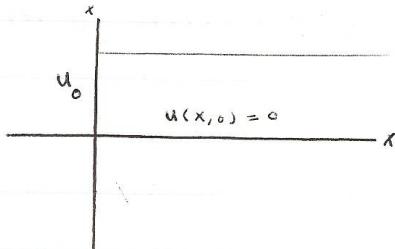


Applications of PDE [1]

Adv. Eng. Math.

- [1] A semi-infinite solid $x > 0$ is initially at temperature zero. At time $t=0$ a constant temperature $U_0 > 0$ is applied and maintained at the surface $x = 0$. Find the temperature at any point of the solid at any later time $t > 0$.

Solution: The boundary-value problem for the determination of the temperature $U(x, t)$ at any point x @ any time $t > 0$ is:



$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2} \quad x > 0, t > 0 \quad \text{and}$$

$$u(x, 0) = 0 \quad u(0, t) = U_0 \quad u(x, t) \text{ is bounded}$$

Apply LT WRT T

$$S U(s) - u(x, 0) = K \frac{d^2 U}{dx^2} \quad U = U(x, s)$$

but $u(x, 0) = 0$

$$\therefore U_{xx} - \frac{S}{K} U = 0$$

$$U(x, s) = A e^{\sqrt{S/K} x} + B e^{-\sqrt{S/K} x}$$

$$\therefore U(x, s) = \lim_{x \rightarrow \infty} \left[A e^{\sqrt{S/K} x} + B e^{-\sqrt{S/K} x} \right]$$

Since $u(x, t)$ is bounded

$$\therefore A = 0 \quad \text{and}$$

$$U(x, s) = B e^{-\sqrt{S/K} x}$$

To evaluate the constant B $U(0, t) = U_0$ (given)

$$\therefore \text{by applying L.T} \quad U(0, s) = \frac{U_0}{s^{\alpha}} \quad \text{and}$$

$$\frac{U_0}{s^{\alpha}} = B e^0 \quad \text{or}$$

$$B = \frac{U_0}{s^{\alpha}}$$

$$\text{and} \quad U(x, s) = U_0 \frac{e^{-\sqrt{S/K} x}}{s^{\alpha}}$$

Applications of PDE [2]

Adv. Eng. Meth.

Apply ILT, to get

$$u(x,t) = u_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right) \quad (\text{Spiegel, MHB})$$

entry 32. III p. 169)

$$\operatorname{erfc}(0) = 1 \quad \operatorname{erfc}(\infty) = 0$$

$$x = 0$$

$$u(0,t) = u_0 \operatorname{erfc}(0)$$

$$= u_0$$

$$t \rightarrow \infty$$

$$\lim_{t \rightarrow \infty} u(x,t) = u_0 \lim_{t \rightarrow \infty} \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right)$$

$$= u_0$$

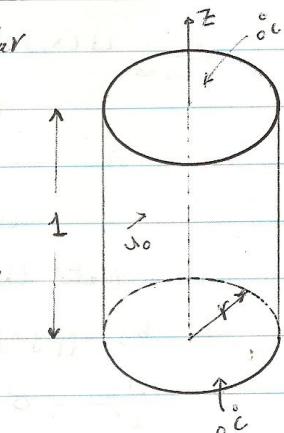
$$\text{also } @ t = 0$$

$$\lim_{t \rightarrow 0} u(x,t) = u_0 \lim_{t \rightarrow 0} \operatorname{erfc}\left(\frac{x}{2\sqrt{2k}}\right)$$

$$= 0$$

- 2- A cylinder of unit radius and height has its circular ends maintained at temp. zero, while its convex surface is maintained @ constant temperature of u_0 . Assuming that the cylinder has its axis coincident with the z -axis find the steady state temperature @ any distance r from the axis and z from one end. Hint: the S.S. equation is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$$



Application of PDE [3]

Adv. Eng. Math.

Solution:

let $u(r, z) = Rz$ where $R = f(r)$ and $Z = f(z)$

then:

$$\frac{\partial u}{\partial r} = R'z, \quad \frac{\partial^2 u}{\partial r^2} = R''z \text{ and } \frac{\partial^2 u}{\partial z^2} = Rz''$$

sub. in a given equation

$$R''z + \frac{1}{r} R'z + Rz'' = 0, \quad \text{div. both sides by } Rz$$

$$\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{z''}{z} = 0$$

or

$$\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} = -\frac{z''}{z} = \lambda^2 \quad [\text{where } \lambda \text{ is constant}]$$

consider

$$\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} = \lambda^2$$

$$R'' + \frac{1}{r} R' - \lambda^2 R = 0; \quad \text{multiply both sides by } r^2$$

$$r^2 R'' + r R' - \lambda^2 r^2 R = 0, \quad \text{which is a modified}$$

Bessel equation of order zero whose general solution

$$R = C_1 I_0(\lambda r) + C_2 K_0(\lambda r)$$

since $I_0(0) = 0$ and $K_0(0) = \infty$ then $C_2 = 0$

$$\therefore R = C_1 I_0(\lambda r)$$

also

$$-\frac{z''}{z} = \lambda^2 \Rightarrow z'' + \lambda^2 z = 0$$

$$\therefore z = a_1 \cos \lambda z + a_2 \sin \lambda z$$

Applications of PDE [4]

Adv Engg. Math.

Therefore

$$u(r, z) = I_0(\lambda r) (A \cos \lambda z + B \sin \lambda z) \quad \text{--- [1]}$$

Boundary Conditions:

$$u(r, 0) = 0 \quad u(1, z) = u_0 \quad u(r, 1) = 0$$

Apply above conditions to evaluate the constants

[a] $u(r, 0) = 0 \quad \text{sub in [1]}$

$$0 = I_0(\lambda r) (A \cos 0 + B \sin 0)$$

~~if~~ \Rightarrow Then $A = 0$

$$\therefore u(r, z) = B I_0(\lambda r) \sin \lambda z \quad \text{--- [2]}$$

[b] $u(r, 1) = 0 \quad \text{sub in [2]}$

$$0 = B I_0(\lambda r) \sin \lambda$$

$$\therefore \lambda = m\pi \quad m = 0, 1, 2, 3, \dots$$

Therefore

$$u(r, z) = B I_0(m\pi r) \sin m\pi z$$

Apply the principle of superposition

$$u(r, z) = \sum_{m=1}^{\infty} B_m I_0(m\pi r) \sin m\pi z$$

Finally

$$u(1, z) = u_0$$

$$u_0 = \sum_{m=1}^{\infty} B_m I_0(m\pi) \sin m\pi z$$

which is a Fourier Sine Series

i.e

$$I_0(m\pi) B_m = \frac{2}{l} \int_0^l u_0 \sin m\pi z \, dz$$

$$I_0(m\pi) B_m = \left(2 u_0 / m\pi\right) \left[\cos m\pi z \right]_0^l$$

Applications of PDE [E]

Adv. Eng. Math.

$$I_0(mn) B_m = \frac{2U_0}{m\pi} [1 - \cos m\pi]$$

if m is even then $B_m = 0$

let $m = 2n-1$

then:

$$B_{2n-1} = \frac{2U_0}{(2n-1)\pi} \frac{[1 - \cos((2n-1)\pi)]}{I_0[(2n-1)\pi]}$$

Finally:

$$1 - \cos((2n-1)\pi) = 2$$

Ansatz

$$u(r, z) = \frac{4U_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \cdot \frac{I_0[(2n-1)\pi r] \sin((2n-1)\pi z)}{I_0[(2n-1)\pi]}$$

$$z=0 \quad z=1 \quad \boxed{u=0}$$

$$\boxed{r=1}$$

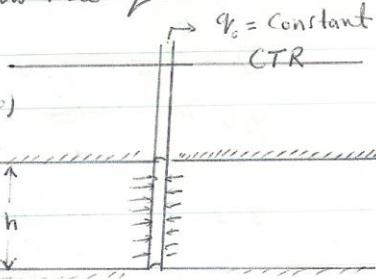
$$u(1, z) = \frac{4U_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin((2n-1)\pi z)$$

- [3] - An oil well completely penetrates a semi-infinite producing reservoir of thickness h , derive an expression for the drawdown distribution $\Delta(r, t)$ where $\Delta = P_i - P(r, t)$, if the well is producing at a constant flow rate q

Hint:

a - apply line-source approach (i.e. $r_w \rightarrow 0$)

b) $\lim_{x \rightarrow 0} x K_1(x) = 1$



c) $\tilde{L}^{-1} \left[\frac{K_0(a\sqrt{s})}{as^2} \right] = -\frac{1}{2} E_i(-u)$

where $u = \frac{1}{4} \frac{a^2}{t}$

Application of PDE [6]

Adv. Eng. Math.

Solution:

$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r} \frac{\partial P}{\partial r} = \frac{1}{\eta} \frac{\partial P}{\partial t} \quad \text{--- (I)}$$

where

$$\eta = \frac{K}{\phi \gamma C} \Rightarrow \text{Hydraulic diffusivity}$$

by definition $\Delta = P_i - P$ and $\Delta = \Delta(r, t)$

then

$$\frac{\partial \Delta}{\partial r} = - \frac{\partial P}{\partial r} \quad \frac{\partial^2 \Delta}{\partial r^2} = - \frac{\partial^2 P}{\partial r^2}$$

$$\frac{\partial \Delta}{\partial t} = - \frac{\partial P}{\partial t} \quad \text{Sub. in (I)}$$

$$\frac{\partial^2 \Delta}{\partial r^2} + \frac{1}{r} \frac{\partial \Delta}{\partial r} = \frac{1}{\eta} \frac{\partial \Delta}{\partial t} \quad \text{--- (II)}$$

Initial and B.C

$$P(r, t) = P_i \quad \text{for } t=0 \quad \text{for all } r$$

$$\Delta(r, 0) = 0 \quad \text{for all } r \quad [\text{initial condition}]$$

B.C:

$$\text{inner B.C} \quad q_o = \text{constant}$$

according to Darcy's law (radial coordinate)

$$q_o = \frac{A K}{\mu} \frac{\partial P}{\partial r} \quad \text{here } A = 2\pi r h$$

apply line source Approach

$$q_o = \lim_{r \rightarrow 0} \frac{2\pi r h K}{\mu} : \frac{\partial P}{\partial r} \quad \text{--- (III)}$$

if we define the term $\frac{Kh}{\mu}$ as transmissivity of the res. then

Application of PDE [7]

Adv. Eng. Math.

eq. [3] became:

$$q = \lim_{r \rightarrow 0} 2\pi r T \frac{\partial P}{\partial r}$$

$$\text{but } \frac{\partial P}{\partial r} = -\frac{\Delta}{\alpha t}$$

$$\therefore q = -\lim_{r \rightarrow 0} 2\pi r T \frac{\Delta}{\alpha t} \quad -[4]$$

Apply L.M.W.M.P for both sides of eq. [2]

$$\frac{d^2 \Delta}{dr^2} + \frac{1}{r} \frac{d\Delta}{dr} = \frac{1}{\eta} [s' \Delta - \Delta(r, 0)] , \quad \Delta = \Delta(r, s)$$

$$\text{but } \Delta(r, 0) = 0 \quad (\text{I.C.})$$

$$\frac{d^2 \Delta}{dr^2} + \frac{1}{r} \frac{d\Delta}{dr} - \frac{s'}{\eta} \Delta = 0 \quad -[5]$$

multiply both sides of eq. [5] by r^2

$$r^2 \frac{d^2 \Delta}{dr^2} + r \frac{d\Delta}{dr} - \frac{s'}{\eta} r^2 \Delta = 0 \quad -[6]$$

which is modified Bessel equation of order zero

$$\therefore \Delta(r, s) = A I_0(\sqrt{\frac{s}{\eta}} r) + B K_0(\sqrt{\frac{s}{\eta}} r) \quad -[7]$$

Since $r \rightarrow \infty$ (semi-infinite ver.) and

$$\lim_{x \rightarrow \infty} I_0(x) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} K_0(x) = 0$$

$\therefore A = 0$

and

$$\Delta(r, s) = B K_0(\sqrt{\frac{s}{\eta}} r) \quad -[8]$$

diff. both sides of eq. [8] W.R.T. r

Application of PDE [8]

Adv Eng Math

$$\frac{d\Delta}{dr} = -B \sqrt{\frac{s}{n}} K_1 \left(\sqrt{\frac{s}{n}} r \right) \quad \square$$

$$(K_n(x) = K_n(x) \text{ and } K'_n(x) = -\frac{1}{2} [K_{n-1}(x) + K_{n+1}(x)])$$

recall eq. 4

$$q = -2\pi T \lim_{r \rightarrow 0} r \frac{d\Delta}{dr} \quad \text{Apply LT for both sides}$$

$$\frac{Q}{n^s} = -2\pi T \lim_{r \rightarrow 0} r \frac{d\Delta}{dr} \quad \square \quad (q = \text{constant})$$

make use of eq [9] in [10], then

$$\frac{Q}{n^s} = -2\pi T \lim_{r \rightarrow 0} r \left(-B \sqrt{\frac{s}{n}} K_1 \left(\sqrt{\frac{s}{n}} r \right) \right)$$

or

$$Q = 2\pi s^s T B \lim_{r \rightarrow 0} r \sqrt{\frac{s}{n}} K_1 \left(\sqrt{\frac{s}{n}} r \right)$$

$$\text{since } \lim_{x \rightarrow 0} x K_1(x) = 1$$

$$\text{or } Q = 2\pi s^s T B$$

or

$$B = \frac{Q}{2\pi s^s T}$$

sub. for [B] in eq [8]

$$\Delta(r, s) = \frac{Q}{2\pi T} \cdot \frac{K_0 \left(\sqrt{\frac{s}{n}} r \right)}{s^s} \quad \square$$

Application of PDE [9]

Adv. Eng. Math.

Apply ILT for both sides of eq. [1] with $a = \frac{r}{\sqrt{\eta}}$

$$\Delta(r,t) = -\frac{q}{4\pi\eta} E_i\left(-\frac{1}{4} \frac{r^2}{\eta t}\right) \quad [12]$$

and $-E_i(-u) = \int_u^\infty \frac{e^{-t}}{t} dt$, Exponential integral function

$$\text{since } \Delta(r,t) = P_i - P(r,t)$$

$$P_i - P(r,t) = -\frac{q}{4\pi\eta} E_i\left(-\frac{1}{4} \frac{r^2}{\eta t}\right)$$

or

$$P(r,t) = P_i + \frac{q}{4\pi\eta} E_i\left(-\frac{1}{4} \frac{r^2}{\eta t}\right)$$