INTRODUCTION TO COMPLEX NUMBERS

1.1 COMPLEX NUMBERS

A complex number is a number that can be expressed in the form *a* + *bi*, where *a* and *b* are real numbers, and *i* is a solution of the equation $x^2 = -1$. Just as the set of all real numbers is denoted *R*, the set of all complex numbers is denoted *C*.

Consider the equation $x^2+1=0$ …… (1) This can be written as

$x^2= -1$

Or $x = \pm\sqrt{-1}$

No real numbers which satisfy $x^2= -1$. In other words, we can say that there is no real numbers whose square is -1, so to solve such equations, let us imagine that there exists a number 'i' which equal to $i = \sqrt{-1}$. Therefore, we can denote the solution of (1) as x = ± i . For ex:

$$-4 = 4(-1)$$

$$\sqrt{-4} = \sqrt{(-1)(4)} = \sqrt{i^2.2^2} = 2i$$

So, we have $\sqrt{-4} = 2i$, $\sqrt{-7} = \sqrt{7}i$

$\sqrt{-4}$, $\sqrt{-7}$ are all examples of complex number

Consider another quadratic equation:

$$x^2 -6x + 13 = 0$$

This can be solved as under:

$$(x -3)^2 + 4 = 0$$

or, $(x -3)^2 = -4$

or, $x - 3 = \pm 2i$

or, $x = 3 \pm 2i$

A complex number is, generally, denoted by the letter z. i.e. z = a+bi , 'a' is called the real part of x and is written as Re (a+bi). 'b' is called the imaginary part of x and is written as Imag (a + bi). –7i, 3i and πi are all examples of purely imaginary numbers, 5, 2.5 and 7 are all examples of real numbers.

Ex1. Simplify each of the following using 'i'.

(i) $\sqrt{-36}$ (ii) $\sqrt{25}.\sqrt{-4}$

1.2 **POSITIVE INTEGRAL POWERS OF i**

Any higher powers of 'i' can be expressed in terms of one of four values i, –1, –i, 1 as:

$i^2 = -1$

$i^3 = i^2.i = -1.i = -i$

$i^4 = (i^2)^2 = (-1)^2 = 1$

$i^5 = (i^2)^2.i = 1.i = i$

$i^6 = (i^2)^3 = (-1)^3 = -1$

$i^7 = (i^2)^3(i) = -i$

$i^8 = (i^2)^4 = 1$

Ex2. Find the value of $1 + i^{10} + i^{20} + i^{30}$?

Solution: $1 + i^{10} + i^{20} + i^{30} = 0$

Ex3. Express

$8i^3 + 6i^{16} - 12i^{11}$ in the form of $a + bi$

Solution

$= 6 + 4i$

1.3 **CONJUGATE OF A COMPLEX NUMBER**

For any complex number $a + bi$ there corresponds a complex number $a - bi$ obtained by changing the sign of the "imaginary part". So, $a - bi$ is the Complex Conjugate of $a + bi$.

Following are some examples of complex conjugates:

(i) If $z = 2 + 3i$, then $\bar{z} = 2 - 3i$

(ii) If $z = 1 - i$, then $\bar{z} = 1 + i$

(iii) If $z = -2 + 10i$, then $\bar{z} = -2 - 10i$

Consider the equation:

$$x^2 - 6x + 25 = 0 \quad \ldots \quad (i)$$

or, $(x - 3)^2 + 16 = 0$

or, $(x - 3)^2 = -16$

or, $(x - 3) = \pm\sqrt{-16} = \pm\sqrt{16.(-1)}$.

or, $x = 3 \pm 4i$

The roots of the above equation (i) are $3 + 4i$ and $3 - 4i$.

Consider another equation:

$$x^2 + 2x + 2 = 0 \quad \ldots \quad (ii)$$

or, $(x + 1)^2 + 1 = 0$

or, $(x + 1)^2 = -1$

or, $(x + 1) = \pm\sqrt{-1} = \pm i$

or, $x = -1 \pm i$

The equations (i) and (ii) have roots of the type a + bi and a – bi. Such roots are known as conjugate roots.

1.3.1 **PROPERTIES OF COMPLEX CONJUGATES**

(1) If z is a real number then z = $\bar{z}$ i.e., the conjugate of a real number is the number itself.

For example, let z = 5

This can be written as :

$$z = 5 + 0i$$

Then

$$\bar{z} = 5 - 0i = 5$$

$$z = 5 = \bar{z}.$$

(2) If z is a purely imaginary number then z = - z For example, if z=3i .This can be written as

$$z = 0 + 3i$$

$$\bar{z} = 0 - 3i = -3i$$

$$= - z$$

$$\bar{z} = - z.$$

(3) Conjugate of the conjugate of a complex number is the number itself.

$$\overline{(\bar{z})} = z$$

For example if $z = a + bi$ then $\bar{z} = a - bi$

Again

$$\overline{(\bar{z})} = \overline{(a - bi)} = a + bi$$

$$= z$$

$$\therefore \quad \overline{(\bar{z})} = z$$

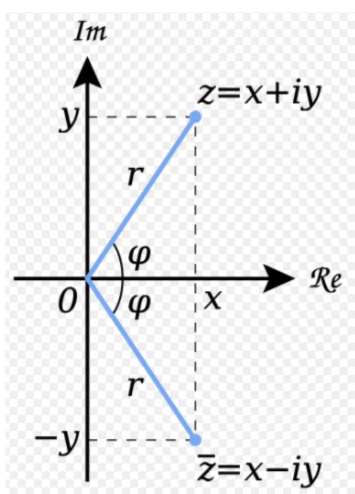
H.w: Find the conjugate of each of the following complex number:

(i) $3 - 4i$ (ii) $2i$

(iii) $(2+i)^2$ (iv) $\frac{i+1}{2}$

1.4 GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER

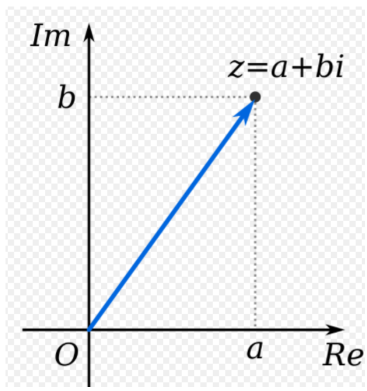
The complex plane or *z*-plane is a geometric representation of the complex numbers established by the real axis and the perpendicular imaginary axis. It can be thought of as a modified Cartesian plane, with the real part of a complex number represented by a displacement along the x-axis, and the imaginary part by a displacement along the y-axis. The complex plane is sometimes known as the **Argand plane** or **Gauss plane**, as shown in figure below:



Two planes are used to represent the complex number z are Cartesian complex plane and Polar complex plane .

Cartesian complex plane

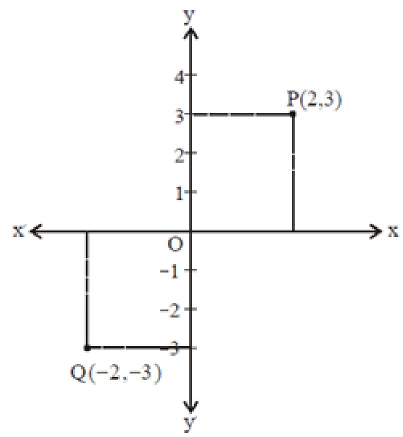
In this plane, the horizontal (*real*) axis is generally used to display the real part, with increasing values to the right, and the imaginary part marks the vertical (*imaginary*) axis, with increasing values upwards.



Ex 4: Represent complex numbers 2 + 3i and –2 –3i in the same Argand Plane.

Solution:
1) 2 + 3i is represented by the point P (2, 3)
2) –2 –3i is represented by the point Q (–2,–3)

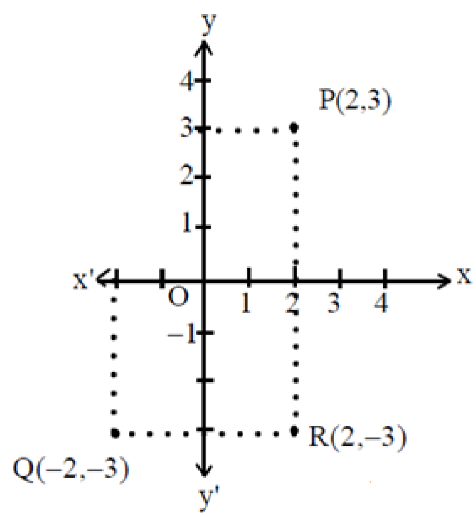
Points P and Q are different and lie in the I quadrant and III quadrant respectively.



Ex 5: Represent complex numbers 2 + 3i, –2–3i, 2–3i in the same Argand Plane.

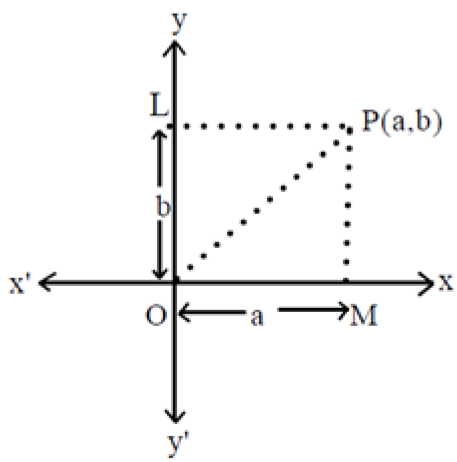
Solution :

1) 2+3i is represented by the point P (2, 3)
2) –2–3i is represented by the point Q (–2,–3)
3) 2–3i is represented by the point R (2, –3)



1.5 MODULUS OF A COMPLEX NUMBER

We have learnt that any complex number z = a + bi can be represented by a point in the Argand Plane. How can we find the distance of the point from the origin? Let P(a, b) be a point in the plane representing a + bi. Draw perpendiculars PM and PL on x-axis and y-axis respectively. Let OM = a and MP = b. We have to find the distance of P from the origin.



$$\therefore \quad OP = \sqrt{OM^2 + MP^2}$$

$$= \sqrt{a^2 + b^2}$$

OP is called the modulus or absolute value of the complex number a + bi. Modulus of any complex number z such that z = a + bi, a∈R, b∈R is denoted by

$|z|$ and is given by $\sqrt{a^2 + b^2}$

$$|z| = |a + ib| = \sqrt{a^2 + b^2}$$

1.5.1 **Properties of Modulus**

a)

$$|z| = 0 \leftrightarrow z = 0$$

Proof : Let $z = a + bi,\ a \in R,\ b \in R$

then $|z| = \sqrt{a^2 + b^2}$

$|z| = 0 \Leftrightarrow a^2 + b^2 = 0$

b)

$$|z| = |\bar{z}|$$

Proof : Let $z = a + bi$

then $|z| = \sqrt{a^2 + b^2}$

Now, $\bar{z} = a - bi$

$\therefore \quad |\bar{z}| = \sqrt{a^2 + (-b^2)} = \sqrt{a^2 + b^2}$

c)

$$|z| = |-z|$$

Proof : Let $z = a + bi$ then $|z| = \sqrt{a^2 + b^2}$

$-z = -a - bi$ then $|-z| = \sqrt{(-a)^2 + (-b)^2}$

$= \sqrt{a^2 + b^2}$

Thus, $|z| = \sqrt{a^2 + b^2} = |-z|$

Finally

$$|z| = |-z| = |\bar{z}|$$

Ex 6. Find the modulus of z and $\bar{z}$ if $z = -4 + 3i$?

Solution:

$z = -4 + 3i$, then $|z| = \sqrt{(-4)^2 + (3)^2}$

$$= \sqrt{16 + 9} = \sqrt{25} = 5$$

and $\quad \bar{z} = -4 - 3i$

then, $\quad |\bar{z}| = \sqrt{(-4)^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$

Thus, $\quad |z| = 5 = |\bar{z}|$

Ex 7. Find the modulus of z and – z if z = 5 + 2i ?

Solution

$z = 5 + 2i$, then $-z = -5 - 2i$

$$|z| = \sqrt{5^2 + 2^2} = \sqrt{29} \text{ and } |-z| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$

Thus, $|z| = \sqrt{29} = |-z|$

Ex 8. Find the modulus of z, –z and $\bar{z}$ where z = 1 + 2i ?

Solution

$$|z| = |-z| = \sqrt{5} = |\bar{z}|$$

H.W

Find the modulus of complex No.s :

(i) $1 + i$ (ii) 2π (iii) 0 (iv) $\frac{1}{2}i$

1.6 **EQUALITY OF TWO COMPLEX NUMBERS**

Let us consider two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ such that $z_1 = z_2$ we have a + bi = c + di or

$$(a - c) + (b - d)i = 0 = 0 + 0i$$

Comparing real and imaginary parts on both sides, we have

$$a - c = 0, \text{ or } a = c$$

And

$$b - d = 0 \qquad \text{or} \ \ b = d$$

Therefore, we can conclude that two complex numbers are equal if and only if their real parts and imaginary parts are respectively equal.
In general a + bi = c + di if and only if a = c and b = d.

Properties: $z_1 = z_2 \Rightarrow |z_1| = |z_2|$

Let $\quad z_1 = a + bi \text{ and } z_2 = c + di$

$$z_1 = z_2 \ \text{ gives } a = c \text{ and } b = d$$

Now $\quad |z_1| = \sqrt{a^2 + b^2} \quad \text{and} \quad |z_2| = \sqrt{c^2 + d^2}$

$$= \sqrt{a^2 + b^2}$$

$$\Rightarrow \qquad |z_1| = |z_2|$$

Ex 9. For what value of x and y, 5x + 6yi and 10 + 18i are equal?

Solution : It is given that

$$5x + 6y\,i = 10 + 18i$$

Comparing real and imaginary parts, we have

$$5x = 10 \qquad \text{or } x = 2$$

and $\quad 6y = 18 \qquad \text{or } y = 3$

For $x = 2$ and $y = 3$, the given complex numbers are equal.

1.7 **Basic Rules of algebra**

1.7.1 ADDITION OF COMPLEX NUMBERS

If $z_1 = a + bi$ and $z_2 = c + di$ are two complex numbers then their sum $z_1 + z_2$ is defined by

$$z_1 + z_2 = (a + c) + (b + d)i$$

For example, if $z_1 = 2 + 3i$ and $z_2 = -4 + 5i$,

then $z_1 + z_2 = [2 + (-4)] + [3 + 5]i$

$$= -2 + 8i.$$

H.W. Simplify:

(i) $(3 + 2i) + (4 - 3i)$

(ii) $(2 + 5i) + (-3 - 7i) + (1 - i)$

Ex 10.

If $z_1 = 2 + 3i$ and $z_2 = 1 + i$,

verify that $| z_1 + z_2 | \leq | z_1 | + | z_2 |$

Solution

$$|z_1 + z_2| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$|z_1| + |z_2| = 3.6 + 1.41 = 5.01$$

$$\therefore \qquad |z_1 + z_2| \leq |z_1| + |z_2|$$

1.7.2 Subtraction of the Complex Numbers

Let two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ be represented by the points (a, b) and (c, d) respectively.

$$(z_1) - (z_2) = (a + bi) - (c + di)$$

$$= (a - c) + (b - d)i$$

which represents a point $(a - c, b - d)$

H.W. Find $z_1 - z_2$ in each of following if:

(a) $z_1 = 3 - 4i, \quad z_2 = -3 + 7i$

(b) $z_1 = -4 + 7i \quad z_2 = -4 - 5i$

PROPERTIES: WITH RESPECT TO ADDITION and SUBTRACTION OF COMPLEX NUMBERS

1. **Commutative** : If $z_1$and z_2 are two complex numbers then

$$z_1 + z_2 = z_2 + z_1$$

For example, if $z_1 = 8 + 7i$ and $z_2 = 9 - 3i$ then

$$z_1 + z_2 = (8 + 7i) + (9 - 3i) \quad \text{and} \quad z_2 + z_1 = (9 - 3i) + (8 + 7i)$$

$$= (8 + 9) + (7 - 3)i \quad \text{and} \quad = (9 + 8) + (-3 + 7)i$$

or $\quad z_1 + z_2 = 17 + 4i \quad$ and $z_2 + z_1 = 17 + 4i$

Hence, addition of complex numbers is commutative. But the subtraction of complex numbers is not commutative !!!

For example, if $z_1 = 8+7i$ and $z_2 = 9-3i$ then

$$z_1 - z_2 = (8 + 7i) - (9 - 3i) \quad \text{and} \quad z_2 - z_1 = (9 - 3i) - (8 + 7i)$$

$$= (8 - 9) + (7 + 3)i \quad \text{and} \quad = (9 - 8) + (-3 - 7)i$$

or $\quad z_1 - z_2 = -1 + 10i \quad$ and $\quad z_2 - z_1 = 1 - 10i$

$$z_1 - z_2 \neq z_2 - z_1$$

2. **Associative** : If $z_1 = a_1 + b_1i$, $z_2 = a_2 + b_2i$ and $z_3 = a_3 + b_3i$ are three complex numbers, then

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$

For example, if $z_1 = 2 + 3i$, $z_2 = 3i$ and $z_3 = 1 - 2i$, then

$$z_1 + (z_2 + z_3) = (2 + 3i) + \{(3i) + (1 - 2i)\}$$

$$= (2 + 3i) + (1 + i)$$

$$= (3 + 4i)$$

and

$$(z_1 + z_2) + z_3 = \{(2 + 3i) + (3i)\} + (1 - 2i)$$

$$= (2 + 6i) + (1 - 2i)$$

$$= (3 + 4i)$$

Thus

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$

Hence, the associativity property holds good in the case of addition of complex numbers. Like commutativity, it can be shown that associativity also does not hold good in the case of subtraction.

3. **Existence of Additive Identitiy**

If x + yi be a complex number, then there exists a complex number (0 + 0i) such that (x + yi) + (0 + 0i) = x + yi.

Let $z_2 = x + yi$ be the additive identity of $z_1 = 2 + 3i$ then

$$z_1 + z_2 = z_1$$

i.e. $(2 + 3i) + (x + yi) = 2 + 3i$

or $(2 + x) + (3 + y)i = 2 + 3i$

or $(2 + x) = 2$ and $3 + y = 3$

or $x = 0$ and $y = 0$

i.e. $z_2 = x + yi = 0 + 0i$ is the additive identity.

$$z_1 - z_2 = (2 + 3i) - (0 + 0i)$$
$$= (2 - 0) + (3 - 0)i$$
$$= 2 + 3i = z_1$$

$z_2 = 0 + 0i$ is the identity w.r.t. subtraction also.

4. **Existence of Additive Inverse**:

For every complex number a + bi there exists a unique complex number –a – bi such that (a + bi) + (–a – bi) = 0 + 0i

e.g. Let $z_1 = 4 + 5i$ and $z_2 = x + yi$ be the additive inverse of z_1

Then, $z_1 + z_2 = 0$

or $(4 + 5i) + (x + yi) = 0 + 0i$

or $(4 + x) + (5 + y)i = 0 + 0i$

or $4 + x = 0$ and $5 + y = 0$

or $x = -4$ and $y = -5$

Thus, $z_2 = -4 - 5i$ is the additive invese of $z_1 = 4 + 5i$

By changing the signs of real and imaginary parts.

Consider $z_1 - z_2 = 0$

or $(4 + 5i) - (x + yi) = 0 + 0i$

or $(4 - x) + (5 - y)i = 0 + 0i$

or $4 - x = 0$ and $5 - y = 0$

or $x = 4$ and $y = 5$

i.e. $z_1 - z_2 = 0$ gives $z_2 = 4 + 5i$

Thus, in subtraction, the number itself is the inverse.

1.7.3 MULTIPLICATION OF TWO COMPLEX NUMBERS

Two complex numbers can be multiplied by the usual laws of addition and multiplication as is done in the case of numbers.

Let $z_1 = (a + bi)$ and $z_2 = (c + di)$ then

$$z_1.z_2 = (a + bi).(c + di)$$

$$= a\,(c + di) + bi\,(c + di)$$

or $= ac + adi + bci + bdi^2$

or $= (ac - bd) + (ad + bc)i.$

Ex. 11 Evaluate the following :

(i) $(1+2i)(1-3i)$, (ii) $\left(\sqrt{3}+i\right)\left(\sqrt{3}-i\right)$ (iii) $(3-2i)^2$

Solution:

(i) $(1+2i)\,(1-3i) = \{1-(-6)\} + (-3+2)i$

$$= 7 - i$$

Ex 12. Find the modulus of the complex number (1 + i) (4 – 3i) ?

Solution

Let $z = (1+i)\,(4-3i)$

$$|z| = |(1+i)\,(4-3i)|$$

$$= |(1+i)|\,.\,|(4-3i)| \qquad (\text{since } |z_1 z_2| = |z_1|.|z_2|)$$

But $|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$

$$|4-3i| = \sqrt{4^2+(-3)^2} = 5$$

$\therefore \quad |z| = \sqrt{2}.5 = 5\sqrt{2}$

$|z_1.z_2| = |z_1|\,.\,|z_2|$ "properties of muliplication"

1.7.4 **DIVISION OF TWO COMPLEX NUMBERS**

Division of complex numbers involves multiplying both numerator and denominator with the conjugate of the denominator.

Let $z_1 = a + bi$ **and** $z_2 = c + di$ **then**.

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di}\left(c+di \neq 0\right)$$

$$\frac{a+bi}{c+di} = \frac{(a+bi)\,(c-di)}{(c+di)(c-di)}$$

(multiplying numerator and denominator with the conjugate of the denominator)

$$= \frac{\left(ac+bd\right)+\left(bc-ad\right)i}{c^2+d^2}$$

$$\frac{a+bi}{c+di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

Ex 13. Divide 3+i by 4–2i ?

$$\frac{3+\mathrm{i}}{4-2\mathrm{i}}=\frac{(3+i)(4+2i)}{(4-2i)(4+2i)}$$

$$=\frac{10+10i}{20}$$

$$=\frac{1}{2}+\frac{1}{2}i$$

$$\frac{3+\mathrm{i}}{4-2\mathrm{i}}=\frac{1}{2}+\frac{1}{2}\mathrm{i}$$

Ex .14 Compute each of the following:

(a) $(58-i)-(2-17i)$

(b) $\dfrac{6+3i}{10+8i}$

(c) $\dfrac{5i}{1-7i}$

Sol.

a)

$$(58-i)-(2-17i)=58-i-2+17i=56+16i$$

b)

$$\frac{6+3i}{10+8i}=\frac{(6+3i)}{(10+8i)}\frac{(10-8i)}{(10-8i)}$$

$$=\frac{60-48i+30i-24i^2}{100+64}$$

$$=\frac{84-18i}{164}$$

$$=\frac{84}{164}-\frac{18}{164}i=\frac{21}{41}-\frac{9}{82}i$$

c)

$$\frac{5i}{1-7i}=\frac{5i}{(1-7i)}\frac{(1+7i)}{(1+7i)}=\frac{-35+5i}{1+49}=-\frac{7}{10}+\frac{1}{10}i$$