

1

A1

Partial Differential Equation [PDE]

PDE arise in connection with various physical and geometrical problems when the functions involved, depend on two or more independent variables, these variables may be time & one or several coordinates in space (x, y & z)

Linear PDE

The general linear PDE of order two in two independent variables has the form

$$A U_{xx} + B U_{yx} + C U_{yy} + D U_x + E U_y + F U = G \quad (1)$$

where A, B, \dots, F may depend on x and y but not on (u) , a 2nd order PDE with independent variable x & y which does not have the form (1) is called non-linear if $G=0$, the equation is called homogeneous, while if $G \neq 0$ it is called non-homogeneous.

Equation (1) can be classified as:

- Elliptic PDE if $B^2 - 4AC < 0$
- Hyperbolic " " $B^2 - 4AC > 0$
- parabolic " " $B^2 - 4AC = 0$

Some important PDE

1- One dimensional heat conduction equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{k} \frac{\partial u}{\partial t}$
 here $u(x,t)$ is the temperature in a solid @ a position x @ time t , the constant k called the thermal diffusivity.

2- One dimensional wave equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 y}{\partial t^2}$

This is applicable to the small transverse vibration of a taut flexible string, initially located on the x -axis and set into motion. The variable $y(x,t)$ is the displacement of any point (x)

2

of the string @ time t . $a^2 = \frac{T}{\rho}$ where

T : Tension in the string

ρ : mass per unit length of the string.

↘ In the case of physical problems giving rise to PDE we have to find solutions which satisfy certain initial & boundary conditions. Such problems are called boundary value problems

Solution of PDE by Laplace Transformation

Given the function $u(x, t)$ defined @ $a \leq x \leq b$, $t > 0$

[L T W R T T]*

$$\mathcal{L}\left[\frac{\partial u}{\partial t}\right] = sU(x, s) - u(x, 0)$$

$$\mathcal{L}\left[\frac{\partial^2 u}{\partial t^2}\right] = s^2 U(x, s) - s' u(x, 0) - \frac{\partial u}{\partial t} \Big|_{t=0}$$

$$\mathcal{L}\left[\frac{\partial u}{\partial x}\right] = \frac{dU(x, s')}{dx}$$

$$\mathcal{L}\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{d^2 U}{dx^2} \quad [U \equiv U(x, s')]$$

The characteristics of the problem that suggest it is worthwhile to try the L.T technique are

- 1- The PDE is linear (necessary)
- 2- The " has constant coefficients (highly desirable)
- 3- At least one independent variable has the range 0 to ∞ (highly desirable)
- 4- There are appropriate initial condition ($t=0$) involving the independent variable (desirable)

* Laplace Transform with respect to time

[3]

[B]

Solve the following boundary value Problem

$$\frac{\partial^2 y}{\partial x^2} = 16 \left[\frac{\partial^2 y}{\partial t^2} \right] \quad \text{Subjected to the following conditions}$$

$y(x, 0) = 0$; $y_t(x, 0) = -1$; $y(0, t) = t^2$ for $t > 0$
 finally y is bounded [$\lim_{x \rightarrow \infty} y(x, t)$ exists]

Solution:-

Apply L.T with respect to time

$$\frac{d^2 Y}{dx^2} = 16 [s^2 Y - s y(x, 0) - y_t(x, 0)]$$

$$\frac{d^2 Y}{dx^2} - 16 s^2 Y = 16 \quad \text{--- [1]} \quad [Y \equiv Y(x, s)]$$

a. Homogeneous Solution

$$Y'' - 16 s^2 Y = 0 \quad r^2 = 16 s^2 \quad \text{or}$$

$$r = \pm 4s$$

$$Y_h = A e^{4s^2 x} + B e^{-4s^2 x} \quad \text{--- [2]}$$

b. particular solution: here $f(x) = 16$ [constant]

$$\text{then assume } Y_p = C \quad Y_p'' = 0$$

Sub in [1]

$$0 - 16 s^2 C = 16 \Rightarrow C = -\frac{1}{s^2}$$

then

$$Y = -\frac{1}{s^2} + A e^{4s^2 x} + B e^{-4s^2 x} \quad \text{--- [3]}$$

$\ll e^{4s^2 x} \rightarrow \infty$ but Y is bounded \gg

Evaluation of constant A, B .Since $y(x, t)$ is bounded as $x \rightarrow \infty$ then $Y(x, s)$ is bounded

4

also then for

$$Y(x, s) = -\frac{1}{s^2} + B e^{-4s^2 x} \quad \text{--- [4]}$$

since $y(0, t) = t^2 \xrightarrow{\mathcal{L.T}} Y(0, s) = \frac{2}{s^3}$

sub in [4] with $x=0$

$$\frac{2}{s^3} = -\frac{1}{s^2} + B \quad [e^0 = 1]$$

$$\therefore B = \frac{2}{s^3} + \frac{1}{s^2}$$

so $Y(x, s) = -\frac{1}{s^2} + \left(\frac{2}{s^3} + \frac{1}{s^2} \right) e^{-4x s^2}$

Take ILT for both sides

$$y(x, t) = -t + \left[\frac{2}{2} (t-4x)^2 + (t-4x) \right] U(t-4x)$$

where U is a unit step function

$$\left[\mathcal{L}^{-1} \left[e^{-as} F(s) \right] = f(t-a) U(t-a) \right]$$

$$U(t-4x) = \begin{cases} 0 & t < 4x \\ 1 & t > 4x \end{cases}$$

5

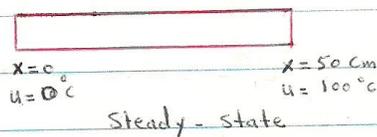
2

P.D.E

A metal rod whose length = 50 cm has its ends kept at 0°C and 100°C respectively, until steady state conditions prevail, then the temperature @ the farthest ends is reduced to 0°C and kept so, while that of the nearest ends is maintained, find the temperature $u(x, t)$ @ any position $x > 0$ @ any time $t > 0$. assume $k = 1$

Solution:

a. Steady-state Distribution



here $\frac{\partial u}{\partial t} = 0$

$$\frac{d^2 u}{dx^2} = 0 \Rightarrow \frac{du}{dx} = C_1 \Rightarrow u = C_1 x + C_2 \quad \text{--- (1)}$$

since $u = 0$ @ $x = 0$ then $0 = 0 + C_2 \Rightarrow C_2 = 0$

or $u = C_1 x$

and $u = 100$ @ $x = 50$ $100 = C_1 (50) \Rightarrow C_1 = 2$

∴ $u(x) = 2x$

When the temperature of the farthest end is lowered to 0°C the following initial & boundary conditions are obtained

$u(x, 0) = 2x$ (initial condition)

$u(0, t) = 0^\circ\text{C}$

$u(50, t) = 0^\circ\text{C}$



b. Unsteady-state distribution.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k} \frac{\partial u}{\partial t}$$

$k = 1$ given, $u \equiv u(x, t)$

6

Apply L.T with respect to time

$$\mathcal{L}\left[\frac{\partial^2 u}{\partial x^2}\right] = \mathcal{L}\left[\frac{\partial u}{\partial t}\right]$$

$$U'' = sU - u(x,0) \quad [U \equiv U(x,s)]$$

or

$$U'' = sU - 2x$$

$$U'' - sU = -2x \quad \text{--- [2]}$$

Homog. part

$$U'' - sU = 0 \quad \text{or} \quad r^2 - s = 0$$

$$r = \pm \sqrt{s}$$

$$\therefore U_h = A e^{x\sqrt{s}} + B e^{-x\sqrt{s}} \quad \text{--- [3]}$$

particular solution

assume $U_p = CX$, $U_p'' = 0$ sub. for U_p, U_p'' in [2]

$$0 - sCX = -2x \quad \Rightarrow \quad C = \frac{2}{s}$$

$$\therefore U(x,s) = \frac{2}{s}x + A e^{x\sqrt{s}} + B e^{-x\sqrt{s}} \quad \text{--- [4]}$$

To evaluate the two constant A and B apply the two B.C

since $u(0,t) = 0 \xrightarrow{\mathcal{L}} U(0,s) = 0$
[i.e. for $x=0$]

7

41

Sub. in [4]

$$0 = \frac{2}{s'}(0) + Ae^0 + Be^0$$

$$A + B = 0 \quad \text{or} \quad B = -A$$

$$\therefore U(x, s') = \frac{2}{s'}x + Ae^{x\sqrt{s'}} - Ae^{-x\sqrt{s'}}$$

$$\text{since } \sinh \theta = \frac{e^\theta - e^{-\theta}}{2} \quad [\text{Euler's identity}]$$

$$\therefore U(x, s') = \frac{2}{s'}x + 2A \sinh x\sqrt{s'} \quad [5]$$

$$\text{The 2nd B.C. } u(50, t) = 0 \xrightarrow{\mathcal{L}} U(50, s') = 0$$

$$0 = \frac{2}{s'}(50) + 2A \sinh 50\sqrt{s'}$$

or

$$A = \frac{-50}{s' \sinh 50\sqrt{s'}}$$

or

$$U(x, s') = \frac{2}{s'}x - \frac{100 \sinh x\sqrt{s'}}{s' \sinh 50\sqrt{s'}}$$

Apply ILT for both sides

$$u(x, t) = 2x - 100 \left[\frac{x}{50} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-\frac{n^2 x^2 t}{2500}} \sin \frac{n\pi x}{50} \right]$$

8

Hint $\mathcal{L}^{-1} \left[\frac{\sinh x \sqrt{s}}{s^2 \sinh a \sqrt{s}} \right] = \frac{x}{a} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-\frac{n^2 \pi^2 t}{a^2}} \sin \frac{n \pi x}{a}$

[See Spiegel Math. H.B entry **32.152**
p. 171]

التحقق من صحة الحل :

$$u(0, t) = 0 - 100 \left[\frac{0}{50} + \frac{2}{\pi} \sum \sin 0 \right] = 0$$

$$u(50, t) = 2(50) - 100 \left[\frac{50}{50} + \frac{2}{\pi} \sum e^{-\frac{n^2 \pi^2 t}{50^2}} \sin \frac{n \pi (50)}{50} \right]$$

$\sin n\pi = 0$ n integer

$$u(50, t) = 0$$

6:18 p.m

16-1-2005

ملاحظة : عند استخدام \mathcal{L}^{-1} بالبرهان الزمن فان اية دوال

تتمتع بـ [Space] تقابل كتابتها !

$$\left[\mathcal{L}[c] = \frac{c}{s} \right]$$

9

3

PDE

A metal bar whose surface is insulated has a length of 1 unit & a thermal diffusivity = 1, if its ends are kept @ temperature zero & its initial temperature = $3 \sin 2\pi x$ find temperature @ any position @ any time $t > 0$

Solution :-

Consider the heat conduction equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\kappa} \frac{\partial u}{\partial t} \quad \text{--- [1] with initial and}$$



$$u(0,t) = 0 \qquad u(1,t) = 0$$

B.C

$$u(x,0) = 3 \sin 2\pi x$$

$$u(x,0) = 3 \sin 2\pi x$$

$$u(0,t) = 0$$

$$u(1,t) = 0$$

Apply L.T with respect to time for both sides of eq. [1]

$$\frac{d^2 U}{dx^2} = s^2 U - u(x,0) \qquad [U = \dot{u}(x,s)]$$

$$\frac{d^2 U}{dx^2} - s^2 U = - 3 \sin 2\pi x \qquad \text{--- [2]}$$

a. Homogeneous solution:

$$U'' - s^2 U = 0 \qquad \text{--- } \gamma^2 - s^2 = 0 \text{ or}$$

$$\gamma = \pm \sqrt{s^2}$$

$$\therefore U_h = C_1 e^{\sqrt{s}x} + C_2 e^{-\sqrt{s}x} \qquad \text{--- [3]}$$

b. particular solution

$$\text{Assume } U_p = A \sin 2\pi x$$

[لاحظ ان المعادلة تحتوي على الجيب فقط لذا $U_p = A \sin + B \cos$]

و فقط ، لكن اذا احتوت المعادلة على الجيب وال cos فالحل يكون $\sin + \cos$

10

differentiate twice $U_p'' = -4\pi^2 A \sin 2\pi x$

sub. for U_p, U_p'' in [2]

$$-4\pi^2 A \sin 2\pi x + (-s) (A \sin 2\pi x) = -3 \sin 2\pi x$$

$$-A \sin 2\pi x (s^2 + 4\pi^2) = -3 \sin 2\pi x$$

$$\therefore A = \frac{3}{s^2 + 4\pi^2}$$

$$\text{Then } U_p = \frac{3}{s^2 + 4\pi^2} \sin 2\pi x$$

The general solution is

$$U(x, s) = \frac{3}{s^2 + 4\pi^2} \sin 2\pi x + C_1 e^{\sqrt{s}x} + C_2 e^{-\sqrt{s}x} \quad [4]$$

Evaluate the constant C_1 & C_2

$$u(0, t) = 0 \xrightarrow{\mathcal{L}} U(0, s) = 0$$

$$u(1, t) = 0 \xrightarrow{\mathcal{L}} U(1, s) = 0$$

From the 1st B.C i.e $x=0$, $U(0, s) = 0$

then

$$0 = \frac{3}{s^2 + 4\pi^2} \sin(0) + C_1 + C_2 \quad ; \quad \sin(0) = 0$$

\therefore

$$C_1 + C_2 = 0$$

Apply the 2nd B.C $U(1, s) = 0$ [i.e $x=1$]

11

14

$$0 = \frac{3}{s^2 + 4\pi^2} \sin 2\pi x + c_1 e^{\sqrt{s}} + c_2 e^{-\sqrt{s}}$$

$$c_1 e^{\sqrt{s}} + c_2 e^{-\sqrt{s}} = 0$$

or

$$e^{\sqrt{s}} \neq e^{-\sqrt{s}} \neq 0$$

Since

$$c_1 = c_2 = 0$$

So

$$U(x, s) = \frac{3}{s^2 + 4\pi^2} \boxed{\sin 2\pi x}$$

Apply ILT :

$$U(x, t) = 3 \sin 2\pi x e^{-4\pi^2 t}$$

$$[\mathcal{L}(e^{at}) = \frac{1}{s-a}]$$

as a check

$$U(x, 0) = 3 \sin 2\pi x$$

$$U(0, t) = 0$$

$$\sin 0 = 0$$

$$U(1, t) = 0$$

$$\sin 2\pi = 0$$

$$\lim_{t \rightarrow \infty} U(x, t) = 0$$

10:54 P.M 16-1-05

Coll. Time!

12

5

PDE

Separation of variable

Remark: If u_1, u_2, \dots, u_n are solutions of a linear homog. PDE then $C_1 u_1 + C_2 u_2 + \dots + C_n u_n$ where $C_1, C_2, C_3, \dots, C_n$ are constants is also a solution.

Solve $\frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2}$ (I) $0 \leq x \leq 3, t > 0$

$u(0, t) = u(3, t) = 0$

$u(x, 0) = 5 \sin 4\pi x - 3 \sin 8\pi x + 2 \sin 10\pi x$

$|u(x, t)| < M$ i.e. the function u is bounded for $0 \leq x \leq 3, t > 0$

Solution:-

let $u = X T$

$u_t = X T'$ $\frac{\partial u}{\partial t} = X T'$ $T = \frac{dT}{dt}$

$u_{xx} = X'' T$

Sub. for u_t, u_{xx} in (I)

$X T' = 2 X'' T$

or

$\frac{X''}{X} = \frac{T'}{2T} = \boxed{-\lambda^2}$

So either

$X'' + \lambda^2 X = 0$

then

$X = A \cos \lambda x + B \sin \lambda x$

or $T' + 2\lambda^2 T = 0 \rightarrow \boxed{T = C_1 e^{-2\lambda^2 t}}$

Since $u = T^1 X$

then $u(x,t) = e^{-2\lambda^2 t} (A \cos \lambda x + B \sin \lambda x)$ [2]

$$A = C_1 A_1 \quad B = C_2 B_1$$

To evaluate A & B

$$u(0,t) = 0$$

$$\therefore 0 = A e^{-2\lambda^2 t} \Rightarrow A = 0$$

i.e. $u(x,t) = B e^{-2\lambda^2 t} \sin \lambda x$

since $u(3,t) = 0$

$$0 = B e^{-2\lambda^2 t} \sin 3\lambda$$

$\therefore B = 0$, the solution is identically zero
(Trivial solution)

\therefore we must have

$$\sin 3\lambda = 0$$

or

$$3\lambda = n\pi \quad [n = 0, \pm 1, \pm 2, \dots, n \text{ is integer}]$$

$$\left\{ \begin{array}{l} \sin 0 = 0 \\ \sin \pi = 0 \\ \sin 2\pi = 0 \end{array} \right\}$$

or

$$\lambda = \frac{n\pi}{3}$$

$$\therefore u(x,t) = B e^{-2 \frac{n^2 \pi^2 t}{9}} \sin\left(\frac{n\pi}{3}\right) x$$

14

8

Also by principle of super position B.C is a sum of three terms

$$u(x,t) = B_1 e^{-2 \frac{n_1^2 \pi^2}{9} t} \sin\left(\frac{n_1 \pi}{3}\right) x + B_2 e^{-2 \frac{n_2^2 \pi^2}{9} t} \sin\left(\frac{n_2 \pi}{3}\right) x$$

$$+ B_3 e^{-2 \frac{n_3^2 \pi^2}{9} t} \sin\left(\frac{n_3 \pi}{3}\right) x$$

Finally apply the last B.C $u(x,0)$
[e⁰ = 1]

$$B_1 \sin\left(\frac{n_1 \pi}{3}\right) x + B_2 \sin\left(\frac{n_2 \pi}{3}\right) x + B_3 \sin\left(\frac{n_3 \pi}{3}\right) x$$

$$=$$

$$5 \sin 4\pi x - 3 \sin 8\pi x + 2 \sin 10\pi x$$

By Comparison

$$\boxed{B_1 = 5}, \quad \boxed{\frac{n_1}{3} = 4}, \quad \boxed{B_2 = -3}, \quad \boxed{\frac{n_2}{3} = 8}$$

$$\boxed{B_3 = 2} \quad \boxed{\frac{n_3}{3} = 10}$$

∴ The required solution is

$$u(x,t) = 5 e^{-32\pi^2 t} \sin 4\pi x - 3 e^{-128\pi^2 t} \sin 8\pi x$$

$$+ 2 e^{-200\pi^2 t} \sin 10\pi x$$

0:15 A.M

17/11/2005

15

Solve:- $\frac{\partial^2 y}{\partial t^2} = 16 \frac{\partial^2 y}{\partial x^2} \quad 0 < x < 2$

Subjected to condition $y(0,t) = 0$, $y_t(x,0) = 0$

$$y(2,t) = 0$$

$$y(x,0) = 6 \sin \pi x - 3 \sin 4\pi x$$

Solution:-

let $y = X T$, $X = f(x)$, $T = f(t)$

then

$$y_{tt} = X T'' \quad y_{xx} = X'' T$$

then

$$X T'' = 16 X'' T$$

or

$$\frac{X''}{X} = \frac{T''}{16 T} = -\lambda^2$$

then

either

$$X = a_1 \cos \lambda x + b_1 \sin \lambda x$$

or

$$T = a_2 \cos 4\lambda t + b_2 \sin 4\lambda t$$

∴ The solution is

$$y(x,t) = (a_1 \cos \lambda x + b_1 \sin \lambda x) (a_2 \cos 4\lambda t + b_2 \sin 4\lambda t)$$

Since $y(0,t) = 0$, then

$$0 = [a_1 \cos(0) + b_1 \sin(0)] [a_2 \cos 4\lambda t + b_2 \sin 4\lambda t]$$

$$\cos(0) = 1 \quad \sin(0) = 0$$

$$0 = a_1 [a_2 \cos 4\lambda t + b_2 \sin 4\lambda t]$$

16

7

So, to obtain a non zero solution we must have $a_1 = 0$

then

$$y(x,t) = b_1 \sin \lambda x [a_2 \cos 4\lambda t + b_2 \sin 4\lambda t]$$

diff. both sides with respect to $[t]$

$$y_t(x,t) = b_1 \sin \lambda x (-4\lambda a_2 \sin 4\lambda t + 4\lambda b_2 \cos 4\lambda t)$$

$\sin \quad y_t(x,0) = 0$

then

$$0 = b_1 \sin \lambda x (-4\lambda a_2 \sin(0) + 4\lambda b_2 \cos(0))$$

or

$$b_1 \sin \lambda x (4\lambda b_2) = 0$$

$$\therefore b_2 = 0 \quad [\text{To avoid trivial solution}]$$

then

$$y_t(x,t) = B \sin \lambda x \cos 4\lambda t \quad [B = b_1 a_2]$$

3rd condition $y(2,t) = 0$

$$B \sin 2\lambda \cos 4\lambda t = 0$$

\therefore We must have

$$\sin 2\lambda = 0$$

$$\text{or } 2\lambda = n\pi \quad [n, \text{ integer}]$$

$$\lambda = \frac{n\pi}{2}$$

$$y(x,t) = B \sin \frac{n\pi}{2} x \cos 2n\pi t$$

To evaluate the constant B apply the principle of superposition:-

17

16213

$$y(x,t) = B_1 \sin \frac{n_1 \pi}{2} x \cos 2n_1 \pi t + B_2 \sin \frac{n_2 \pi}{2} x$$

$$\cos 2n_2 \pi t$$

finally apply the last condition $y(x,0) =$

$$B_1 \sin \frac{n_1 \pi}{2} x + B_2 \sin \frac{n_2 \pi}{2} x = 6 \sin \pi x - 3 \sin 4\pi x$$

so

$$B_1 = 6$$

$$\frac{n_1 \pi}{2} = \pi$$

$$\Rightarrow n_1 = 2$$

$$B_2 = -3$$

$$\frac{n_2 \pi}{2} = 4\pi$$

$$n_2 = 8$$

so

$$y(x,t) = 6 \sin \pi x \cos 4\pi t - 3 \sin 4\pi x \cos 16\pi t$$

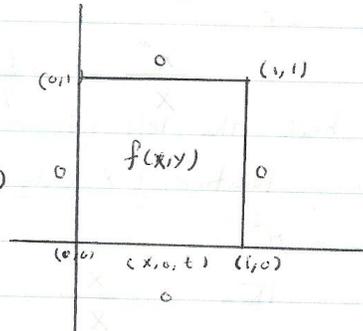
18

18

A square plate with sides of unit length has its faces insulated and its sides kept at 0°C . If the initial temperature is specified, determine the sub-sequent temperature at any point of the plate. Hint: assume $k=1$

Solution:

The equation of temperature $u(x,y,t)$ at any point (x,y) at time t is



$$\frac{\partial u}{\partial t} = k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \text{--- (1)}$$

The boundary Conditions are:-

$$u(0,y,t) = u(1,y,t) = u(x,0,t) = u(x,1,t) = 0$$

$u(x,y,t)$ is bounded; $0 < x < 1, 0 < y < 1$.

$$u(x,y,0) = f(x,y) \quad \text{[I.C]}$$

finally $u(x,y,t)$ is bounded

[$u(0,y,t)$: left side, $u(1,y,t)$: Right side, $u(x,0,t)$: Bottom
 $u(x,1,t)$: Top]

Solution: let $u = XYT$ [$X = f(x), Y = f(y), T = f(t)$]

Then eq. (1) becomes:

$$XYT' = k [X''YT + XY''T]$$

Dividing by $kXYT$ yields

$$\frac{T'}{kT} = \frac{X''}{X} + \frac{Y''}{Y}$$

here The left side is a function of t alone, while the right side is a function of x and y . we see that each side must be a constant, say $-\lambda^2$ (which is needed for boundedness) Thus

$$\nabla' + k\lambda^2 \nabla = 0 \quad \text{--- [2]}$$

and

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda^2$$

or

$$\frac{X''}{X} = -\frac{Y''}{Y} - \lambda^2$$

here the left side is a function of x only and the right side is a function of y only, \therefore each side must be a constant, say

$-\mu^2$

$$\text{Then } \frac{X''}{X} = -\frac{Y''}{Y} - \lambda^2 = -\mu^2$$

or

$$X'' + \mu^2 X = 0 \quad \text{--- [3]}$$

$$Y'' + (\lambda^2 - \mu^2) Y = 0 \quad \text{--- [4]}$$

From [2]

$$\nabla = a e^{-k\lambda^2 t} \quad \text{--- (5)}$$

From [3]

$$X = b \cos \mu x + c \sin \mu x \quad \text{--- (6)}$$

finally

$$Y = d \cos \sqrt{\lambda^2 - \mu^2} y + e \sin \sqrt{\lambda^2 - \mu^2} y \quad \text{--- (7)}$$

where a, b, c, d, e, μ are arbitrary constants

It follows the solution of eq. (1) is

$$u(x, y, t) = a e^{-k\lambda^2 t} (b \cos \mu x + c \sin \mu x) (d \cos \sqrt{\lambda^2 - \mu^2} y + e \sin \sqrt{\lambda^2 - \mu^2} y)$$

To evaluate the constant, apply the given conditions

$$u(0, y, t) = 0 \quad \text{[i.e. } x=0]$$

$$0 = a e^{-k\lambda^2 t} (b + 0) (d \cos \sqrt{\lambda^2 - \mu^2} y + e \sin \sqrt{\lambda^2 - \mu^2} y)$$

20

(11)

∴ b = 0

∴

U(x,y,t) = (c sin μx + d cos √λ² - μ² y + e sin √λ² - μ² y) e^{-κλ²t}

U(x,y,t) = a e^{-κλ²t} (c sin μx) (d cos √λ² - μ² y + e sin √λ² - μ² y)

and U(x,0,t) = 0 (y=0)

U(x,0,t) = 0 = a e^{-κλ²t} (c sin μx) (d + e)

∴ d = 0

i.e

U(x,y,t) = A e^{-κλ²t} (sin μx sin √λ² - μ² y)

[A = ace]

U(1,y,t) = 0

0 = A e^{-κλ²t} (sin μ sin √λ² - μ² y)

i.e

sin μ = 0 or μ = nπ (n: integer)

∴

U(x,y,t) = A e^{-κλ²t} (sin nπx sin √λ² - μ² y)

U(x,1,t) = 0

0 = A e^{-κλ²t} (sin nπx sin √λ² - μ²)

Then

sin √λ² - μ² = 0 or √λ² - μ² = mπ

(m: integer)

∴ λ = √(m² + n²) π [μ = nπ]

21

It follows that the solution satisfying all the conditions except $u(x, y, 0) = f(x, y)$ is given by:

$$u(x, y, t) = A e^{-k(m^2+n^2)\pi^2 t} \sin n\pi x \sin m\pi y$$

Apply the principle of superposition

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} e^{-k(m^2+n^2)\pi^2 t} \sin n\pi x \sin m\pi y$$

Since $u(x, y, 0) = f(x, y)$

Then:

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin n\pi x \sin m\pi y$$

[which is a double Fourier ~~sine series~~ sine series]

where:

$$B_{nm} = \frac{4}{L_1 L_2} \int_0^{L_1} \int_0^{L_2} f(x, y) \sin \frac{n\pi x}{L_1} \sin \frac{m\pi y}{L_2} = A_{nm}$$

but here $L_1 = 1$ $L_2 = 2$

$$A_{nm} = 4 \int_0^1 \int_0^1 f(x, y) \sin n\pi x \sin m\pi y$$

Q. 4

PDE

22

8

EX:- Solve The B. Value Prob.

a. $\frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2}$; $u(0,t) = u(4,t) = 0$

$u(x,0) = 25x$

b. Interpret physically the B. value prob. in a

Solution:

Assume $u(x,t) = X T$, $X = X(x)$, $T = T(t)$

The

$X T' = 2 X'' T$

$\frac{T'}{2T} = \frac{X''}{X} = -\lambda^2$ [λ is constant]

$X'' + \lambda^2 X = 0 \Rightarrow X = A \sin \lambda x + B \cos \lambda x$

$T' + 2\lambda^2 T = 0 \Rightarrow T = C e^{-2\lambda^2 t}$

Then:

$u(x,t) = C e^{-2\lambda^2 t} [A \sin \lambda x + B \cos \lambda x]$

or

$u(x,t) = D e^{-2\lambda^2 t} \sin \lambda x + E e^{-2\lambda^2 t} \cos \lambda x$

[$D = CA$; $E = CB$]

Apply The 1st B.C $u(0,t) = 0$ [i.e $x=0$ for any t]

$0 = D e^{-2\lambda^2 t} \sin 0 + E e^{-2\lambda^2 t} \cos(0)$

$\therefore E = 0$

or

$u(x,t) = D e^{-2\lambda^2 t} \sin \lambda x$

H.A. Baki
2008

2nd B.C : u(4,t) = 0

or $0 = D e^{-2\lambda^2 t} \sin 4\lambda$ [if D=0, Trivial sol.]

or $\sin 4\lambda = 0$ or $4\lambda = n\pi$

i.e $\lambda = \frac{n\pi}{4}$

$u(x,t) = \sum_{n=1}^{\infty} D_n e^{-2(\frac{n\pi}{4})^2 t} \sin \frac{n\pi x}{4}$

Finally $u(x,0) = 25x$

$25x = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi}{4} x$

[which is a fourier sine series]

here $D_n \equiv b_n$, $L=4$

$[f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x]$

By definition $b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx$
but $f(x) = 25x$

Then

$b_n = \frac{2}{4} \int_0^4 25x \sin \frac{n\pi}{4} x dx$

$b_n = \frac{25}{2} \left[\frac{\sin \frac{n\pi}{4} x}{(\frac{n\pi}{4})^2} - \frac{x \cos \frac{n\pi}{4} x}{\frac{n\pi}{4}} \right]_0^4$

$= \frac{25}{2} \left[0 - \frac{4 \cos n\pi}{\frac{n\pi}{4}} \right]$

24

PPE

4

$$b_n = \frac{-200}{n\pi} \cos n\pi$$

$$i.e. \quad U(x,t) = \frac{-200}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\pi}{n} e^{-2\left(\frac{n\pi}{4}\right)^2 t} \sin \frac{n\pi}{4} x$$

b. We have a solid bar whose length = 4 units
 , the temp. at each end of the bar kept constant ($T=0$)
 while the initial temp. = $25x$

Ex:- Show that the solution of B. Value prob.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u_x(0,t) = u_x(\pi,t) = 0$$

$$u(x,0) = f(x) \quad 0 < x < \pi, t > 0$$

is given by:-

$$u(x,t) = \frac{1}{\pi} \int_0^{\pi} f(x) dx + \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} \cos nx \int_0^{\pi} f(x) \cos nx dx$$

Solution:

$$\text{let } u(x,t) = X T \quad \text{by same procedure}$$

$$u(x,t) = e^{-\lambda^2 t} [A \cos \lambda x + B \sin \lambda x]$$

$$u_x = \frac{\partial u}{\partial x} \Big|_{\pi}$$

$$u_x = e^{-\lambda^2 t} [-A \lambda \sin \lambda x + B \lambda \cos \lambda x]$$

Apply B.C $u_x = 0$

$$0 = e^{-\lambda^2 t} (-\lambda A \sin 0 + \lambda B \cos 0) \quad \therefore B = 0$$

H.A. Baku
2008

25

$$\therefore U_x(x,t) = e^{-\lambda^2 t} (-\lambda A \sin \lambda x)$$

$$\text{also } U_x(\pi, t) = 0$$

$$0 = e^{-\lambda^2 t} \lambda A \sin \lambda \pi$$

So:

$$\sin \lambda \pi = 0$$

$$\text{or } \lambda \pi = n\pi \Rightarrow \lambda = n$$

$$\text{So } e^{-n^2 t}$$

$$U(x,t) = A e^{-n^2 t} \cos n x$$

[Fourier Cosine Series where:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}]$$

Evaluate The Constant A

$$\text{Apply I.C } U(x,0) = f(x)$$

$$\therefore f(x) = \sum_{n=0}^{\infty} A_n \cos n x$$

$$a_0 = \frac{2}{L} \int_0^L f(x) dx$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos n x dx$$

$$\therefore U(x,t) = \frac{1}{\pi} \int_0^{\pi} f(x) dx + \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} \cos n x \int_0^{\pi} f(x) \cos n x dx$$

H.A. Palle
2008