Cramer's Rule: (Using Determinants)

$$a1x + b1y + c1z = m1$$

$$a2x + b2y + c2\,z = m2$$

$$a3x + b3y + c3z = m3$$

$$\begin{bmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} m1 \\ m2 \\ m3 \end{bmatrix}$$ where

$$X = \frac{|Dx|}{|D|} \qquad y = \frac{Dy}{|D|} \qquad z = \frac{Dz}{|D|}$$

$$\Downarrow$$

$$\frac{\begin{vmatrix} m1 & b1 & c1 \\ m2 & b2 & c2 \\ m3 & b3 & c3 \end{vmatrix}}{\begin{vmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{vmatrix}} \qquad \frac{\begin{vmatrix} a1 & m1 & c1 \\ a2 & m2 & c2 \\ a3 & m3 & c3 \end{vmatrix}}{\begin{vmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{vmatrix}} \qquad \frac{\begin{vmatrix} a1 & b1 & m1 \\ a2 & b2 & m2 \\ a3 & b3 & m3 \end{vmatrix}}{\begin{vmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{vmatrix}}$$

Solve the following system by using determinants.

$$\begin{cases} 4x - 3y = -14 \\ 3x - 5y = -5 \end{cases}$$

To solve this system, three determinants are created. One is called the **denominator determinant**, labeled *D*; another is the ***x*-numerator determinant**, labeled D_x; and the third is the ***y*-numerator determinant**, labeled D_y

.

The denominator determinant, *D*, is formed by taking the coefficients of *x* and *y* from the equations written in standard form.

$$D = \begin{vmatrix} 4 & -3 \\ 3 & -5 \end{vmatrix}$$

$$= (4)(-5) - (3)(-3)$$

$$= -20 - (-9)$$

$$= -20 + 9$$

$$= -11$$

The *x*-numerator determinant is formed by taking the constant terms from the system and placing them in the *x*-coefficient positions and retaining the *y*-coefficients.

$$D_x = \begin{vmatrix} -14 & -3 \\ -5 & -5 \end{vmatrix}$$

$$= (-14)(-5) - (-5)(-3)$$

$$= 70 - 15$$

$$= 55$$

The y-numerator determinant is formed by taking the constant terms from the system and placing them in the y-coefficient positions and retaining the x-coefficients.

$$D_y = \begin{vmatrix} 4 & -14 \\ 3 & -5 \end{vmatrix}$$
$$= (4)(-5) - (3)(-14)$$
$$= -20 - (-42)$$
$$= -20 + 42$$
$$= 22$$

The answers for x and y are as follows:

$$x = \frac{D_x}{D} = \frac{55}{-11} = -5 \quad y = \frac{D_y}{D} = \frac{22}{-11} = -2$$

Use Cramer's Rule to solve this system.

$$\begin{cases} 4x+6y=3 \\ 8x-3y=1 \end{cases}$$

$$D=\begin{vmatrix} 4 & 6 \\ 8 & -3 \end{vmatrix}=-12-48=-60$$

$$D_x=\begin{vmatrix} 3 & 6 \\ 1 & -3 \end{vmatrix}=-9-6=-15$$

$$D_y=\begin{vmatrix} 4 & 3 \\ 8 & 1 \end{vmatrix}=4-24=-20$$

$$x=\frac{D_x}{D}=\frac{-15}{-60}=\frac{1}{4},\quad y=\frac{D_y}{D}=\frac{-20}{-60}=\frac{1}{3}$$

$$\begin{cases} x+2y+3z=-5 \\ 3x+y-3z=4 \\ -3x+4y+7z=-7 \end{cases}$$

From the given system of linear equations, I will construct the four matrices that will be used to solve for the values of x, y, and z.

- **coefficient matrix**

$$D = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & -3 \\ -3 & 4 & 7 \end{bmatrix}$$

- **X – matrix**

$$D_x = \begin{bmatrix} -5 & 2 & 3 \\ 4 & 1 & -3 \\ -7 & 4 & 7 \end{bmatrix}$$

- **Y – matrix**

$$D_y = \begin{bmatrix} 1 & -5 & 3 \\ 3 & 4 & -3 \\ -3 & -7 & 7 \end{bmatrix}$$

- **Z – matrix**

$$D_z = \begin{bmatrix} 1 & 2 & -5 \\ 3 & 1 & 4 \\ -3 & 4 & -7 \end{bmatrix}$$

The values of the determinants are listed below.

Determinants of each matrix:

$$|D| = 40$$

$$|D_x| = -40$$

$$|D_y| = 40$$

$$|D_z| = -80$$

Solved values for x, y, and z.

$$x = \frac{|D_x|}{|D|} = \frac{-40}{40} = -1$$

$$y = \frac{|D_y|}{|D|} = \frac{40}{40} = 1$$

$$z = \frac{|D_z|}{|D|} = \frac{-80}{40} = -2$$

The final answer written in point notation is $(x, y, z) = (-1, 1, -2)$.

Definition of eigenvalues and eigenvectors of a matrix

Let $\mathbf{A}$ be any square matrix. A non-zero vector $\mathbf{v}$ is an **eigenvector** of $\mathbf{A}$ if

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

for some number λ, called the corresponding **eigenvalue**.

NOTE: The German word "*eigen*" roughly translates as "own" or "belonging to". Eigenvalues and eigenvectors correspond to each other (are paired) for any particular matrix $\mathbf{A}$.

How to find the eigenvalues and eigenvectors of a 2x2 matrix

1. Set up the **characteristic equation**, using $|\mathbf{A} - \lambda\mathbf{I}| = 0$
2. **Solve** the characteristic equation, giving us the **eigenvalues** (2 eigenvalues for a 2x2 system)
3. **Substitute** the eigenvalues into the two equations given by $\mathbf{A} - \lambda\mathbf{I}$
4. Choose a convenient value for x_1, then find x_2
5. The resulting values form the corresponding **eigenvectors** of $\mathbf{A}$ (2 eigenvectors for a 2x2 system)

There is no single **eigenvector formula** as such - it's more of a sset of steps that we need to go through to find the eigenvalues and eigenvectors.

Example 1

We start with a system of two equations, as follows:

$$y_1 = -5x_1 + 2x_2$$

$$y_2 = -9x_1 + 6x_2$$

We can write those equations in matrix form as:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

In general we can write the above matrices as:

$$\mathbf{y} = \mathbf{A}\mathbf{v}$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}, \text{ and}$$

$$\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Step 1. Set up the characteristic equation, using $|\mathbf{A} - \lambda \mathbf{I}| = 0$

Our task is to find the **eigenvalues** λ, and **eigenvectors** $\mathbf{v}$, such that:

$$\mathbf{y} = \lambda \mathbf{v}$$

We are looking for **scalar values** λ (numbers, not matrices) that can replace the matrix $\mathbf{A}$ in the expression $\mathbf{y} = \mathbf{Av}.$

That is, we want to find λ such that :

$$-5x_1 + 2x_2 = \lambda x_1$$

$$-9x_1 + 6x_2 = \lambda x_2$$

Rearranging gives:

$$\begin{aligned} -(5 - \lambda)x_1 + 2x_2 &= 0 \\ -9x_1 + (6 - \lambda)x_2 &= 0 \end{aligned} \qquad (1)$$

This can be written using matrix notation with the identity matrix $\mathbf{I}$ as:

$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = 0$, that is:

$$\left(\mathbf{A} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\mathbf{v} = 0$$

$$\left(\mathbf{A} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)\mathbf{v} = 0$$

Step 2. Solve the characteristic equation, giving us the eigenvalues (2 eigenvalues for a 2x2 system)

In this example, the coefficient determinant from equations (1) is:

$$
\begin{aligned}
|\mathbf{A} - \lambda\mathbf{I}| &= \begin{vmatrix} -5-\lambda & 2 \\ -9 & 6-\lambda \end{vmatrix} \\
&= (-5-\lambda)(6-\lambda) - (-9)(2) \\
&= -30 - \lambda + \lambda^2 + 18 \\
&= \lambda^2 - \lambda - 12 \\
&= (\lambda+3)(\lambda-4)
\end{aligned}
$$

Now this equals 0 when:

$$(\lambda+3)(\lambda-4) = 0$$

That is, when:

$$\lambda = -3 \quad \text{or} \quad 4.$$

These two values are the **eigenvalues** for this particular matrix $\mathbf{A}$.

Step 3. Substitute the eigenvalues into the two equations given by $\mathbf{A} - \lambda\mathbf{I}$

Case 1: $\lambda_1 = -3$

When $\lambda = \lambda_1 = -3$, equations (1) become:

$$[-5 - (-3)]x_1 + 2x_2 = 0$$

$$-9x_1 + [6 - (-3)]x_2 = 0$$

That is:

$$\begin{aligned} -2x_1 + 2x_2 &= 0 \\ -9x_1 + 9x_2 &= 0 \end{aligned} \qquad (2)$$

Dividing the first line of Equations (2) by -2 and the second line by -9 (not really necessary, but helps us see what is happening) gives us the identical equations:

$$x_1 - x_2 = 0$$

$$x_1 - x_2 = 0$$

Step 4. Choose a convenient value for x_1, then find x_2

There are infinite solutions of course, where $x_1 = x_2$. We choose a convenient value for x_1 of, say 1, giving $x_2 = 1$.

Step 5. The resulting values form the corresponding eigenvectors of A (2 eigenvectors for a 2x2 system)

So the corresponding eigenvector is:

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

NOTE: We could have easily chosen $x_1 = 3$, $x_2 = 3$, or for that matter, $x_1 = -100$, $x_2 = -100$. These values will still "work" in the matrix equation.

Is it correct?

We can check by substituting:

$$\begin{aligned}\mathbf{A}\mathbf{v}_1 &= \begin{bmatrix}-5 & 2\\ -9 & 6\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}\\ &= \begin{bmatrix}-3\\ -3\end{bmatrix}\\ &= -3\begin{bmatrix}1\\ 1\end{bmatrix}\\ &= \lambda_1\mathbf{v}_1\end{aligned}$$

We have found an **eigenvalue** $\lambda_1 = -3$ and an **eigenvector** $\mathbf{v}_1 = \begin{bmatrix}1\\ 1\end{bmatrix}$ for the matrix $\mathbf{A} = \begin{bmatrix}-5 & 2\\ -9 & 6\end{bmatrix}$ such that $\mathbf{A}\mathbf{v}_1 = \lambda_1\mathbf{v}_1$.

How many eigenvalues and eigenvectors?

In the above example, we were dealing with a 2×2 system, and we found 2 eigenvalues and 2 corresponding eigenvectors.

If we had a 3×3 system, we would have found 3 eigenvalues and 3 corresponding eigenvectors.

In general, a $n \times n$ system will produce n eigenvalues and n corresponding eigenvectors.

The matrix $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ corresponds to the linear equations:

$$y_1 = 3x_1 + 2x_2$$

$$y_2 = x_1 + 4x_2$$

We want to find λ such that :

$$3x_1 + 2x_2 = \lambda x_1$$

$$x_1 + 4x_2 = \lambda x_2$$

Rearranging gives:

$$\begin{aligned} (3 - \lambda)x_1 + 2x_2 &= 0 \\ x_1 + (4 - \lambda)x_2 &= 0 \end{aligned} \qquad (4)$$

The **characterstic equation** $|\mathbf{A} - \lambda\mathbf{I}| = 0$ for this example is given by:

$$
\begin{aligned}
|\mathbf{A} - \lambda\mathbf{I}| &= \begin{vmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{vmatrix} \\
&= 12 - 7\lambda + \lambda^2 - 2 \\
&= \lambda^2 - 7\lambda + 10 \\
&= 0
\end{aligned}
$$

This has value 0 when $(\lambda - 5)(\lambda - 2) = 0$.

Case 1: $\lambda = 5$

With $\lambda_1 = 5$, equations (4) become:

$$(3-5)x_1 + 2x_2 = 0$$

$$x_1 + (4-5)x_2 = 0$$

That is:

$$-2x_1 + 2x_2 = 0$$

$$x_1 - x_2 = 0$$

We choose a convenient value $x_1 = 1$, giving $x_2 = 1$. So the corresponding eigenvector is:

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Multiplying to check our answer, we would find:

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$, that is $\mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$.

Case 2: $\lambda = 2$

With $\lambda_2 = 2$, equations (4) become:

$$(3-2)x_1 + 2x_2 = 0$$

$$x_1 + (4-2)x_2 = 0$$

That is:

$$x_1 + 2x_2 = 0$$

$$x_1 + 2x_2 = 0$$

We choose a convenient value $x_1 = 2$, giving $x_2 = -1$. So the corresponding eigenvector is:

$$\mathbf{v_2} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Multiplying to check our answer, we would find:

$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, that is $\mathbf{A}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$.

3×3 matrices and their eigenvalues and eigenvectors

The process for finding the eigenvalues and eigenvectors of a 3×3 matrix is similar to that for the 2×2 case.

Example 4: 3×3 case

Find the eigenvalues and eigenvectors for the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

The **characterstic equation** $|\mathbf{A} - \lambda\mathbf{I}| = 0$ for this example is given by:

$$
\begin{aligned}
|\mathbf{A} - \lambda\mathbf{I}| &= \begin{vmatrix} 0-\lambda & 1 & 0 \\ 1 & -1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} \\
&= -\lambda^3 - \lambda^2 + 2\lambda \\
&= -\lambda(\lambda^2 + \lambda - 2) \\
&= -\lambda(\lambda+2)(\lambda-1) \\
&= 0
\end{aligned}
$$

This occurs when $\lambda_1 = 0$, $\lambda_2 = -2$, or $\lambda_3 = 1$.

Case 1: $\lambda = 0$

Equations (5) become:

$$x_2 = 0$$

$$x_1 - x_2 + x_3 = 0$$

$$x_2 = 0$$

Clearly, $x_2 = 0$ and we'll choose $x_1 = 1$, giving
$x_3 = -1$.

So for the eigenvalue $\lambda_1 = 0$, the correspondin
eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Case 2: $\lambda = -2$

Equations (5) become:

$$2x_1 + x_2 = 0$$

$$x_1 + x_2 + x_3 = 0$$

$$x_2 + 2x_3 = 0$$

Choosing $x_1 = 1$ gives $x_2 = -2$ and then $x_3 = 1$.

So for the eigenvalue $\lambda_2 = -2$, the corresponding eigenvector is $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Case 3: $\lambda = 1$

Equations (5) become:

$$-x_1 + x_2 = 0$$

$$x_1 - 2x_2 + x_3 = 0$$

$$x_2 - x_3 = 0$$

Choosing $x_1 = 1$ gives $x_2 = 1$ and then $x_3 = 1$.

So for the eigenvalue $\lambda_3 = 1$, the corresponding eigenvector is $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.