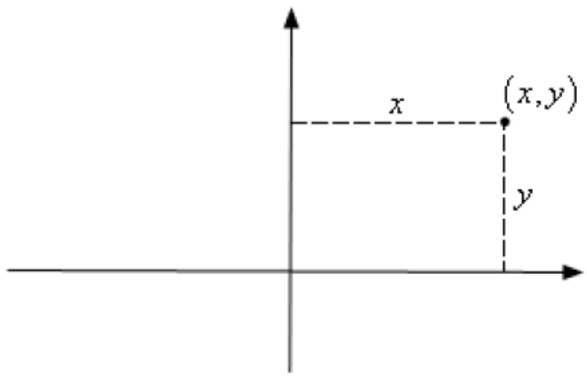
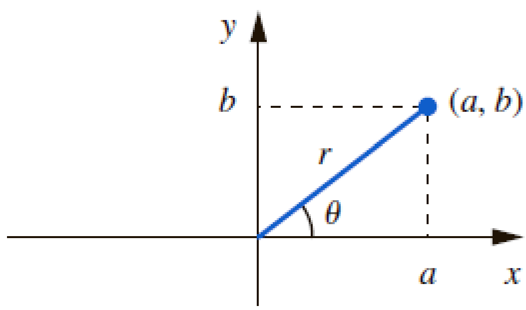
Polar Coordinates:

Up to this point we've dealt exclusively with the Cartesian (or Rectangular, or *x*-*y*) coordinate system. Coordinate systems are really nothing more than a way to define a point in space. For ex. Z=x+iy



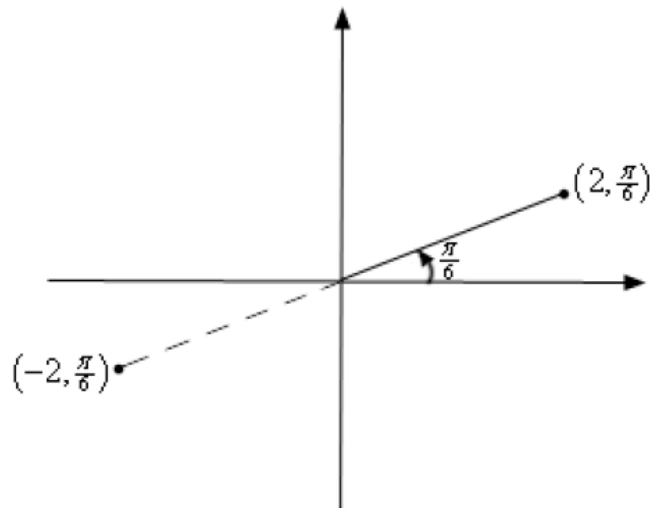
In polar coordinates, we go straight out of the origin until we hit the point and then determine the angle this line makes with the positive *x*-axis then use the distance of the point from the origin and the amount of rotation from the positive *x*-axis as the coordinates of the point. For example consider the figure below:



Coordinates in this form are called **polar coordinates** for ex Z= r $\langle\theta$.

$$z = r\angle\theta$$
$$= r\cos\theta + \mathrm{i}r\sin\theta = r(\cos\theta + \mathrm{i}\sin\theta)$$

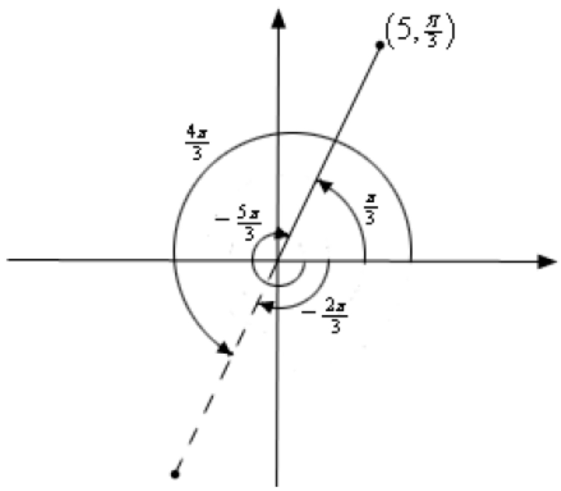
Below is a sketch of the two points (2,π/6) , (-2, π/6):



r is called the **modulus** of the complex number z , denoted |z|. The angle θ is called the **argument** of z, denoted arg(z).

If *r* is positive the point will be in the same quadrant as θ. On the other hand if *r* is negative the point will end up in the quadrant exactly opposite θ. Notice as well that the coordinates (-2, π/6) describe the same point as the coordinates (-2,7π/6) do.

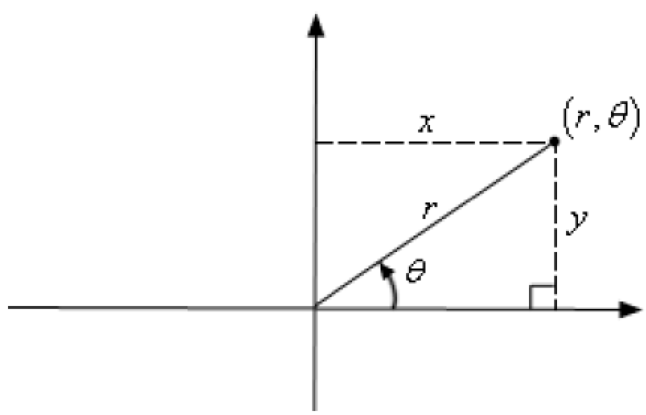
In Cartesian coordinates there is exactly one set of coordinates for any given point. With polar coordinates this isn't true. In polar coordinates there is literally an infinite number of coordinates for a given point. For instance, the following four points are all coordinates for the same point.



$$\left(5,\frac{\pi}{3}\right)=\left(5,-\frac{5\pi}{3}\right)=\left(-5,\frac{4\pi}{3}\right)=\left(-5,-\frac{2\pi}{3}\right)$$

In polar coordinates the origin is often called the **pole**, the coordinates of the origin/pole are (0,Ө).

Now to convert between the two coordinate systems, consider the following sketch,



1) **Polar to Cartesian Conversion Formulas**

$$x = r\cos\theta \qquad y = r\sin\theta$$

So, in Cartesian coordinates the point is (x,y).

2) **Cartesian to Polar Conversion Formulas**

$$x^2+y^2=(r\cos\theta)^2+(r\sin\theta)^2 \qquad =r^2\left(\cos^2\theta+\sin^2\theta\right)=r^2$$

$$=r^2\cos^2\theta+r^2\sin^2\theta \qquad r=\sqrt{x^2+y^2}$$

Getting an equation for θ is

$$\frac{y}{x}=\frac{r\sin\theta}{r\cos\theta}=\tan\theta$$

$$\theta=\tan^{-1}\left(\frac{y}{x}\right)$$

Note: ϴ is the reference angle. There is a second possible angle and that the second angle is given by (ϴ+π). The relation between radians and degrees is given by:

$$\pi \text{ radians } = 180^\circ.$$

For example, 45° in radian measure is

$$45\cdot\frac{\pi}{180}=\frac{\pi}{4}\text{ rad},$$

and $\pi/6$ radians is

$$\frac{\pi}{6}\cdot\frac{180}{\pi}=30^\circ.$$

To find the angle according to the quadrant its belong so,

Q1=Qref
Q2=180-Qref
Q3=180+Qref
Q4=360-Qref

Ex.1:
Convert each of the following points into the given coordinate system.

(a) $\left(-4, \frac{2\pi}{3}\right)$ into Cartesian coordinates.

(b) (-1,-1) into polar coordinates.

Sol. a)

$$x = -4\cos\left(\frac{2\pi}{3}\right) = -4\left(-\frac{1}{2}\right) = 2$$

$$y = -4\sin\left(\frac{2\pi}{3}\right) = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$$

this point is $\left(2, -2\sqrt{3}\right)$.

b)

$$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

The point is given in the third quadrant. The actual angle is,

$$\theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$$

So, in polar coordinates the point is ($\sqrt{2}, \frac{5\pi}{4}$) .

Ex.2 :
Express z = −1 − i in polar form.

Sol.

$$|z| = r = \sqrt{a^2 + b^2},$$

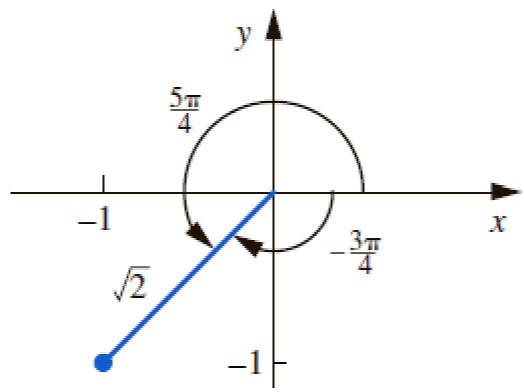
$$r = \sqrt{2}.$$

$\tan\theta = b/a,$ $\tan\theta = -1/-1 = 1$ $\theta = \pi/4.$

$$\theta = -3\pi/4.$$

$$z = -1 - j = \sqrt{2}\angle -3\pi/4,$$

Then graph z in polar form as



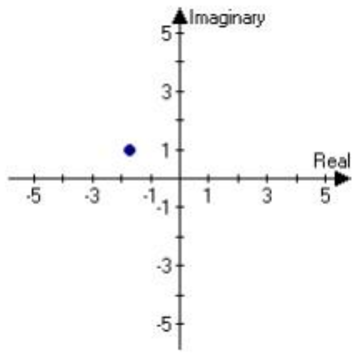
Example 3: Plot the complex number $z = -\sqrt{3} + i$ in the complex plane and then write it in its polar form.

Solution:

Find r

$$r = \sqrt{a^2 + b^2}$$
$$r = \sqrt{\left(-\sqrt{3}\right)^2 + (1)^2}$$
$$r = \sqrt{3+1}$$
$$r = \sqrt{4}$$
$$r = 2$$

Plot the complex number to determine the quadrant in which it lies



The angle θ would be in quadrant II

Find θ

$$\tan\theta = \frac{b}{a}$$
$$\tan\theta = \frac{1}{-\sqrt{3}}$$
$$\tan\theta = -\frac{\sqrt{3}}{3}$$

$\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$ so the reference angle of $\frac{\pi}{6}$ would be subtracted from π to get the value of θ.

$$\theta = \pi - \frac{\pi}{6}$$
$$\theta = \frac{5\pi}{6}$$

Write the complex number in its polar form

$$z = r(\cos\theta + i\sin\theta)$$
$$z = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

Multiplication and division in polar form:

For example, suppose we want to multiply the complex numbers z_1 , z_2

$$z_1 = r_1(\cos\theta_1 + \mathrm{j}\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + \mathrm{j}\sin\theta_2)$$

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + \mathrm{j}\sin\theta_1) r_2(\cos\theta_2 + \mathrm{j}\sin\theta_2) \\ &= r_1 r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + \mathrm{j}(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)\} \end{aligned}$$

which can be written as

$$r_1 r_2\{\cos(\theta_1 + \theta_2) + \mathrm{j}\sin(\theta_1 + \theta_2)\}$$

To multiply two complex numbers we multiply their moduli and add their arguments, that is:

$$z_1 z_2 = r_1 r_2 \angle(\theta_1 + \theta_2)$$

Ex.4:

If $z_1 = 3\angle\pi/3$ and $z_2 = 4\angle\pi/6$ find $z_1 z_2$.

Sol.

Multiplying the moduli we find $r_1 r_2 = 12$, and adding the arguments we find $\theta_1 + \theta_2 = \pi/2$. Therefore $z_1 z_2 = 12\angle\pi/2$.

To divide two complex numbers we divide their moduli and subtract their arguments, that is

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle (\theta_1 - \theta_2)$$

Ex.5:

If $z_1 = 3\angle \pi/3$ and $z_2 = 4\angle \pi/6$ find z_1/z_2.

Sol.

Dividing the respective moduli, we find $r_1/r_2 = 3/4$ and subtracting the arguments, $\pi/3 - \pi/6 = \pi/6$. Hence $z_1/z_2 = 0.75\angle \pi/6$.

A comparison between these two forms:

Rectangular	Polar
a,b	r,Θ
z=a+bi	$z = r(\cos\theta + i\sin\theta)$