\tan\sqrt{x} + c$

$d. \int \sin x \cos^2 x\,dx = \frac{-\cos^3 x}{3} + c$

$e. \int \frac{\cos x}{\sqrt{\sin x}}\,dx = \frac{\sin^{\frac{1}{2}} x}{\frac{1}{2}} + c = 2\sqrt{\sin x} + c$

$f. \int \frac{\cos^3 x\,dx}{(\sin x)^{\frac{1}{2}}} = \int \frac{(1-\sin^2 x)\cos x}{(\sin x)^{\frac{1}{2}}} = \int \cos x\,(\sin x)^{-\frac{1}{2}}\,dx - \int \cos x\,(sinx)^{\frac{3}{2}}\,dx$

$$= \frac{(\sin x)^{\frac{1}{2}}}{\frac{1}{2}} - \frac{(\sin x)^{\frac{5}{2}}}{\frac{5}{2}} + c$$

Example 4: Evaluate the integral

$$\int \frac{dx}{\sqrt{1-x^2}}$$

Solution

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$$

Example 5: Evaluate the integral

$$\int \frac{dx}{\sqrt{1-4x^2}}$$

Solution

$$\int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\int \frac{2\,dx}{\sqrt{1-(2x)^2}} = \frac{1}{2}\sin^{-1} 2x + c$$

Example 6: Evaluate the integral

$$\int \frac{x\,dx}{\sqrt{1-4x^2}}$$

Solution

$$\int \frac{x\,dx}{\sqrt{1-4x^2}} = -\frac{1}{8}\int \frac{-8x\,dx}{\sqrt{1-4x^2}} = -\frac{1}{8}\cdot\frac{(1-4x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

Example 7: Evaluate the integral

$$\int \frac{x\,dx}{\sqrt{4-9x^4}}$$

Solution

$$\int \frac{x\,dx}{\sqrt{4-9x^4}} = \frac{1}{6}\int \frac{6x\,dx}{\sqrt{4-(3x^2)^2}} = \frac{1}{6}\sin^{-1}\frac{3x^2}{2} + c$$

Example 8: Evaluate the integral

$$\int \frac{x^3\,dx}{\sqrt{4-9x^4}}$$

Solution

$$\int \frac{x^3\,dx}{\sqrt{4-9x^4}} = -\frac{1}{36}\int \frac{-36\,x^3\,dx}{(4-9x^4)^{\frac{1}{2}}} = -\frac{1}{36}\cdot\frac{(4-9x^4)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

Example 9: Evaluate the integral

$$\int \frac{dx}{9+4x^2}$$

Solution

$$\int \frac{dx}{9+4x^2} = \frac{1}{2}\int \frac{2dx}{9+(2x)^2} = \frac{1}{6}\tan^{-1}\frac{2x}{3} + c$$

Example 10: Evaluate the integral

$$\int \frac{dx}{(2x-1)\sqrt{(2x-1)^2-4}}$$

Solution

$$\int \frac{dx}{(2x-1)\sqrt{(2x-1)^2-4}} = \frac{1}{4}\sec^{-1}\left|\frac{2x-1}{2}\right| + c$$

Example 11: Evaluate the integral

$$\int \frac{\sin t \cos t}{\sqrt{3\sin t+5}}dx$$

Solution

$$\int \frac{\sin t\cos t}{\sqrt{3\sin t+5}}dt = \int \frac{\frac{u-5}{3}\cdot\frac{du}{3}}{\sqrt{u}} = \frac{1}{9}\int u^{-\frac{1}{2}}(u-5)du = \frac{1}{9}\int (u^{\frac{1}{2}} - 5u^{-\frac{1}{2}})\,du$$

$$let\ u = 3\sin t + 5 \ \Rightarrow sint = \frac{u-5}{3} \quad , \ du = 3costdt \Rightarrow \cos t\ dt = \frac{du}{3}$$

$$= \frac{1}{9}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{5u^{\frac{1}{2}}}{\frac{1}{2}}\right) + c = \frac{2}{27}(3sint+5)^{\frac{3}{2}} - \frac{10}{9}(3sint+5)^{\frac{1}{2}} + c$$

Example 12: Evaluate the integral

$$\int \frac{\tan 2x}{\sqrt{\sec^2 2x - 4}}dx$$

Solution

$$\int \frac{\tan 2x}{\sqrt{\sec^2 2x-4}}dx = \int \frac{\sec 2x\tan 2x}{\sec 2x\sqrt{\sec^2 2x-4}}dx$$

$$letu = \sec 2x\ , \quad du = 2\sec 2x\tan 2x\ dx$$

$$= \frac{1}{2}\int \frac{du}{u\sqrt{u^2-4}} = \frac{1}{2}\cdot\frac{1}{2}\sec^{-1}\left|\frac{u}{2}\right| + c = \frac{1}{4}\sec^{-1}\left|\frac{\sec 2x}{2}\right| + c$$

Example 13: Evaluate the integral

$$\int \frac{dx}{(1+x^2)(1+(\tan^{-1}x)^2)}$$

Solution

$$let\ u = \tan^{-1}x,\ \ du = \frac{dx}{1+x^2}$$

$$\int \frac{dx}{(1+x^2)(1+(\tan^{-1}x)^2)} = \int \frac{du}{1+u^2} = \tan^{-1}u + c = \tan^{-1}(\tan^{-1}x) + c$$

Example 14: Evaluate the integral

$$\int \frac{\sin x}{16+\cos^2 x}\,dx$$

Solution

$let\ u = \cos x,\ \ du = -\sin x\ dx$

$$\int \frac{\sin x}{16+\cos^2 x}\,dx = \int \frac{-du}{16+u^2} = -\frac{1}{4}\tan^{-1}\left(\frac{u}{4}\right) + c = -\frac{1}{4}\tan^{-1}\left(\frac{cos\ x}{4}\right) + \boldsymbol{c}$$

Example 15(Completing the Square): Evaluate the integral

$$\int \frac{dx}{\sqrt{8x-x^2}}$$

Solution

$$8x - x^2 = -(x^2-8x) = -(x^2-8x+16-16) = -(x^2-8x+16)+16 = 16-(x-4)^2$$

$$\int \frac{dx}{\sqrt{8x-x^2}} = \int \frac{dx}{\sqrt{16-(x-4)^2}} = \int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + c = \sin^{-1}\left(\frac{x-4}{4}\right) + c$$

Example 16: Evaluate the integral

$$\int \frac{dx}{x^2+2x+5}$$

Solution

$x^2+2x+5 = x^2+2x+1+4 = (x+1)^2+4$

$$\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{(x+1)^2+4} = \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c$$

Example 17(Expanding a Power and Using a Trigonometric Identity): Evaluate the integral

$$\int (\sec x + \tan x)^2\, dx$$

Solution

$\int (\sec x + \tan x)^2\, dx = \int (\sec^2 x + 2\sec x \tan x + \tan^2 x)\, dx$

$= \int \sec^2 x\, dx + 2\int \sec x \tan x\, dx + \int \tan^2 x\, dx$

$= \int \sec^2 x\, dx + 2\int \sec x \tan x\, dx + \int (\sec^2 x - 1) dx$

$= tanx + 2\sec x + tanx - x + c = 2\tan x + 2\sec x - x + c$

Example 18: Evaluate the integral

$$\int \sin^2\left(\frac{x}{2}\right) dx$$

Solution

$\int \sin^2\left(\frac{x}{2}\right) dx = \frac{1}{2}\int (1 - \cos x)\, dx = \frac{1}{2}(x - \sin x) + c$

Example 19: Evaluate the integral

$$\int \cos^4 2x\ \ dx$$

Solution

$\int \cos^4 2x\ \ dx = \int \left(\frac{1 + \cos 4x}{2}\right)^2 dx = \frac{1}{4}\int (1 + 2\cos 4x + \cos^2 4x)\, dx$

$= \frac{1}{4}\left(x + \frac{1}{2} sin4x + \frac{1}{2}\int (1 + \cos 8x)\, dx\right) = \frac{x}{4} + \frac{1}{8}\sin 4x + \frac{x}{8} + \frac{1}{64}\sin 8x + c$

Example 20: Evaluate the integral

$$\int \sin^3 2x\ \ dx$$

Solution

$= \int (1 - \cos^2 2x)\sin 2x\, dx = \int \sin 2x\, dx - \int \cos^2 2x \sin 2x\ \ dx = \frac{1}{2}\cos 2x + \frac{1}{2}\cdot\frac{\cos^3 2x}{2} + c$

Example 21: Evaluate the integral

$$\int \sec^4\left(\frac{x}{2}\right)\ dx$$

Solution

$$= \int (1 + \tan^2 \frac{x}{2}) \sec^2 \frac{x}{2} dx = \int \sec^2 \frac{x}{2} dx + \int \tan^2 \frac{x}{2} \sec^2 \frac{x}{2} dx = 2 \tan \frac{x}{2} + 2 \cdot \frac{\tan^3 \frac{x}{2}}{3} + c$$

Products of Sines and Cosines

The integrals

$$\int \sin ax \cos bx \ \ dx \qquad \int \cos ax \cos bx \ \ dx \qquad \int \sin ax \sin bx \ \ dx$$

arise in many places where trigonometric functions are applied to problems in mathematics and science. We can evaluate these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the identities

$$\int \sin ax \cos bx \ \ dx = \frac{1}{2} \int (\sin(a-b)x + \sin(a+b)\, x)\, dx$$

$$\int \cos ax \cos bx \ \ dx = \frac{1}{2} \int (\cos(a-b)x + \cos(a+b)\, x)\, dx$$

$$\int \sin ax \sin bx \ \ dx = \frac{1}{2} \int (cos(a-b)x - cos \ \ (a+b)\, x)\, dx$$

Example 22: Evaluate the integral

$$\int \sin\left(\frac{x}{2}\right) \cos x \, dx$$

Solution

$$= \frac{1}{2}\left(\int \sin\left(\frac{1}{2} - 1\right) x \, dx + \int \sin\left(\frac{1}{2} + 1\right) x \, dx\right) = \frac{1}{2} \int \sin\left(-\frac{x}{2}\right) dx + \frac{1}{2} \int \sin\left(\frac{3x}{2}\right) dx$$

$$= \cos\left(\frac{x}{2}\right) - \frac{1}{2} \cdot \frac{2}{3} \cos\left(\frac{3x}{2}\right) + c$$

Example 23: Evaluate the integral

$$\int \frac{dx}{1 + \cos x}$$

Solution

$$\int \frac{dx}{1 + \cos x} = \int \frac{1}{1 + \cos x} \cdot \frac{1 - \cos x}{1 - \cos x} dx = \int \frac{1 - \cos x}{\sin^2 x} dx = \int \frac{1}{\sin^2 x} - \int \frac{cos\ x}{\sin^2 x} dx$$

$$= \int \csc^2 x - \int \cot x \csc x = -\cot x + \csc x + c$$

Example 24: Evaluate the integral

$$\int \sqrt{1-\sin x}\ dx$$

Solution

$$\int \sqrt{1-\sin x}\cdot\frac{\sqrt{1+\sin x}}{\sqrt{1+\sin x}}dx = \int \frac{\cos x}{(1+\sin x)^{\frac{1}{2}}}dx = \frac{(1+\sin x)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

Example25 : Evaluate the integral

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx$$

Solution

To eliminate the square root try to use the following identities

$$\cos^2 x = \frac{1+\cos 2x}{2} \Rightarrow 2\cos^2 x = 1+\cos 2x$$

Then

$$1+\cos 4x = 2\cos^2 2x$$

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\, dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx = \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}$$

Example 26(Reducing an Improper Fraction): Evaluate the integral

$$\int \frac{3x^2-7x}{3x+2}dx$$

Solution

$$\frac{3x^2-7x}{3x+2} = x-3+\frac{6}{3x+2}$$

$$\int \frac{3x^2-7x}{3x+2}dx = \int \left(x-3+\frac{6}{3x+2}\right)dx$$

$$= \frac{x^2}{2} - 3x + 2\ln|3x+2| + c$$

$x-3$

$3x+2\ \big)\ 3x^2-7x$

$3x^2+2x$

$-9x$

$-9x-6$

6

Example 27: Evaluate the integral

x^2-1

$x^2+4\ \big)\ x^4+3x^2-5$

x^4+4x^2

$-x^2-5$

$$\int \frac{x^4+3x^2-5}{x^2+4}\,dx$$

Solution

$$\int \frac{x^4+3x^2-5}{x^2+4}\,dx = \int \left(x^2-1+\frac{-1}{x^2+4}\right)dx$$

$$= \frac{x^3}{3} - x - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$$

Example 28(Separating a Fraction) : Evaluate the integral

$$\int \frac{3x+2}{\sqrt{1-x^2}}\,dx$$

Solution

$$\int \frac{3x+2}{\sqrt{1-x^2}}\,dx = 3\int \frac{x}{\sqrt{1-x^2}}\,dx + 2\int \frac{dx}{\sqrt{1-x^2}} = -3\frac{\sqrt{1-x^2}}{2\cdot\frac{1}{2}} + 2\sin^{-1}x + c$$

Trigonometric Substitutions:

1) $a^2-u^2 \Longrightarrow \ let\ u = a\sin\theta$
2) $a^2+u^2 \Longrightarrow \ let\ u = a\tan\theta$
3) $u^2-a^2 \Longrightarrow \ let\ u = a\sec\theta$

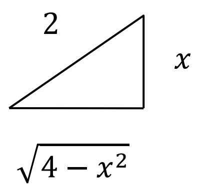
Example 29: Evaluate the integral

$$\int \frac{dx}{(4-x^2)^{\frac{3}{2}}}\,.$$

Solution

$let\ x = 2\ sin\theta$ $\qquad \sin\theta = \frac{x}{2}$

$dx = 2\cos\theta\, d\theta$



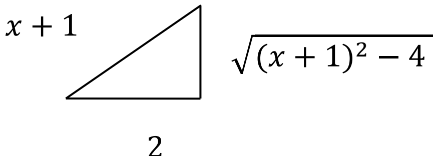
$$= \int \frac{2\cos\theta}{(4-4\sin^2\theta)^{\frac{3}{2}}}\,d\theta = \frac{2}{8}\int \frac{\cos\theta}{\cos^3\theta}\,d\theta = \frac{1}{4}\int \sec^2\theta\, d\theta = \frac{1}{4}\tan\theta + c = \frac{1}{4}\cdot\frac{x}{\sqrt{4-x^2}} + c$$

Example 30: Evaluate the integral

$$\int \frac{dx}{(x^2+2x-3)^{\frac{3}{2}}}.$$

Solution

$x^2+2x-3 = x^2+2x+1-1-3 = (x+1)^2-4$



$let\ x+1 = 2\sec\theta$

$dx = 2\sec\theta\tan\theta$

$$= \int \frac{dx}{((x+1)^2-4)^{\frac{3}{2}}} = \int \frac{2\sec\theta\tan\theta\, d\theta}{(4\sec^2\theta-4)^{\frac{3}{2}}} = \frac{2}{8}\int \frac{\sec\theta\tan\theta}{\tan^3\theta}d\theta = \frac{1}{4}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}d\theta = \frac{1}{4}\int \csc\theta\cot\theta\, d\theta = -\frac{1}{4}\csc\theta + c = \frac{-(x+1)}{4\sqrt{(x+1)^2-4}} + c$$

Example 31: Evaluate the integral

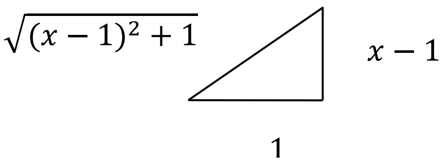
$$\int \frac{dx}{(x^2-2x+2)^{\frac{3}{2}}}\ .$$

Solution

$x^2-2x+2 = x^2-2x+1+1 \Longrightarrow (x-1)^2+1$

$let\ x-1 = \tan\theta$

$dx = \sec^2\theta\, d\theta$



$$\int \frac{\sec^2\theta}{(\tan^2\theta+1)^{3/2}}d\theta = \int \frac{\sec^2\theta}{\sec^3\theta}d\theta = \int \frac{1}{\sec\theta}d\theta = \int \cos\theta\, d\theta$$

$$= \sin\theta + c = \frac{x-1}{\sqrt{(x-1)^2+1}} + c$$