

F.T [1]

Fourier Transform

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The Fourier transform is the operation that decompose a signal into its constituent frequencies. The original signal depends on time, and therefore is called the time domain representation of the signal, whereas the Fourier transform depends on frequency and is called the frequency domain representation of the signal.

Mathematically F.T defined as.

$$\mathcal{F}_t [f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \quad \text{--- [1]}$$

and

$$\mathcal{F}_\omega^{-1} [F(\omega)] = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \quad \text{--- [2]}$$

in eq. 1 Integrate with respect to time, i.e the frequency ω treated as constant.

in eq. 2 Integrate with respect to frequency at t is constant.

Equations [1] and [2], together are called a transform pair and their relationship is often represented in mathematical notation as:

$$f(t) \xrightarrow{\mathcal{F}_t} F(\omega)$$

Applications: Engineers use the F.T. to simplify the mathematical analysis of isolated signals [single pulse] and systems and for explaining physical phenomena mathematically. It is widely used in the field of electrical engineering, especially in the study of electronic communication signals and system.

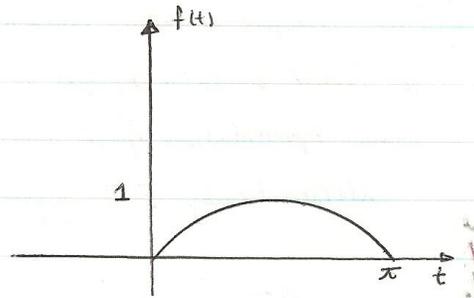
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EX:- Find the F.T of the sinusoidal pulse show in figure below

Solution.

here $f(t) = \sin t$ and

$$f(t) = \begin{cases} 0 & -\infty < t < 0 \\ \sin t & 0 < t < \pi \\ 0 & \pi < t < \infty \end{cases}$$



$$\therefore \sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\therefore F(\omega) = \int_0^{\pi} \left(\frac{e^{j\theta} - e^{-j\theta}}{2j} \right) e^{-j\omega t} dt \quad ; \quad \frac{1}{j} = -j$$

$$F(\omega) = \frac{1}{2} \left[\frac{e^{j(1-\omega)t}}{1-\omega} + \frac{e^{-j(1+\omega)t}}{1+\omega} \right]_0^{\pi}$$

since $e^{j\pi} = -1$ and $e^{-j\pi} = -1$

Then

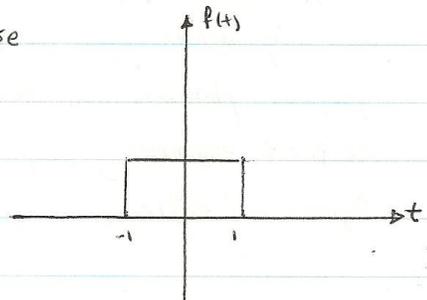
$$F(\omega) = \frac{1 + e^{-j\omega\pi}}{1 - \omega^2}$$

and

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1 + e^{-j\omega\pi}}{1 - \omega^2} \right) e^{j\omega t} d\omega$$

EX:- Find the F.T of the single pulse

$$f(t) = \begin{cases} 1 & -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$F(\omega) = \int_{-1}^1 (1) e^{-j\omega t} dt$$

$$\begin{aligned}
 F(\omega) &= \left[\frac{e^{-j\omega}}{2j\omega} \right]_{-1}^1 \\
 &= \frac{2}{\omega} \cdot \left(\frac{e^{j\omega} - e^{-j\omega}}{2j} \right) \\
 &= 2 \frac{\sin \omega}{\omega} \quad , \text{ by definition } \text{Sinc } \theta = \frac{\sin \theta}{\theta}
 \end{aligned}$$

$$\therefore F(\omega) = 2 \frac{\sin \omega}{\omega} = 2 \text{Sinc } \omega$$

$$f(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \text{Sinc } \omega e^{j\omega t} d\omega$$

By definition of the signal

$$f(-1) = \frac{1+0}{2} \quad f(1) = \frac{1}{2} \quad f(0) = 1$$

$$\therefore \int_{-\infty}^{\infty} \text{Sinc } \omega e^{j\omega t} d\omega = \begin{cases} 0 & t > 1 \text{ or } t < -1 \\ \pi & -1 < t < 1 \\ \frac{\pi}{2} & t = \pm 1 \text{ (i.e. } t = -1 \text{ or } t = 1) \end{cases}$$

Fourier Cosine and Sine Transform

1. Fourier cosine transform: For an even function $f(t)$, the Fourier integral is the Fourier cosine integral

$$f(t) = \int_0^{\infty} A(\omega) \cos \omega t d\omega \quad \text{where } A(\omega) = \frac{2}{\pi} \int_0^{\infty} f(t) \cos \omega t dt$$

We now set $A(\omega) = \sqrt{\frac{2}{\pi}} F_c(\omega)$ where F_c suggest "cosine"

then

$$F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos \omega t dt \quad \text{--- (1) } (\omega: \text{ constant})$$

and

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$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(\omega) \cos t\omega \, d\omega \quad \text{--- [2]} \quad (t: \text{constant})$$

Eq. [1] and [2] are called Fourier Cosine transform

2- Fourier Sine Transforms: Similarly, for an odd function $f(t)$ the Fourier Sine integral is

$$f(t) = \int_0^{\infty} B(\omega) \sin t\omega \, d\omega \quad \text{where}$$

$$B(\omega) = \frac{2}{\pi} \int_0^{\infty} f(t) \sin \omega t \, dt$$

Set $B(\omega) = \sqrt{\frac{2}{\pi}} F_s(\omega)$ where F_s suggests "sine" then

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin \omega t \, dt \quad (\text{integrate w.r.t } t)$$

where F_s suggests "sine" then

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin \omega t \, dt \quad \text{--- [3]}$$

is called The Fourier sine transform of $f(t)$ and

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(\omega) \sin t\omega \, d\omega \quad \text{--- [4]}$$

Equation [3] and [4] are called the Fourier Sine Transform pair

$\Leftarrow f(t)$ is called the IFS'T of $F_s(\omega)$ \Rightarrow

IFS'T: Inverse Fourier Sine Transform

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Hint: If the function is neither even nor odd then

$$f(t) = \int_0^{\infty} (A(\omega) \cos \omega t + B(\omega) \sin \omega t) d\omega \quad \text{i.e. we}$$

integrate with respect to (ω) , where ω is the frequency and

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t dt$$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \omega t dt$$

} here we integrate
wrt (t)

The main use of the Fourier Integral is in solving D.E.

Examples

1. Single pulse: Find the Fourier integral representing the function

$$f(t) = \begin{cases} 1 & |t| < 1 \\ 0 & |t| > 1 \end{cases}$$

Solution

$$f(t) = \begin{cases} 1 & -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

Since the function is even

$$\therefore f(t) = \int_0^{\infty} A(\omega) \cos \omega t d\omega \quad \text{and}$$

$$A(\omega) = \frac{2}{\pi} \int_0^{\infty} f(t) \cos \omega t dt = \frac{2}{\pi} \int_0^1 \cos \omega t dt$$

$$A(\omega) = \frac{2}{\pi} \frac{\sin \omega}{\omega} \quad \text{or} \quad A(\omega) = \frac{2}{\pi} \sin(\omega)$$

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$$\therefore f(t) = \frac{2}{\pi} \int_0^{\infty} \sin w \cos tw \, dw$$

By definition $f(0) = 1$ $f(1) = \frac{1+0}{2} = \frac{1}{2}$

$$\therefore \int_0^{\infty} \sin w \cos tw \, dw = \begin{cases} \frac{\pi}{2} & 0 \leq t < 1 \\ \frac{\pi}{4} & t = 1 \\ 0 & t > 1 \end{cases}$$

$\int_{t=0}^{\infty} \frac{\sin w}{w} \, dt = \frac{\pi}{2}$, This integral is the limit
of so-called Sine Integral

$$Si(u) = \int_0^u \frac{\sin w}{w} \, dw \quad \text{as } u \rightarrow \infty$$

3- Find the Fourier cosine and sine Integral of $f(t) = e^{-kt}$
 k is constant and

$$\int e^{ax} \cos bx \, dx = e^{ax} \left[\frac{a \cos bx + b \sin bx}{a^2 + b^2} \right]$$

$$\int e^{ax} \sin bx \, dx = e^{ax} \left[\frac{a \sin bx - b \cos bx}{a^2 + b^2} \right]$$

Solution:

a- Fourier Cosine Transform

$$A(\omega) = \frac{2}{\pi} \int_0^{\infty} e^{-kt} \cos \omega t \, dt$$

$$A(\omega) = \frac{2k}{\pi} \left(\frac{1}{k^2 + \omega^2} \right)$$

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$$\therefore f(t) = \frac{2K}{\pi} \int_0^{\infty} \frac{\cos tw}{K^2 + w^2} dw$$

$$\text{but } f(t) = e^{-Kt}$$

$$\therefore \int_0^{\infty} \frac{\cos tw}{K^2 + w^2} dw = \frac{\pi}{2K} e^{-Kt} \quad \text{--- [1]}$$

b. Fourier Sine Integral

$$B(w) = \frac{2}{\pi} \int_0^{\infty} e^{-Kt} \sin wt dt$$

$$B(w) = \frac{2w}{\pi} \frac{1}{K^2 + w^2} \quad \text{but } f(t) = e^{-Kt}$$

$$\therefore f(t) = \frac{2}{\pi} \int_0^{\infty} \frac{w \sin tw}{K^2 + w^2} dw \quad \text{and}$$

$$\int_0^{\infty} \frac{w \sin tw}{K^2 + w^2} dw = \frac{\pi}{2} e^{-Kt} \quad \text{--- [2]}$$

Remark: The integrals [1] and [2] are called the Laplace Integrals

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[4] - Find the Fourier Cosine and sine transform of the function

$$f(t) = \begin{cases} k & \text{if } 0 < t < a \\ 0 & \text{if } t > a \end{cases}$$

Solution:

a -

$$F_c(\omega) = \sqrt{\frac{2}{\pi}} k \int_0^a \cos \omega t dt$$

$$F_c(\omega) = \sqrt{\frac{2}{\pi}} a k \operatorname{Sinc}(a\omega)$$

b - $F_s(\omega) = \sqrt{\frac{2}{\pi}} k \int_0^a \sin \omega t dt$

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} k \left[\frac{1 - \cos a\omega}{\omega} \right]$$

[5] - Solve the Integral equation

$$\int_0^{\infty} f(t) \sin \omega t dt = \begin{cases} 1 - \omega & 0 \leq \omega \leq 1 \\ 0 & \omega > 1 \end{cases}$$

i.e. find $f(t)$; Hint $\int x \sin ax dx = \frac{\sin ax}{a^2} - \frac{x \cos ax}{a}$

Solution:

By Definition:

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin \omega t dt$$

$$\therefore \int_0^{\infty} f(t) \sin \omega t dt = \sqrt{\frac{\pi}{2}} F_s(\omega) \quad \text{or}$$

$$\sqrt{\frac{\pi}{2}} F_s(\omega) = \begin{cases} 1 - \omega & 0 \leq \omega \leq 1 \\ 0 & \omega > 1 \end{cases}$$

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$$F_s(\omega) = \begin{cases} \sqrt{\frac{2}{\pi}} (1-\omega) & 0 \leq \omega \leq 1 \\ 0 & \omega > 1 \end{cases}$$

$$\text{But: } f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(\omega) \sin \omega t \, d\omega$$

$$\therefore f(t) = \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{2}{\pi}} \int_0^{\infty} (1-\omega) \sin \omega t \, d\omega \quad [\text{int. w.r.t } \omega]$$

$$\therefore f(t) = \frac{2(t - \sin t)}{t^2}$$

Finite Fourier Sine Transform

Given a function $f(x, y, z, t)$ defined in the interval $0 < x < a$, its finite Fourier sine transform with respect to x is given by

$$\mathcal{F}_{fs} [f(x, y, z, t)] \equiv F_{fs} \equiv F(n, y, z, t) = \int_0^a f(x, y, z, t) \sin \frac{n\pi}{a} x \, dx$$

where $n = \text{integer}$

and the Inversion formula is

$$\mathcal{F}_{fs}^{-1} [F(n, y, z, t)] = \frac{2}{a} \sum_{n=1}^{\infty} F_{fs}(n, y, z, t) \sin \frac{n\pi}{a} x$$

FFST of derivatives: Given a function $f(x, y, z, t)$ find the

$$\text{FFST w.r.t } x \text{ of } \frac{\partial^2 f}{\partial x^2}$$

$$\mathcal{F}_{fs} \left(\frac{\partial^2 f}{\partial x^2} \right) = -\left(\frac{n\pi}{a}\right)^2 F(n, y, z, t) + \frac{n\pi}{a} [f(0, y, z, t) - (-1)^n f(a, y, z, t)]$$

$n: \text{Integer}$

The finite Sine transform is useful for problems involving Boundary condition of Potential (pressure, temperature ---) distribution on two parallel boundaries

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Finite Fourier Cosine Transform

Given a function $f(x, y, z, t)$ defined in the interval $0 < z < b$, its finite transform WRT (Z) is given by

$$F_{ffc} = F(x, y, n, t) = \int_0^b f(x, y, z, t) \cos \frac{n\pi}{b} z \, dz$$

$n = \text{integer}$

The inverse formula is

$$\mathcal{F}_{ffc}^{-1} [F_{ffc}] = f(x, y, z, t) = \frac{1}{b} F(x, y, 0, t) + \frac{2}{b} \left[\sum_{n=1}^{\infty} F(x, y, n, t) \cos \frac{n\pi}{b} z \right]$$

The FFCIT of the z^{nd} derivative of $f(x, y, z, t)$ WRT (Z) is given by

$$\mathcal{F}_{fc} \left[\frac{\partial^2 f}{\partial z^2} \right] = -\left(\frac{n\pi}{b}\right)^2 F(x, y, n, t) + (-1)^n \frac{\partial}{\partial z} f(x, y, b, t) - \frac{\partial}{\partial z} f(x, y, 0, t)$$

The finite cosine transform is useful for problem in which the derivative of the function (or the velocities normal to two parallel boundaries) are among the boundary condition of the problem.

Important Results

1- The Fourier sine transform of derivatives

$$F_s \left[\frac{df}{dt} \right] = -\omega F_c(\omega)$$

$$F_s \left[\frac{d^2f}{dt^2} \right] = -\omega^2 F_s(\omega) + \sqrt{\frac{2}{\pi}} \omega f(0)$$

2. The Fourier Cosine Transform of Derivatives

$$F_c \left(\frac{df}{dt} \right) = \omega F_s(\omega) - \sqrt{\frac{2}{\pi}} f(0)$$

$$F_c \left(\frac{d^2f}{dt^2} \right) = -\omega^2 F_c(\omega) - \sqrt{\frac{2}{\pi}} f'(0)$$

Summary

1. The Complex Form of Fourier transform

$$\mathcal{F}_0[f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \quad [\omega: \text{constant}]$$

$$\mathcal{F}_0^{-1}[F(\omega)] = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \quad [t: \text{constant}]$$

2. Fourier Transform (or Fourier Integral)

a. General Form

$$f(t) = \int_0^{\infty} (A(\omega) \cos \omega t + B(\omega) \sin \omega t) d\omega$$

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t dt$$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \omega t dt$$

b. Fourier Sine Transform

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin \omega t dt$$

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and
$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(\omega) \sin t \omega d\omega$$

c- Fourier Cosine Transform

$$F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos \omega t dt$$

and

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(\omega) \cos t \omega d\omega$$

d. Fourier sine Transform of Derivatives

$$F_s\left(\frac{df}{dt}\right) = -\omega F_c(\omega)$$

$$F_s\left(\frac{d^2f}{dt^2}\right) = -\omega^2 F_s(\omega) + \sqrt{\frac{2}{\pi}} \omega f(0)$$

e. Fourier Cosine Transform of Derivatives

$$F_c\left(\frac{df}{dt}\right) = \omega F_s(\omega) - \sqrt{\frac{2}{\pi}} f(0)$$

$$F_c\left(\frac{d^2f}{dt^2}\right) = -\omega^2 F_c(\omega) - \sqrt{\frac{2}{\pi}} f'(0)$$

Finite Fourier Transform

1- Finite Fourier Sine Transform (transform WRT x)

$$F_{ffs} = F(n, y, z, t) = F(n, y, z, t) = \int_0^a f(x, y, z, t) \sin \frac{n\pi}{a} x dx$$

$n = \text{Integer}$

$$\text{and } \mathcal{F}_{fs}^{-1} [F(n, y, z, t)] = f(x, y, z, t) \\ = \frac{2}{a} \sum_{n=1}^{\infty} F_{fs}(n, y, z, t) \sin \frac{n\pi}{a} x$$

$$\mathcal{F}_{fs} \left(\frac{\partial^2 f}{\partial x^2} \right) = - \left(\frac{n\pi}{a} \right)^2 F(n, y, z, t) + \frac{n\pi}{a} [f(0, y, z, t) \\ - (-1)^n f(a, y, z, t)]$$

2. Finite Fourier Cosine Transform

$$F_{ffc} = F(x, y, n, t) = \int_0^b f(x, y, z, t) \cos \frac{n\pi}{b} z \, dz$$

n - Integer

The Inverse formula

$$\mathcal{F}_{ffc}^{-1} [F_{ffc}] = f(x, y, z, t) = \frac{1}{b} F(x, y, 0, t) + \frac{2}{b} \left[\sum_{n=1}^{\infty} F(x, y, n, t) \right. \\ \left. \cos \frac{n\pi}{b} z \right]$$

$$\mathcal{F}_{fc} \left[\frac{\partial^2 f}{\partial z^2} \right] = - \left(\frac{n\pi}{b} \right)^2 F(x, y, n, t) + (-1)^n \frac{\partial}{\partial z} f(x, y, b, t) \\ - \frac{\partial}{\partial z} f(x, y, 0, t)$$

