

Fourier Analysis

Introduction:

Signal characteristics : a - Even and odd function

By definition, the function $f(t)$ is even if $f(t) = f(-t)$, an even function has symmetry with respect to the vertical axis i.e. the signal for $t < 0$ is a mirror image of the signal for $t > 0$. The function $f(t) = \cos wt$ is even since $\cos wt = \cos(-wt)$, another example of an even function is given in fig. 1

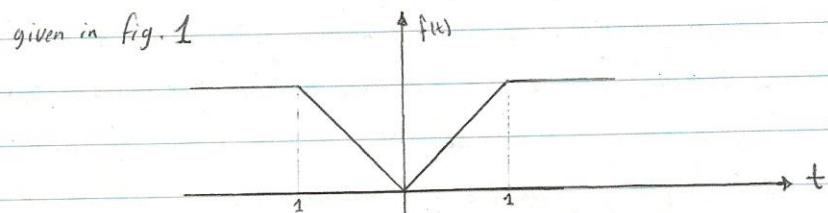


Fig. 1: An example of even function

A function $f(t)$ is odd if $f(t) = -f(-t)$, an odd function has symmetry with respect to the origin. The function $f(t) = \sin wt$ is odd since $\sin wt = -\sin(-wt)$. Another example of an odd function is given in

Fig. 2.

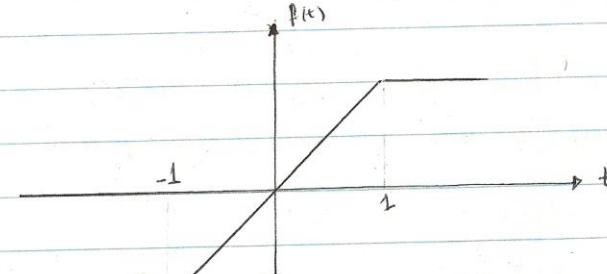


Fig. 2. odd function

The function shown in fig. 3 is neither odd nor even

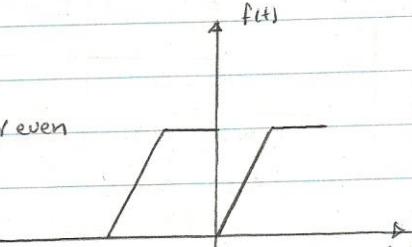


Fig. 3

F.A. [2]

Periodic Function: A continuous time signal $f(t)$ is periodic if

$f(t) = f(t + T)$ $T > 0$ — (1), for all t , where the constant T is the period. A signal that is not periodic is said to be aperiodic.

Result: The periodic function satisfies the equation

$$f(t) = f(t + nT)$$

where n is any integer i.e. a periodic signal with $T > 0$ is also periodic with period nT . The minimum value of the period $T > 0$ that satisfies the definition $f(t) = f(t + T)$ is called the fundamental period of the signal and is denoted as T_0 . with T_0 in seconds, the fundamental frequency in Hertz (the number of period per second (f_0) and the fundamental frequency in rad/sec; ω_0 are given by

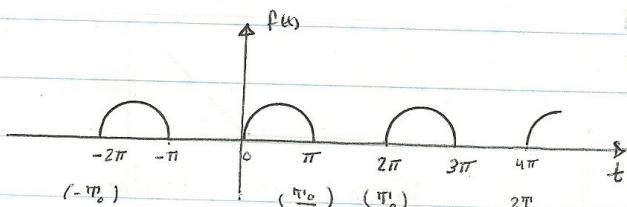
$$f_0 = \frac{1}{T_0} \text{ (Hz)} ; \quad \omega_0 = 2\pi f_0 \Rightarrow \omega_0 = \frac{2\pi}{T_0} \frac{\text{rad}}{\text{sec}}$$

Examples of Periodic Signals

1- Half-wave rectified signal (sinusoidal voltage passed through a half wave rectifier which clips the negative portion of the wave)

$$f(t) = \begin{cases} \sin t & 0 \leq t < \pi \\ 0 & \pi \leq t < 2\pi \end{cases}$$

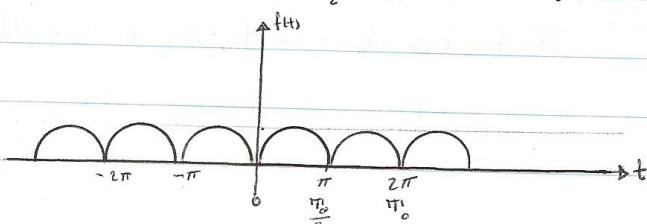
$$\text{here } T_0 = 2\pi$$



2- Full-wave rectified Signal

$$f(t) = |\sin t| \quad -\pi < t < \pi$$

$$T_0 = 2\pi$$



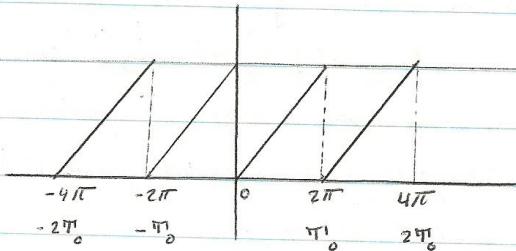
F.A. [3]

Adv. Math. 2010-2011

3- Saw Tooth Wave Signal

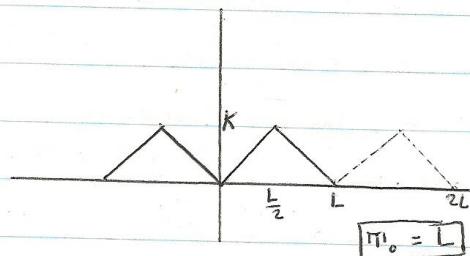
$$f(t) = t \quad 0 < t < 2\pi$$

$$[T_0 = 2\pi]$$



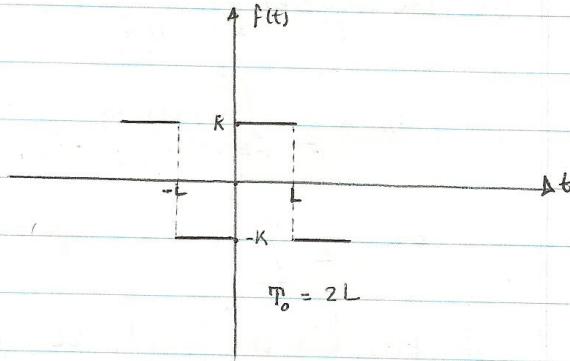
4- Triangular Wave (Triangular pulse)

$$f(t) = \begin{cases} \frac{2Kt}{L} & 0 < t < \frac{L}{2} \\ \frac{2K(L-t)}{L} & \frac{L}{2} < t < L \end{cases}$$



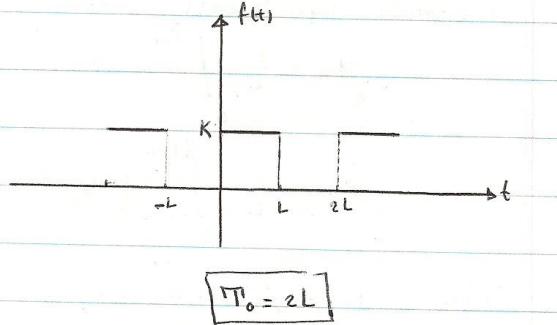
5- Square wave

$$f(t) = \begin{cases} K & 0 < t < L \\ -K & -L < t < 0 \end{cases}$$



6- Rectangular Wave

$$f(t) = \begin{cases} 0 & -L < t < 0 \\ K & 0 < t < L \end{cases}$$



Fourier Series

We will consider one of the most important procedure in signal and linear time invariant [LTI] system analysis; this procedure is used to express a complicated signal as a sum of simpler signals. The simpler signals are sinusoids and the resulting sum is called The Fourier Series or Fourier expansion.

Definition of Fourier Series: Let $f(t)$ be defined in the interval $(-L, L)$ and determined outside of this interval by:

$$f(t+2L) = f(t), \text{ i.e. assume that } f(t) \text{ has the period}$$

$T_0 = 2L$, the Fourier series or Fourier expansion corresponding to $f(t)$ is defined to be:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi}{L} t + b_n \sin \frac{n\pi}{L} t) \quad \boxed{1}$$

where the Fourier coefficients a_n and b_n are

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{n\pi}{L} t dt \quad \boxed{2}$$

$n = 0, 1, 2, 3, \dots$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{n\pi}{L} t dt \quad \boxed{3}$$

If $f(t)$ has the period $2L$, the coefficients a_n and b_n can be determined equivalently from

$$a_n = \frac{1}{L} \int_0^{c+2L} f(t) \cos \frac{n\pi}{L} t dt \quad \boxed{2-a}$$

$$b_n = \frac{1}{L} \int_c^{c+2L} f(t) \sin \frac{n\pi}{L} t dt \quad \boxed{3-a}$$

where c is any real number, in the special case $c = -L$

eq $\boxed{2-a}$ and $\boxed{3-a}$ become $\boxed{2}$ and $\boxed{3}$

F.A. [5]

Adv. Eng. Math. 2010-2011

The coefficients a_n and b_n sometimes are called Euler coefficients or Euler-Fourier coefficients.

* The constant term in eq. [1] i.e. $\frac{a_0}{2}$ is equal to

$$\frac{a_0}{2} = \frac{1}{2L} \int_{-L}^L f(t) dt \text{ which is the mean of } f(t) \text{ over a period}$$

Dirichlet Condition: If $f(t)$ is a bounded periodic function which in any one period has at most a finite number of local maximum and minimum and a finite number of points of discontinuity, then the Fourier series of $f(t)$ converge to $f(t)$ at all points where $f(t)$ is continuous and converge to the average of the right and left hand limits of $f(t)$ at each points where $f(t)$ is discontinuous; i.e.

((Fig. shown a periodic function defined by different formulas over different portion of a period))

Since,

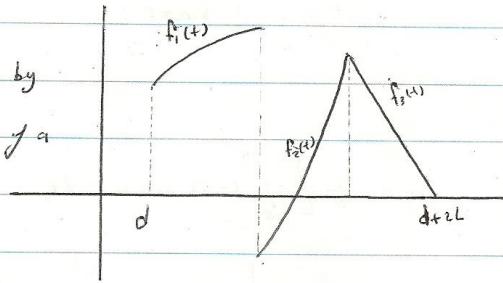
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi}{L} t + b_n \sin \frac{n\pi}{L} t) \quad - [4]$$

or

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

where ω_0 : principle frequency defined as $\omega_0 = \frac{2\pi}{T}$

However if $f(t)$ is a point of discontinuity then the left side of eq. [4] is replaced by $\frac{1}{2} [f(t)^+ + f(t)^-]$ so that the series converge to the mean value of $f(t)^+$ and $f(t)^-$



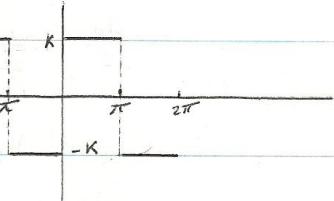
F.A. [6]

Adv Eng. Math. 2010-2011

Examples

- II) Square wave: Find the Fourier coefficients of the periodic function $f(t)$ whose analytic representation is

$$f(t) = \begin{cases} -K & -\pi < t < 0 \\ K & 0 < t < \pi \end{cases} \quad (\text{Period } T_0 = 2\pi)$$



Solution:

$$a_0 = \frac{1}{\pi} \left[\int_{-\pi}^0 (-K) dt + \int_0^\pi K dt \right] \Rightarrow [a_0 = 0]$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \left[\int_{-\pi}^0 (-K) \cos nt dt + \int_0^\pi K \cos nt dt \right] \\ &= \frac{-K}{\pi} \left[\frac{1}{n} \sin nt \right]_{-\pi}^0 + \frac{K}{\pi} \left[\frac{1}{n} \sin nt \right]_0^\pi \Rightarrow [a_n = 0] \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \left[\int_{-\pi}^0 (-K) \sin nt dt + \int_0^\pi K \sin nt dt \right] \\ &= \frac{1}{\pi} \left[\frac{K}{n} \cos nt \right]_{-\pi}^0 - \left[\frac{K}{n} \cos nt \right]_0^\pi \\ &= \frac{K}{n\pi} \left[\cos(0) - \cos(-n\pi) - \cos(n\pi) + \cos(0) \right] \end{aligned}$$

$$\therefore b_n = \frac{2K}{n\pi} [1 - \cos n\pi] \quad [\text{Hint } \cos(n\pi) = \cos(-n\pi)]$$

also $\cos n\pi = (-1)^n = \begin{cases} -1 & \text{if } n \text{ odd} \\ 1 & \text{if } n \text{ even} \end{cases}$

$$b_n = \begin{cases} \frac{4K}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

F.A. [7]

Adv. Eng. Math. 2010 - 2011

$$f(t) = \frac{4K}{\pi} \left[\sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \frac{1}{7} \sin 7t + \dots \right]$$

and

$$f(0) = 0$$

$$f\left(\frac{\pi}{2}\right) = \frac{4K}{\pi} \left[\sin \frac{\pi}{2} + \frac{1}{3} \sin \frac{3\pi}{2} + \frac{1}{5} \sin \frac{5\pi}{2} + \frac{1}{7} \sin \frac{7\pi}{2} \right]$$

$$\text{since } f\left(\frac{\pi}{2}\right) = K \quad (\text{By definition})$$

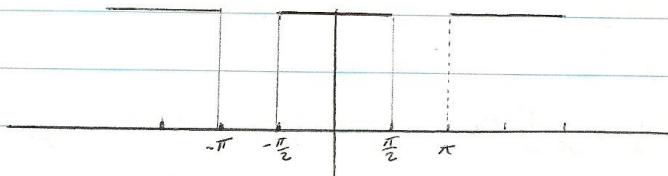
∴

$$K = \frac{4K}{\pi} \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots \right]$$

or

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \dots$$

2- Find the Fourier Series of the function whose graph is shown below, also find $f\left(\frac{\pi}{4}\right)$ by expansion of the series. Compare your answer with the exact value



$$T_0 = 2\pi, L = \pi$$

Solution: By examination of the graph, the periodic function is defined as follows:

$$f(t) = \begin{cases} 0 & -\pi < t < -\frac{\pi}{2} \\ 1 & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < t < \pi \end{cases}$$

F.A [8]

Adv. Eng. Meth. 2010 - 2011

$$a_0 = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t) dt \Rightarrow a_0 = 1$$

$\frac{a_0}{2} = \frac{1}{2}$, is the average of the function over period.

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos nt dt \\ &= \frac{1}{n\pi} \left[\sin nt \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \quad \left[\sin \frac{n\pi}{2} = -\sin \frac{n\pi}{2} \right] \end{aligned}$$

$$a_n = \frac{2}{n\pi} \sin \frac{n\pi}{2}$$

$$a_n = \begin{cases} 0 & n \text{ is even} \\ \frac{2}{n\pi} & n = 1, 5, 9, 13 \\ -\frac{2}{n\pi} & n = 3, 7, 11, 15 \end{cases}$$

$$b_n = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin nt dt \Rightarrow b_n = 0$$

$$\therefore f(t) = \frac{1}{2} + \frac{2}{\pi} \left[\cos t - \frac{1}{3} \cos 3t + \frac{1}{5} \cos 5t - \frac{1}{7} \cos 7t + \frac{1}{9} \cos 9t - \dots \right]$$

graphically, The exact value of $f(\frac{\pi}{4}) = 1$

By expansion of series

$$f(\frac{\pi}{4}) = \frac{1}{2} + \frac{2}{\pi} \left[\cos \frac{\pi}{4} - \frac{1}{3} \cos \frac{3\pi}{4} + \frac{1}{5} \cos \frac{5\pi}{4} - \dots + \frac{1}{17} \cos \frac{17\pi}{4} \right]$$

$$= 0.999$$

$$\text{Absolute \% Error} = \left| \frac{1 - 0.999}{1} \right| (100) = 0.2 \%$$

F.A. [9]

Adv. Eng. Math.

3- Obtain the Fourier Series of the function whose definition in one period is

$$f(t) = \begin{cases} 0 & -1 \leq t < 0 \\ 1 & 0 \leq t < \frac{1}{2} \\ 0 & \frac{1}{2} \leq t < 1 \end{cases} \quad T_0 = 2, L = 1$$

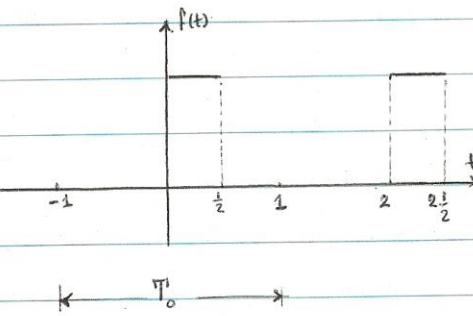
Solution: Graph the function $f(t)$

$$a_0 = \int_0^{\frac{1}{2}} dt \Rightarrow a_0 = \frac{1}{2} \Rightarrow \frac{a_0}{2} = \frac{1}{4}$$

$$a_n = \int_0^{\frac{1}{2}} \cos n\pi t dt \Rightarrow \boxed{a_n = \frac{1}{n\pi} \sin \frac{n\pi}{2}}$$

$$b_n = \int_0^{\frac{1}{2}} \sin n\pi t dt \Rightarrow b_n = \frac{1}{n\pi} \left[1 - \cos \frac{n\pi}{2} \right]$$

$$\therefore f(t) = \frac{1}{4} + \frac{1}{\pi} \left[\sum_{n=1}^{\infty} \frac{1}{n} \left\{ \sin \frac{n\pi}{2} \cos n\pi t + \left(1 - \cos \frac{n\pi}{2} \right) \sin n\pi t \right\} \right]$$



Odd and Even Function, Half-Range Expansion

((Half-Range Fourier Sine or Cosine Series))

A half-range Fourier Sine or Cosine Series is a Series in which only Sine terms or only cosine terms are presents respectively. When a half-range series corresponding to a given function is desired the function is generally defined in the interval $(0, L)$ which is half of the interval $(-L, L)$, thus accounting for the name half-range, and the function is specified as odd or even, so that it is clearly defined in the other half of the interval, namely $(-L, 0)$. In such case we have:-

$$\boxed{1-1}$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{2}{L} \int_0^L f(t) \sin \frac{n\pi}{L} t dt$$

F.A [10]

Adv. Eng. Maths.

$$\text{and } f(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} t$$

For half-range sine series or half-range sine expansion, such expansion is specified for odd function provided that the function passes the Dirichlet Condition.

[2] For half-Range Cosine Series (or expansion)

$$b_n = 0 \quad a_n = \frac{2}{L} \int_0^L f(t) \cos \frac{n\pi}{L} t \, dt$$

$$\therefore f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} t \quad (\text{i.e. only cosine term appear in the series})$$

Ex:- Expand $f(t) = t$ for $t < 2$ in a half-range

a. Sine Series

b. Cosine Series

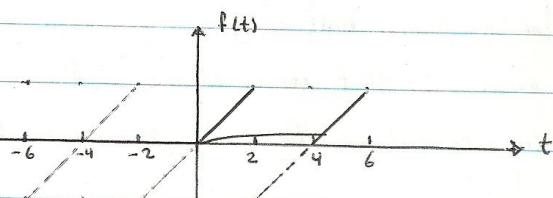
Hint:

$$\int x \sin ax \, dx = \frac{\sin ax}{a^2} - \frac{x \cos ax}{a}$$

$$\int x \cos ax \, dx = \frac{\cos ax}{a^2} + \frac{x \sin ax}{a}$$

Solution:

a - Half range Sine expansion; Extend the definition of given function to that of the odd function of period $T_o = 4$ as shown below



here $f(t) = -f(-t)$ i.e. $f(t)$ is odd function

$$T_o = 4$$

$$\therefore L = 2$$

$$[T_o = 2L]$$

$$a_n = 0$$

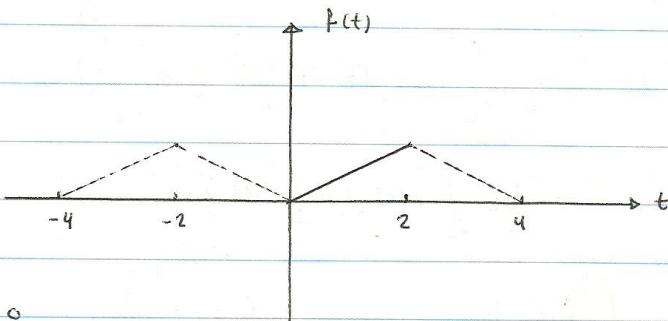
$$b_n = \frac{2}{2} \int_0^2 t \sin \frac{n\pi}{2} t dt$$

$$b_n = -\frac{4}{n\pi} \cos n\pi \quad \text{since } \cos n\pi = (-1)^n$$

$$\therefore b_n = \begin{cases} \frac{4}{n\pi} & n \text{ is odd} \\ -\frac{4}{n\pi} & n \text{ is even} \end{cases}$$

$$\therefore f(t) = \frac{4}{\pi} \left[\sin \frac{\pi}{2} t - \frac{1}{2} \sin \pi t + \frac{1}{3} \sin \frac{3\pi}{2} t - \dots \right]$$

b- Half Range Cosine expansion : Extend the definition of $f(t)$ to that of the even function of period 4



$$\text{here } b_n = 0$$

$$a_0 = \frac{2}{2} \int_0^2 t dt \Rightarrow a_0 = 2$$

$$a_n = \frac{2}{2} \int_0^2 t \cos \frac{n\pi}{2} t dt \Rightarrow a_n = \frac{4}{n^2\pi^2} (\cos n\pi - 1)$$

$$a_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-8}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}$$

$$f(t) = 1 - \frac{8}{\pi^2} \left[\cos \frac{\pi}{2} t + \frac{1}{3^2} \cos \frac{3\pi}{2} t + \frac{1}{5^2} \cos \frac{5\pi}{2} t + \dots \right]$$

F.A [2]

Parseval's Identity : Assume that the Fourier series corresponding to $f(t)$ converges uniformly to $f(t)$ in $(-L, L)$ prove Parseval's Identity

$$\frac{1}{L} \int_{-L}^L [f(t)]^2 dt = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

where the integral is assumed to exist

Solution: Since the Fourier series of $f(t)$ is given by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi}{L} t + b_n \sin \frac{n\pi}{L} t) \quad \text{--- (I)}$$

Multiply both sides of eq.(I) by $f(t)$ and integrating term by term from $-L$ to L we obtain

$$\begin{aligned} \int_{-L}^L [f(t)]^2 dt &= \frac{a_0}{2} \int_{-L}^L f(t) dt + \sum_{n=1}^{\infty} \left(a_n \int_{-L}^L f(t) \cos \frac{n\pi}{L} dt \right. \\ &\quad \left. + b_n \int_{-L}^L f(t) \sin \frac{n\pi}{L} dt \right) \end{aligned}$$

but

$$a_0 = \frac{1}{L} \int_{-L}^L f(t) dt \quad \text{or} \quad a_0 L = \int_{-L}^L f(t) dt$$

$$\text{similarly } La_n = \int_{-L}^L f(t) \cos \frac{n\pi}{L} dt$$

and

$$L b_n = \int_{-L}^L f(t) \sin \frac{n\pi}{L} dt$$

$$\therefore \int_{-L}^L [f(t)]^2 dt = \frac{L}{2} a_0^2 + \sum_{n=1}^{\infty} (La_n^2 + Lb_n^2)$$

$$\text{so: } \frac{1}{L} \int_{-L}^L [f(t)]^2 dt = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

F.A. [3]

Adv. Eng. Math

Exam. Let $f(t) = t$ be a periodic function where $L = 2$ $a_0 = 2$

$a_n = \frac{4}{n^2\pi^2} (\cos n\pi - 1)$, $n \neq 0$ if $b_n = 0$ write the Parseval's identity corresponding to the Fourier series of $f(t)$

Solution:

$$\frac{1}{2} \int_{-2}^2 t^2 dt = \frac{(2)^2}{2} + \sum_{n=1}^{\infty} \frac{16}{n^4\pi^4} (\cos n\pi - 1)^2$$

$$[(\cos n\pi - 1)^2 = 4 \text{ if } n \text{ is odd and } \cos n\pi - 1 = 0 \text{ if } n \text{ is even}$$

then

$$\frac{8}{3} = 2 + \frac{64}{\pi^4} \cdot \frac{1}{n^4} \quad n \text{ is odd}$$

or

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots + \frac{1}{n^4} \quad (n \text{ is odd})$$

Complex Form of the Fourier Series

The Fourier Series corresponding to a periodic function $f(t)$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi}{L} t + b_n \sin \frac{n\pi}{L} t) \quad \square$$

can be written in complex form, which some time simplifies calculation. This is done using Euler's formula [or Euler's identity] that is

$$\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \quad \cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

where $j = \sqrt{-1}$ with the property $\frac{1}{j} = -j$

for $\theta = \frac{n\pi}{L} t$, eq. 1 can be written as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{jn\pi t/L} - e^{-jn\pi t/L}}{2} \right) + b_n \left(\frac{e^{jn\pi t/L} + e^{-jn\pi t/L}}{2j} \right) \right] \quad \square$$

F.A. [14]

Collecting the like terms with the property $\frac{1}{j} = -j$, we obtain

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ \left(\frac{a_n - jb_n}{2} \right) e^{\frac{jn\pi t}{L}} + \left(\frac{a_n + jb_n}{2} \right) e^{-\frac{jn\pi t}{L}} \right\} \quad [3]$$

if we define $C_0 = \frac{a_0}{2}$, $C_n = \frac{a_n - jb_n}{2}$

$$C_n = \frac{a_n + jb_n}{2}$$

so eq. [3] can be written as:

$$f(t) = C_0 + \sum_{n=1}^{\infty} (C_n e^{\frac{jn\pi t}{L}} + C_n e^{-\frac{jn\pi t}{L}}) \quad [4]$$

The coefficients C_0 , C_n , C_{-n} are determined as follows

$$\begin{aligned} C_0 &= \frac{1}{2L} \int_{-L}^{L} f(t) dt \\ C_n &= \frac{1}{2L} \int_{-L}^{L} f(t) e^{-\frac{jn\pi t}{L}} dt \\ C_{-n} &= \frac{1}{2L} \int_{-L}^{L} f(t) e^{\frac{jn\pi t}{L}} dt \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{Complex Fourier Coefficients}$$

In more symmetrical form the complex series can be written as,

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{jn\pi t}{L}}$$

Useful Relationships

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{jn\pi} = \cos n\pi - j\sin n\pi = (-1)^n$$

F.A. [15]

Adv. Eng. Math.

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$e^{-jn\pi} = \cos(-n\pi) = (-1)^n$$

$$\left. \begin{aligned} a_n &= C_n + \bar{C}_n \\ b_n &= j(C_n - \bar{C}_n) \end{aligned} \right\}$$

$$\sinh\theta = \frac{e^\theta - e^{-\theta}}{2}$$

$$\cosh\theta = \frac{e^\theta + e^{-\theta}}{2}$$

Ex: Find the Complex form of the Fourier Series of the periodic function whose definition in one period is $f(t) = e^t$ $-1 < t < 1$

Solution: here $L = 1$

$$C_n = \frac{1}{2} \int_{-1}^1 e^{-t} e^{-jn\pi t} dt$$

$$C_n = \frac{(-1)^n (1 - jn\pi)}{i + n^2\pi^2} \sinh(1)$$

$$\therefore f(t) = \sum_{n=-\infty}^{n=\infty} (-1)^n \frac{1 - jn\pi}{i + n^2\pi^2} \sinh(1) e^{jn\pi t}$$

Above expansion can be converted to the real trigonometry without difficulty

$$a_n = C_n + \bar{C}_n$$

$$C_n = (-1)^n \frac{1 + jn\pi}{i + n^2\pi^2} \sinh(1) \quad [\text{by sub. } (-n) \text{ instead of } n]$$

and $(-1)^n = (-1)^{-n}$

$$\therefore a_n = \frac{2(-1)^n}{i + n^2\pi^2} \sinh(1)$$

F.A. [16]

$$b_n = j (C_n - \bar{C}_n)$$

$$b_n = \frac{(-1)^n 2n\pi}{1+n^2\pi^2} \operatorname{Sinh}(1)$$

$$C_0 = \operatorname{Sinh}(1) \quad \text{and} \quad \frac{dC_0}{2} = \operatorname{Sinh}(1)$$

Finally:

$$f(t) = \operatorname{Sinh}(1) + 2 \operatorname{Sinh}(1) \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{1+n^2\pi^2} \cos n\pi t + \frac{(-1)^n n\pi}{1+n^2\pi^2} \operatorname{Sinh} n\pi t \right)$$

EX: Find the Complex Form of Fourier Series for the following periodic function whose definition in one period is given below, then convert to real trigonometry also find $f(a)$:

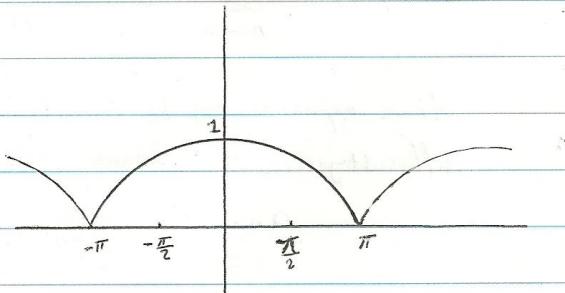
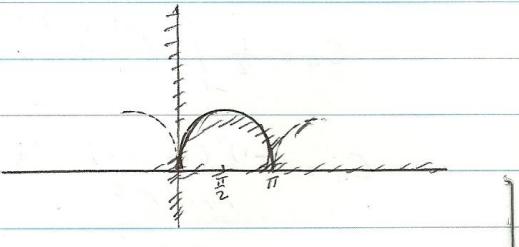
$$f(t) = \cos\left(\frac{t}{2}\right)$$

Solution:

$$\text{here } T_0 = 2\pi \quad L = \pi$$

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos \frac{t}{2} e^{-jnt} dt$$

$$\text{since } \cos \frac{t}{2} = \frac{e^{j\frac{t}{2}} - e^{-j\frac{t}{2}}}{2}$$



$$C_n = \frac{1}{4\pi} \left[\int_{-\pi}^{\pi} e^{j\frac{t}{2} - jnt} dt + \int_{-\pi}^{\pi} e^{-j\frac{t}{2} - jnt} dt \right]$$

$$C_n = \frac{2(-1)^n}{\pi} \left[-\frac{1}{1-4n^2} \right]$$

F.A. [17]

$$f(t) = \frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1-4n^2} e^{jnt}$$

Adv. Eng. Math.

Converting to real trigonometry

$$a_n = C_n + S_n \Rightarrow a_n = \frac{4(-1)^n}{\pi} \left(\frac{1}{1-4n^2} \right)$$

$$b_n = j(C_n - S_n) = 0 \quad \{ f(t) \text{ is even function} \}$$

$$a_0 = \frac{4}{\pi}$$

$$\therefore f(t) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{1-4n^2} \cos nt$$

$$\text{Since } f(t) = \cos\left(\frac{t}{2}\right) \quad -\pi < t < \pi$$

$$\therefore f(0) = 1$$

Expand the Fourier Series

$$f(0) = \frac{2}{\pi} + \frac{4}{\pi} \left[\frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \frac{1}{63} + \frac{1}{99} \right] -$$

$$f(0) = 1.005 \quad \% E = 0.5 \%$$

\therefore The expansion gave a value which is very close to the exact value of the function.

Ex:- The complex Fourier series of a periodic function is given below:

$$I(t) = \pi + \sum_{n=-\infty}^{\infty} \frac{j}{n} e^{jnt}$$

a- Does the function odd or even?

b- what is the value of fundamental period T_0 ?

F.A. [8]

Solution: Convert to real trigonometry

$$\text{here } C_0 = \pi \quad \therefore \frac{a_0}{2} = \pi$$

$$C_n = \frac{j}{n} \quad C_{-n} = -\frac{j}{n} \quad \therefore [a_n = 0]$$

$$b_n = -\frac{2}{n}$$

\therefore The function is neither even nor odd

The value of the fundamental period $T_0 = 2\pi$, the fundamental frequency $\omega_0 = 1$.

Summary

In mathematics, a Fourier Series decomposes any periodic function or signal into the sum of a set of simple oscillating functions, namely sines and cosines (or complex exponentials). The study of Fourier series is a branch of Fourier analysis. Fourier series were introduced by Joseph Fourier (1768-1830) for the purpose of solving the heat equation in a square metal plate.