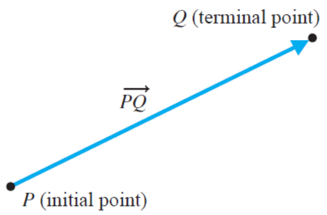
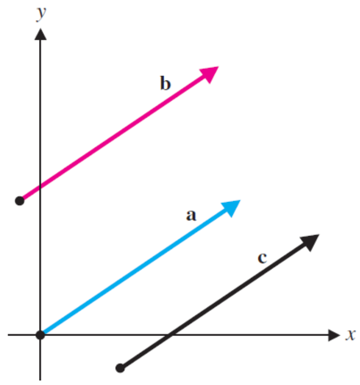
Chapter 3 Vectors and Analytical Geometry in Space (ch11)

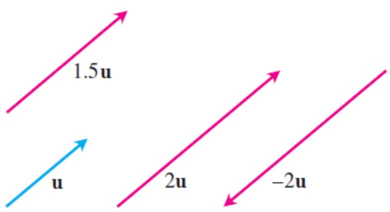


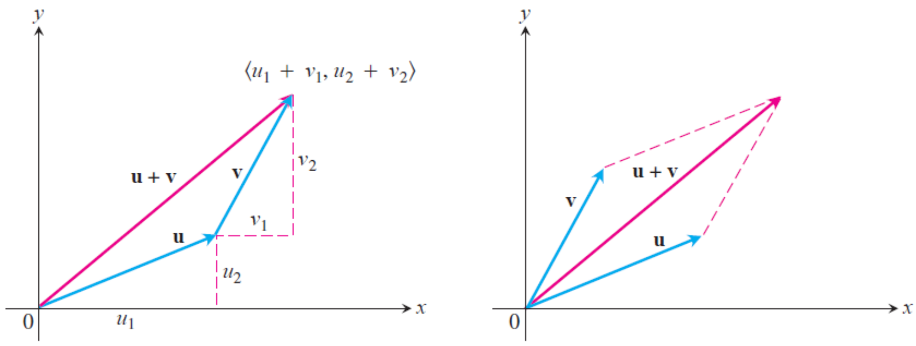


A **Vector** is an object with length (magnitude) and direction. The direction of a vector is determined by the vector's start (initial) point and end (terminal) point. Two vectors are equal if they have the same length and direction. Changing the position of the vector's start point does not change the vector, therefore, vectors are typically drown with their start point at origin. Vectors with start points at origin are called **Position Vectors**.

Scaling of a Vector

A vector can be scaled by a number called **Scalar**, the length of the vector will be changed (scaled) accordingly, i.e., the vector will be stretched of the scaler is greater than 1 or shrinks if the scalar is less than 1. The direction will not be affected if the scalar is +ve and the direction will be reversed if the scaler is –ve.



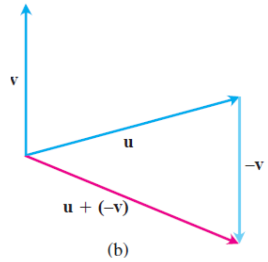


(a) (b)

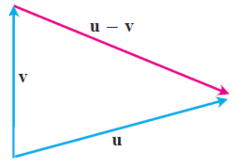
Vector Addition and Subtraction

Vectors ***u*** and ***v*** can be added by placing the start point of the ***v*** on the endpoint of the ***u***; the resulting vector (***u***+***v***) will have the start point of the ***u*** and the endpoint of the ***v***. An alternative way is to place the starting point of both vectors ***u*** and ***v*** together and completing the parallelogram; the sum of the vectors (***u***+***v***) will be the diagonal of the parallelogram with same start point.

The subtraction of vectors ***u***–***v*** can be interpreted as addition of the vectors ***u*** and –***v***, where the vector –***v*** is obtained from ***v*** by reversing its direction.

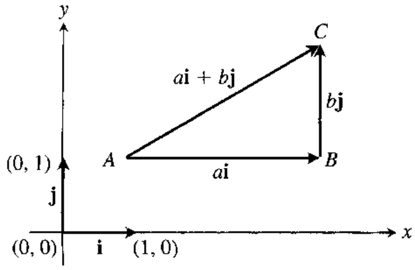


(b)



(a)

Vector Components

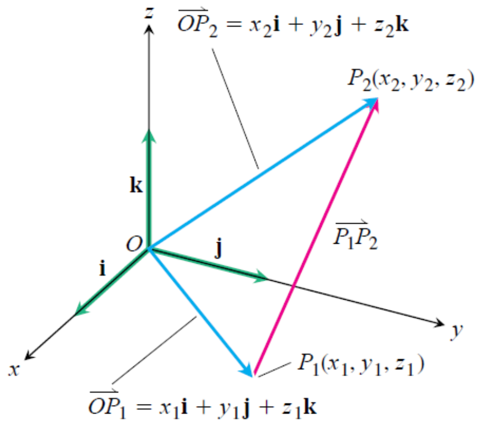


Any vector (in the plane) can be decomposed in terms of the two unit vectors ***i*** and ***j*** where the vector ***i*** is in the direction of the +ve x-axis and the vector ***j*** is in the direction of the +ve y-axis. By scaling and adding these two vectors, we can compose any vector in the plane. ($\mathbf{v}=a\mathbf{i}+b\mathbf{j}$)

In other words, any vector in the plane is a linear combination of these two vectors. In case of the vectors in space (3 dimensions) in addition to the vectors ***i*** and ***j*** there is a unit vector k in the direction of the +ve z-axis.

($\mathbf{v}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$)

The values of a, b and c are called the components of the vector v. A vector ***v*** can be represented in terms of its components as $\mathbf{v}=\langle a, b, c\rangle$



A vector can be represented in term of the coordinates of its start point $P_1(x_1, y_1, z_1)$ and endpoint $P_2(x_2, y_2, z_2)$

$$\mathbf{V}=\mathbf{P_1P_2}=(x_2-x_1)\mathbf{i}+(y_2-y_1)\mathbf{j}+(z_2-z_1)\mathbf{k}$$

Vectors can be added, subtracted and scaled (scalar multiplication) in terms of their components. Let

$\boldsymbol{V}_1 = a_1\boldsymbol{i}+b_1\boldsymbol{j}+c_1\boldsymbol{k}$, $\boldsymbol{V}_2 = a_2\boldsymbol{i}+b_2\boldsymbol{j}+c_2\boldsymbol{k}$ then

$\boldsymbol{V}_1+\boldsymbol{V}_2=(a_1+a_2)\boldsymbol{i}+(b_1+b_2)\boldsymbol{j}+(c_1+c_2)\boldsymbol{k}$

$\boldsymbol{V}_1-\boldsymbol{V}_2=(a_1-a_2)\boldsymbol{i}+(b_1-b_2)\boldsymbol{j}+(c_1-c_2)\boldsymbol{k}$

$r\boldsymbol{V}_1=ra_1\boldsymbol{i}+rb_1\boldsymbol{j}+rc_1\boldsymbol{k}$

Length of a Vector

Let $\boldsymbol{V}= a\boldsymbol{i}+b\boldsymbol{j}+c\boldsymbol{k}$ be a vector, the length of $\boldsymbol{V}$ is given by

$$|\boldsymbol{V}| = \sqrt{a^2 + b^2 + c^2}$$

In the vector V is scaled by the scalar r, then the new vector $r\boldsymbol{V}$ will have the length

$$|r\boldsymbol{V}| = \sqrt{(ra)^2 + (rb)^2 + (rc)^2} = \sqrt{r^2(a^2 + b^2 + c^2)} = |r|\sqrt{a^2 + b^2 + c^2} = |r||\boldsymbol{V}|$$

Unit Vector is a vector of unit length, like

$$\boldsymbol{u} = \boldsymbol{i},\ \boldsymbol{u} = \boldsymbol{j},\ \boldsymbol{u} = \cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j}$$

If the vector $\boldsymbol{V}\neq\boldsymbol{0}$ then the **Direction** of $\boldsymbol{V}$ is the unit vector created by that vector. To make a vector with unit length we simply divide by its length.

$$\boldsymbol{u}_{\boldsymbol{V}} = \frac{\boldsymbol{V}}{|\boldsymbol{V}|}$$

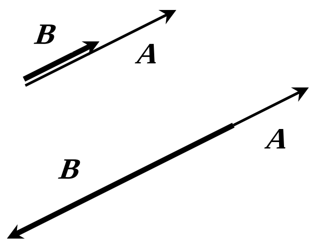
Two vector $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same direction if and only if they have the same unit vector

$$\frac{\boldsymbol{A}}{|\boldsymbol{A}|} = \frac{\boldsymbol{B}}{|\boldsymbol{B}|} \Rightarrow \boldsymbol{A} = \frac{|\boldsymbol{A}|}{|\boldsymbol{B}|}\boldsymbol{B} = k\boldsymbol{B}$$

$\boldsymbol{A}$ is a positive scalar multiple of $\boldsymbol{B}$.

Two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ point in opposite direction if

$$\frac{\boldsymbol{A}}{|\boldsymbol{A}|} = -\frac{\boldsymbol{B}}{|\boldsymbol{B}|} \Rightarrow \boldsymbol{A} = -\frac{|\boldsymbol{A}|}{|\boldsymbol{B}|}\boldsymbol{B} = -k\boldsymbol{B}$$



Ex:

a) Same direction: $\boldsymbol{A} = 3\boldsymbol{i} - 4\boldsymbol{j} + 6\boldsymbol{k}$, $B = \frac{3}{2}\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k} = \frac{1}{2}\boldsymbol{A}$

b) Opposite direction: $\boldsymbol{A} = 3\boldsymbol{i} - 4\boldsymbol{j} + 6\boldsymbol{k}$, $B = -9\boldsymbol{i} + 12\boldsymbol{j} - 18\boldsymbol{k} = -3\boldsymbol{A}$

The Dot Product (11.3)

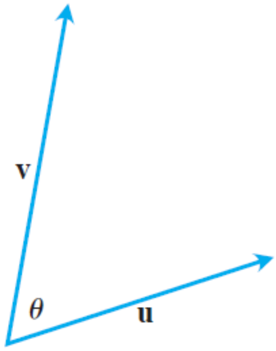
The **Dot Product** or the **Scalar Produce** of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined by the number

$\boldsymbol{u}.\boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}|\cos\theta$

θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

As a special case $\boldsymbol{u}.\boldsymbol{u} = |\boldsymbol{u}||\boldsymbol{u}|\cos(0) = |\boldsymbol{u}|^2$ or $|\boldsymbol{u}| = \sqrt{\boldsymbol{u}.\boldsymbol{u}}$

$\cos\theta = \frac{\boldsymbol{u}.\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$ or $\theta = \cos^{-1}\left(\frac{\boldsymbol{u}.\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}\right)$



The Dot Product in terms of vector components

Let $\boldsymbol{A} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{B} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ then

$\boldsymbol{A}.\boldsymbol{B} = (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}).(b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k})$

$= a_1b_1(\boldsymbol{i}.\boldsymbol{i}) + a_1b_2(\boldsymbol{i}.\boldsymbol{j}) + a_1b_3(\boldsymbol{i}.\boldsymbol{k})$

$+a_2b_1(\boldsymbol{j}.\boldsymbol{i}) + a_2b_2(\boldsymbol{j}.\boldsymbol{j}) + a_2b_3(\boldsymbol{j}.\boldsymbol{k})$

$+a_3b_1(\boldsymbol{k}.\boldsymbol{i}) + a_3b_2(\boldsymbol{k}.\boldsymbol{j}) + a_3b_3(\boldsymbol{k}.\boldsymbol{k})$

Since the vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are orthogonal unit vectors, their dot product is 0 for different vector and 1 for similar vectors, therefore

$\boldsymbol{A}.\boldsymbol{B} = a_1b_1 + a_2b_2 + a_3b_3$

Ex: Find the angle between the vectors $\boldsymbol{A} = \boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{B} = 6\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$

Sol.: $\boldsymbol{A}.\boldsymbol{B} = (1)(6) + (-2)(3) + (-2)(2) = -4$

$|\boldsymbol{A}| = \sqrt{1 + 4 + 4} = 3$

$|\boldsymbol{B}| = \sqrt{36 + 9 + 4} = 7$

$$\cos\theta = \frac{\boldsymbol{u}.\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|} = \frac{-4}{(3)(7)} = \frac{-4}{21} \Rightarrow \theta = \cos^{-1}\frac{-4}{21} \approx 101^o$$

Rules of Multiplication

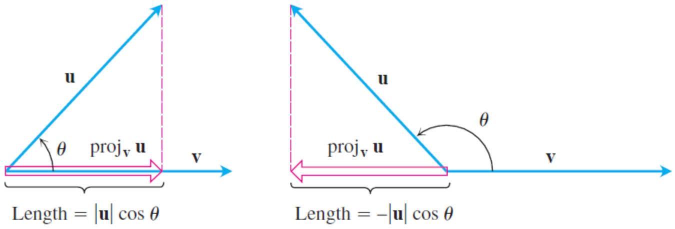
- $\boldsymbol{A}.\boldsymbol{B} = \boldsymbol{B}.\boldsymbol{A}$ (Commutative)
- $(c\boldsymbol{A}).\boldsymbol{B} = \boldsymbol{A}.(c\boldsymbol{B}) = c(\boldsymbol{A}.\boldsymbol{B})$ (Associative scalar multiplication)
- $\boldsymbol{A}.(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A}.\boldsymbol{B} + \boldsymbol{A}.\boldsymbol{C}$ (Associative)

Vectors A and B are **Orthogonal** if $\boldsymbol{A}.\boldsymbol{B} = \boldsymbol{0}$

Ex: The vectors $\boldsymbol{A} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{B} = 2\boldsymbol{j} + 4\boldsymbol{k}$ are orthogonal because

$$\boldsymbol{A}.\boldsymbol{B} = (3)(0) + (-2)(2) + (1)(4) = 0$$

Vector Projection and Scalar Components



The scalar projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$ is

$$Proj_v u = |\boldsymbol{u}| \cos\theta = \frac{\boldsymbol{u}.\boldsymbol{v}}{|\boldsymbol{v}|}$$

The scalar projection of $\boldsymbol{v}$ onto $\boldsymbol{u}$ is

$$Proj_{\boldsymbol{u}} v = |\boldsymbol{v}| \cos\theta = \frac{\boldsymbol{u}.\boldsymbol{v}}{|\boldsymbol{u}|}$$

The vector projection is obtained by multiplying the scalar projection by the direction of the vector

$$Proj_v \boldsymbol{u} = \frac{\boldsymbol{u}.\boldsymbol{v}}{|\boldsymbol{v}|}\frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{\boldsymbol{u}.\boldsymbol{v}}{\boldsymbol{v}.\boldsymbol{v}}\boldsymbol{v}$$

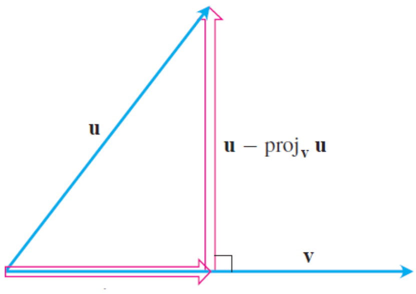
$$Proj_{\boldsymbol{u}} \boldsymbol{v} = \frac{\boldsymbol{u}.\boldsymbol{v}}{|\boldsymbol{u}|}\frac{\boldsymbol{u}}{|\boldsymbol{u}|} = \frac{\boldsymbol{u}.\boldsymbol{v}}{\boldsymbol{u}.\boldsymbol{u}}\boldsymbol{u}$$

Writing a vector as the sum of two orthogonal vectors

$$\boldsymbol{u} = \frac{\boldsymbol{u}.\boldsymbol{v}}{\boldsymbol{v}.\boldsymbol{v}}\boldsymbol{v} + \left(\boldsymbol{u} - \frac{\boldsymbol{u}.\boldsymbol{v}}{\boldsymbol{v}.\boldsymbol{v}}\boldsymbol{v}\right)$$

The second component is orthogonal to $\boldsymbol{v}$ because

$$\left(\boldsymbol{u} - \frac{\boldsymbol{u}.\boldsymbol{v}}{\boldsymbol{v}.\boldsymbol{v}}\boldsymbol{v}\right).\boldsymbol{v} = \boldsymbol{u}.\boldsymbol{v} - \frac{\boldsymbol{u}.\boldsymbol{v}}{\boldsymbol{v}.\boldsymbol{v}}\boldsymbol{v}.\boldsymbol{v} = 0$$



Ex: Express the vector $\boldsymbol{B} = 2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ as the sum of a vector parallel to $\boldsymbol{A} = 3\boldsymbol{i} - \boldsymbol{j}$ and a vector orthogonal to $\boldsymbol{A}$

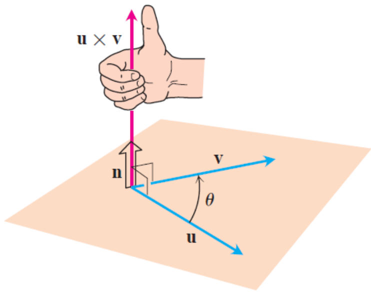
Sol.: $\boldsymbol{A}.\boldsymbol{B} = 6 - 1 = 5$, $\boldsymbol{A}.\boldsymbol{A} = 9 + 1 = 10 \neq 0$

$$\boldsymbol{B} = \left(\frac{5}{10}(3\boldsymbol{i} - \boldsymbol{j})\right) + \left((2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) - \frac{5}{10}(3\boldsymbol{i} - \boldsymbol{j})\right)$$

$$\boldsymbol{B} = \left(\frac{3}{2}\boldsymbol{i} - \frac{1}{2}\boldsymbol{j}\right) + \left(\frac{1}{2}\boldsymbol{i} + \frac{3}{2}\boldsymbol{j} - 3\boldsymbol{k}\right)$$

Cross Product (11.4)

$\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{n}|\boldsymbol{u}||\boldsymbol{v}| \sin\theta$



$\boldsymbol{n}$ is the unit vector normal to the plane determined by $\boldsymbol{u}$ and $\boldsymbol{v}$, θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$

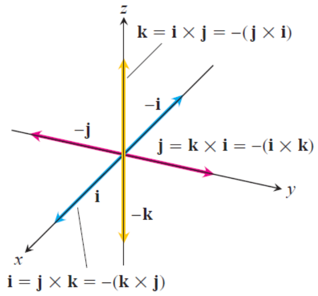
$\boldsymbol{n}$ is determined by the right hand rule

$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$$

$$\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$$

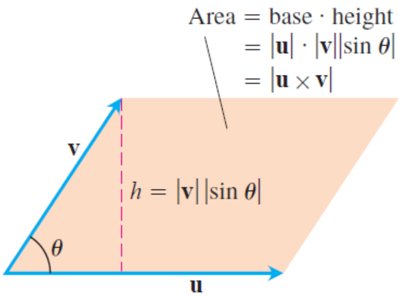
$$\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$$

$$\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = 0$$



$$|\boldsymbol{u} \times \boldsymbol{v}| = |\boldsymbol{n}||\boldsymbol{u}||\boldsymbol{v}|| \sin\theta\,|$$

The magnitude of the cross product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ equals the area of the parallelogram determined by $\boldsymbol{u}$ and $\boldsymbol{v}$



Cross Product Rules

$$(r\boldsymbol{u}) \times (s\boldsymbol{v}) = (rs)\boldsymbol{u} \times \boldsymbol{v}$$

$$\boldsymbol{u} \times (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{w}$$

$$(\boldsymbol{v} + \boldsymbol{w}) \times \boldsymbol{u} = \boldsymbol{v} \times \boldsymbol{u} + \boldsymbol{w} \times \boldsymbol{u}$$

$$\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u}$$

The determinant formula of $\boldsymbol{A} \times \boldsymbol{B}$

If $\boldsymbol{A} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{B} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ Then

$$\boldsymbol{A} \times \boldsymbol{B} = (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) \times (b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k})$$

$$= a_1b_1(\boldsymbol{i} \times \boldsymbol{i}) + a_1b_2(\boldsymbol{i} \times \boldsymbol{j}) + a_1b_3(\boldsymbol{i} \times \boldsymbol{k})$$

$$+a_2b_1(\boldsymbol{j} \times \boldsymbol{i}) + a_2b_2(\boldsymbol{j} \times \boldsymbol{j}) + a_2b_3(\boldsymbol{j} \times \boldsymbol{k})$$

$$+a_3b_1(\boldsymbol{k} \times \boldsymbol{i}) + a_3b_2(\boldsymbol{k} \times \boldsymbol{j}) + a_3b_3(\boldsymbol{k} \times \boldsymbol{k})$$

$= (a_2b_3 - a_3b_2)\boldsymbol{i} + (a_3b_1 - a_1b_3)\boldsymbol{j} + (a_1b_2 - a_2b_1)\boldsymbol{k}$

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Ex: Find the vector perpendicular to the plane containing the points $P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$.

Sol.:

$\boldsymbol{PQ} = (2-1)\boldsymbol{i} + (1+1)\boldsymbol{j} + (-1-0)\boldsymbol{k} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

$\boldsymbol{PR} = (-1-1)\boldsymbol{i} + (1+1)\boldsymbol{j} + (2-2)\boldsymbol{k} = -2\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$

$$\boldsymbol{PQ} \times \boldsymbol{PR} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = (4+2)\boldsymbol{i} - (2-2)\boldsymbol{j} + (2+4)\boldsymbol{k} = 6\boldsymbol{i} + 6\boldsymbol{k}$$

Ex: Find the area of the triangle with vertices $P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$.

Area of the parallelogram= $|\boldsymbol{PQ} \times \boldsymbol{PR}| = |6\boldsymbol{i} + 6\boldsymbol{j}| = \sqrt{6^2 + 6^2} = 6\sqrt{2}$

Area of a triangle= $\frac{1}{2}$ area of parallelogram = $\frac{1}{2}(6\sqrt{2}) = 3\sqrt{2}$

Triple, Scalar or Box Product

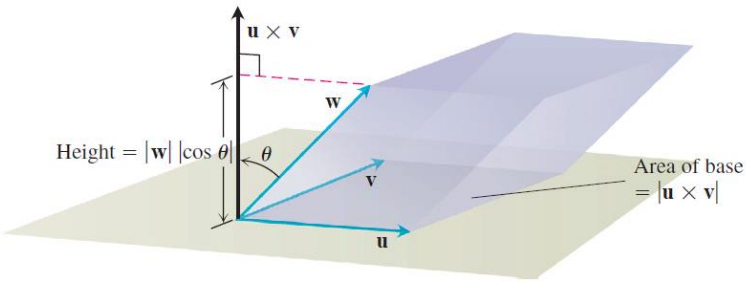
The product $\boldsymbol{A} \times \boldsymbol{B}.\boldsymbol{C}$ is called triple product

$\boldsymbol{A} \times \boldsymbol{B}.\boldsymbol{C} = |\boldsymbol{A} \times \boldsymbol{B}||\boldsymbol{C}| \cos\theta$ (scalar quantity)

It is equal to the parallelogram-sided box determined by $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$

$\boldsymbol{A} \times \boldsymbol{B}.\boldsymbol{C} = \boldsymbol{B} \times \boldsymbol{C}.\boldsymbol{A} = \boldsymbol{C} \times \boldsymbol{A}.\boldsymbol{B}$

$\boldsymbol{A} \times \boldsymbol{B}.\boldsymbol{C} = \boldsymbol{A}.\boldsymbol{B} \times \boldsymbol{C}$



$$\boldsymbol{A} \times \boldsymbol{B}.\boldsymbol{C} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Ex: $\boldsymbol{A} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{B} = -2\boldsymbol{i} + 3\boldsymbol{k}$ and $\boldsymbol{C} = 7\boldsymbol{j} - 4\boldsymbol{k}$

Sol.: $\boldsymbol{A} \times \boldsymbol{B}.\boldsymbol{C} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = 2(-8+7) - 3(7-0) = -2 - 21 = -23$

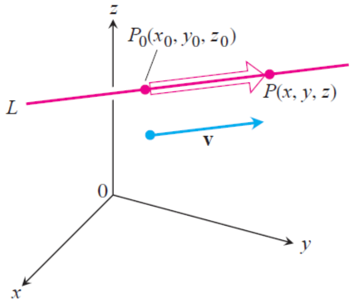
Lines and Planes in the Space (11.5)

Equations for lines and line segments

If L is a line in space passes through $P_0(x_0, y_0, z_0)$ and parallel to the vector $\boldsymbol{v} = A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}$, then L is the set of all points $P(x, y, z)$ for which $\boldsymbol{P_0P}$ is parallel to $\boldsymbol{v}$, or

$$\boldsymbol{P_0P} = t\boldsymbol{v}$$

$$(x - x_0)\boldsymbol{i} + (y - y_0)\boldsymbol{j} + (z - z_0)\boldsymbol{k} = t(A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k})$$



Or

$$x - x_0 = At,\ y - y_0 = Bt,\ z - z_0 = Ct$$

$$x = x_0 + At$$

$$y = y_0 + Bt$$

$$z = z_0 + Ct$$

These are the parametric equation of a line through the point $P_0(x_0, y_0, z_0)$ parallel to the vector

$$\boldsymbol{v} = A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}$$

Ex: Find the line equation through the points $P(-3,2,-3)$ and $Q(1,-1,4)$

Sol.: $\boldsymbol{PQ} = (1 + 3)\boldsymbol{i} + (-1 - 2)\boldsymbol{j} + (4 + 3)\boldsymbol{k} = 4\boldsymbol{i} - 3\boldsymbol{j} + 7\boldsymbol{k}$

Choosing point P

$$x = -3 + 4t,\ y = 2 - 3t,\ z = -3 + 7t$$

Ex: Find the equation of the line segment from the point $P(-3,2,-3)$ to point $Q(1,-1,4)$

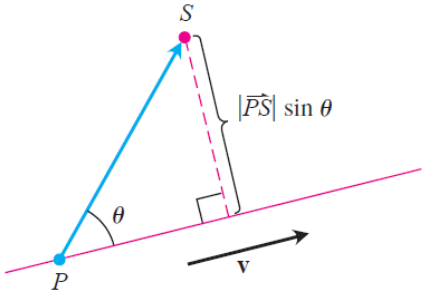
Sol.:

$$x = -3 + 4t,\ y = 2 - 3t,\ z = -3 + 7t,\ 0 \le t \le 1$$

Distance from a Point to a Line

The distance between a point and a line is the perpendicular distance and it represents the minimum distance between the given point and any point on the line. Therefore, the distance between point and a line can be found as

$$d = \frac{|\boldsymbol{PS} \times \boldsymbol{v}|}{|\boldsymbol{v}|}$$



Ex: Find the distance between the point $S(1,1,5)$ and line L:

$$x = 1 + t,\ y = 3 - t,\ z = 2t$$

Sol.:

First we find a point *P* on the line by sybstituing any value for *t*. We can choose *t*=0, therfore *P*(1,3,0)

$$\boldsymbol{PS} = (1-1)\boldsymbol{i} + (1-3)\boldsymbol{j} + (5-0)\boldsymbol{k} = -2\boldsymbol{j} + 5\boldsymbol{k}$$

The vector v parallel to the line can be found from the coefficients of t in the parametric equations of the line.

$$\boldsymbol{v} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$$

$$\boldsymbol{PS} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = (-4+5)\boldsymbol{i} - (0-5)\boldsymbol{j} + (0+2)\boldsymbol{k} = \boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k}$$

$$|\boldsymbol{PS} \times \boldsymbol{v}| = \sqrt{1+25+4} = \sqrt{30}$$

$$|\boldsymbol{v}| = \sqrt{1+1+4} = \sqrt{6}$$

$$d = \frac{|\boldsymbol{PS} \times \boldsymbol{v}|}{|\boldsymbol{v}|} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}$$

Another method to for finding the distance between the point S and the line L is by finding the minimum distance. Note that the parametric equations of the line represent the coordinates of a general point *P* on the line, therefore we can find the general distance between *S* and the general point *P* as a function of *t*.

$$D^2 = f(t) = (1+t-1)^2 + (3-t-1)^2 + (2t-5)^2 = t^2 + (2-t)^2 + (2t-5)^2$$

Finding the minimum of the distance is the same as finding the minimum of the square of the distance, this avoids working with square roots. Therefore, we differentiate *f*(*t*) and set to zero

$$\frac{df}{dt} = 2t + 2(2-t)(-1) + 2(2t-5)(2) = 0$$

$$\frac{df}{dt} = t(1) + (2-t)(-1) + (2t-5)(2) = 0$$

The above equation looks like a dot product between the vector ***PS*** and the vector *v*. The dot product is equal to zero. This means that we are trying to find the value of *t* that makes the vector ***PS*** orthogonal to ***v***.

By solving for *t*

$$t + t + 4t = 2 + 10 \Rightarrow 6t = 12 \Rightarrow t = 2$$

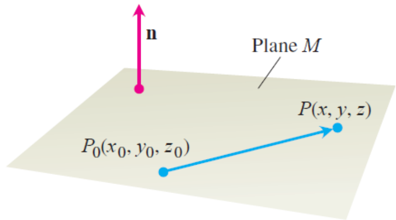
Substituting *t*=2 in the parametric equations gives the coordinates of the point P such that PS is perpendicular to L. P(3,0,4)

$$d = |\boldsymbol{PS}| = \sqrt{(3-1)^2 + (1-1)^2 + (4-5)^2} = \sqrt{4+1} = \sqrt{5}$$

Note: the point *P* in the first solution is different from the point *P* in the second solution.

Equations for Planes

If $P_0(x_0, y_0, z_0)$ is a point in the plane M and $\boldsymbol{n} = A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}$ is a vector normal to the plane M. The vector between P_0 and the general point $P(x, y, z)$ $\boldsymbol{PS} = (x - x_0)\boldsymbol{i} + (y - y_0)\boldsymbol{i} + (z - z_0)\boldsymbol{k}$ will be orthogonal to $\boldsymbol{n}$.



$$\boldsymbol{n}.\boldsymbol{P_0P} = 0$$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$Ax + By + Cz = Ax_0 + By_0 + Cz_0 = D$$

Ex: Find the equation of the plane through the points A(0,0,1), B(2,0,0) and C(0,3,0)

Sol.:

In order to find the vector normal to the plane we need to find two vectors in the plane and perform cross product

$$\boldsymbol{AB} = 2\boldsymbol{i} - \boldsymbol{k},\ \boldsymbol{AC} = 3\boldsymbol{j} - \boldsymbol{k}$$

$$\boldsymbol{n} = \boldsymbol{AB} \times \boldsymbol{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = (0+3)\boldsymbol{i} - (-2-0)\boldsymbol{j} + (6-0)\boldsymbol{k} = 3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}$$

Use point A

$$3x + 2y + 6z = (3)(0) + (2)(0) + (6)(1) = 6$$

$$3x + 2y + 6z = 6$$

Ex: Find the point of intersection of the line $x = \frac{8}{3} + 2t,\ y = -2t,\ z = 1 + t$ and the plane

$$3x + 2y + 6z = 6$$

Sol.:

Substitute x, y, z from the line equation into the plane equation

$$3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) = 6$$

$$8 + 6t - 4t + 6 + 6t = 6$$

$$8t = -8 \Rightarrow t = -1$$

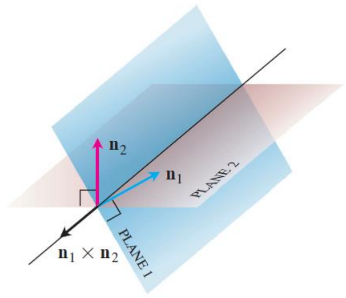
$$(x, y, z)|_{t=-1} = \left(\frac{8}{3} - 2, 2, 1 - 1\right) = \left(\frac{2}{3}, 2, 0\right)$$

Ex: Find the equation of the line of intersection of the planes $3x - 6y - 2z = 15$ and

$2x + y - 2z = 5$

Sol.:

Since it is required to find the equation of a line, we need to find a vector ***v*** parallel to the line and a point on the line. To find the parallel vector we note that a vector normal to the plane is normal to every line in the plane. Therefore the line of intersection is normal to $\boldsymbol{n}_1$ (normal vector of the first plane) and $\boldsymbol{n}_2$ (normal vector of the second plane). This means that the required parallel vector ***v*** is normal to both $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, so, it can be found by cross product of $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$.



$$\boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = (12+2)\boldsymbol{i} - (-6+4)\boldsymbol{j} + (3+12)\boldsymbol{k} = 14\boldsymbol{i} + 2\boldsymbol{j} + 15\boldsymbol{k}$$

To find a point on the line of intersection, we note that for a line we have one independent (free) variable, so, we chose a value for *z* (for example) and the two planes' equations will reduce to two equations of two variables (*z* and *y*) and we can solve for *x* and *y*.

Let *z*=0 then we get

$3x - 6y = 15 \Rightarrow x - 2y = 5$

$2x + y = 5 \Rightarrow y = 5 - 2x$

$x - 2(5 - 2x) = 5 \Rightarrow x - 10 + 4x = 5 \Rightarrow 5x = 15 \Rightarrow x = 3$

$y = 5 - 2(3) = -1$

Therefore, we have the point $P(3, -1, 0)$ that satisfy the equations of both planes which means it lies on the line of intersection. Therefore the line equations are

$x = 3 + 14t$

$y = -1 + 2t$

$z = 15t$

Distance from a Point to a Plane

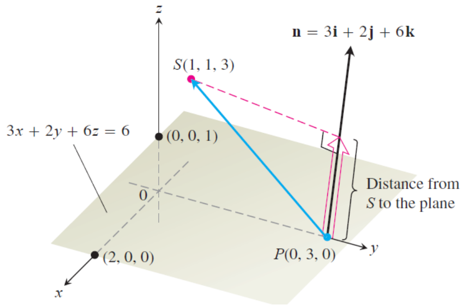
If P is a point on a plane with normal ***n***, then the distance from any point *S* to the plane is the length of the vector projection of ***PS*** onto ***n***. That is, the distance from *S* to the plane is

$$d = \left| \boldsymbol{PS}.\frac{\boldsymbol{n}}{|\boldsymbol{n}|} \right|$$

Ex: Find the distance from the point $S(1, 1, 3)$ and the plane $3x + 2y + 6z = 6$

Sol.:

First we find a point P on the plane and this can be made by fixing two variable in the plane equation, so, by setting x=0 and z=0 we get y=3, then $P(0, 3, 0)$



$$\boldsymbol{PS} = (1-0)\boldsymbol{i} + (1-3)\boldsymbol{j} + (3-0)\boldsymbol{k} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$$

$$\boldsymbol{n} = 3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}$$

$$|\boldsymbol{n}| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{9 + 4 + 36} = \sqrt{49} = 7$$

$$d = \left|\boldsymbol{PS}.\frac{\boldsymbol{n}}{|\boldsymbol{n}|}\right| = \frac{1}{7}(\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}).(\boldsymbol{n} = 3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}) = \frac{1}{7}(3 - 4 + 18) = \frac{17}{7}$$

Ex: For the points $A(1,2,0)$, $B(4,1,1)$, $C(0,4,3)$, $D(1,1,0)$ and $E(0,1,1)$, let M_1 be the plane that contains the points A, B and C, and let M_2 be the plane that contain the points C, D and E. Find the equation of the line of intersection of M_1 and M_2.

Sol.:

Since it is required to find line equation then we need to find a point on the line and a parallel vector. The parallel vector can be found by cross product of the normal vectors of the two planes n_1 and n_2, each normal vector can be found as cross product of two vectors on the panes that can be found from the given points. The point on the line is common to the two planes M_1 and M_2 and it can be seen that C is on M_1 and M_2.

$$\boldsymbol{AB} = (4-1)\boldsymbol{i} + (1-2)\boldsymbol{j} + (1-0)\boldsymbol{k} = 3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$$

$$\boldsymbol{AC} = (0-1)\boldsymbol{i} + (4-2)\boldsymbol{j} + (3-0)\boldsymbol{k} = -\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$$

$$\boldsymbol{AB} \times \boldsymbol{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & -1 & 1 \\ -1 & 2 & 3 \end{vmatrix} = (-3-2)\boldsymbol{i} - (9+1)\boldsymbol{j} + (6-1)\boldsymbol{k} = -5\boldsymbol{i} - 10\boldsymbol{j} + 5\boldsymbol{k}$$
$$= -5(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$$

$$\boldsymbol{n_1} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$$

$$\boldsymbol{CD} = (1-0)\boldsymbol{i} + (1-4)\boldsymbol{j} + (0-3)\boldsymbol{k} = \boldsymbol{i} - 3\boldsymbol{j} - 3\boldsymbol{k}$$

$$\boldsymbol{CE} = (0-0)\boldsymbol{i} + (1-4)\boldsymbol{j} + (1-3)\boldsymbol{k} = -3\boldsymbol{j} - 2\boldsymbol{k}$$

$$\boldsymbol{CD} \times \boldsymbol{CE} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -3 & 3 \\ 0 & -3 & -2 \end{vmatrix} = (6-9)\boldsymbol{i} - (-2+0)\boldsymbol{j} + (-3-0)\boldsymbol{k} = -3\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$$

$$\boldsymbol{n_2} = -3\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$$

$$\boldsymbol{n_1} \times \boldsymbol{n_2} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & -1 \\ -3 & 2 & -3 \end{vmatrix} = (-6+2)\boldsymbol{i} - (-3-3)\boldsymbol{j} + (2-6)\boldsymbol{k} = -4\boldsymbol{i} - 6\boldsymbol{j} - 4\boldsymbol{k}$$
$$= -2(2\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k})$$

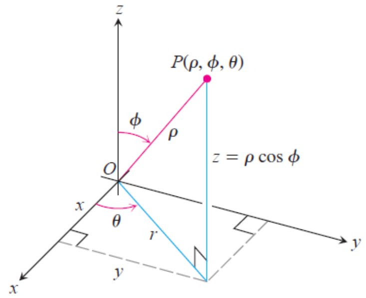
$\boldsymbol{v} = 2\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$

Since point C is on the line of intersection then the line parametric equations are

$x = 2t,\ y = 4 + 3t,\ z = 3 - 2t$

Cylindrical and Spherical Coordinates (11.7)

Cylindrical Coordinates



$r^2 = x^2 + y^2,\ \tan\theta = \frac{y}{x},\ 0 \le \theta \le 2\pi$

$x = r\cos\theta,\ y = r\sin\theta,\ z=z$

Ex:

$r = 2$ is a cylinder of radius 2 and axis is the z-axis.

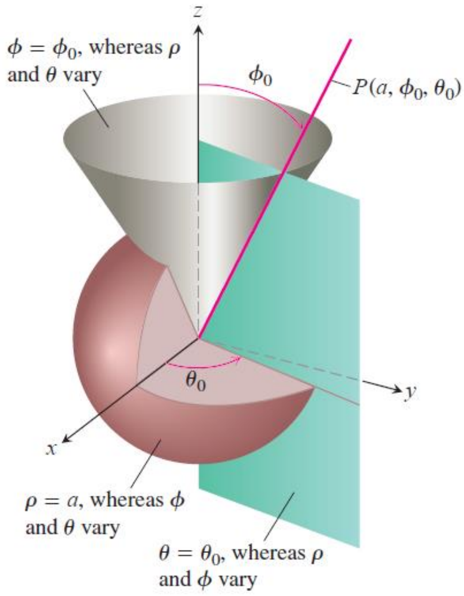
$\theta = \pi/4$, is a plane with $y=x$, $x \ge 0,\ y \ge 0$

$\tan\theta = \frac{y}{x} = \tan\frac{\pi}{4} = 1$

$z=0$ is the xy-plane

$z=2$ is a plane paralle1 to the xy-plane

Spherical Coordinates



$\rho^2 = x^2 + y^2 + z^2 = r^2 + z^2$

$z = \rho\cos\phi$

$r = \rho\sin\phi$

$\tan\phi = \frac{r}{z},\ 0 \le \theta \le \pi$

$x = r\cos\theta = \rho\sin\phi\cos\theta$

$y = r\sin\theta = \rho\sin\phi\sin\theta$

$\tan\theta = \frac{y}{x},\ 0 \le \theta \le 2\pi$

Ex:

$\rho = 2$ is a sphere with radius equals 2 and center at origin

$x^2 + y^2 + z^2 = 4$

$\phi = \pi/4$ is a cone open upward

$\phi = \pi/2$ is the *xy*-plane

$\theta = \pi/4$, is a plane with $y=x$, $x \geq 0$, $y \geq 0$

	(x, y, z)	(r, θ, z)	(ρ, ϕ, θ)
1	(0, 0, 0)	$(0, \theta, 0)$	$(0, \phi, \theta)$
2	(1, 0, 0)	(1, 0, 0)	$(1, \pi/2, 0)$
3	(0, 1, 0)	$(1, \pi/2, 0)$	$(1, \pi/2, \pi/2)$
4	(0, 0, 1)	$(0, \theta, 1)$	$(1, \theta, 0)$
5	$(\sqrt{2}, 0, 1)$	$(\sqrt{2}, 0, 1)$	$(\sqrt{3}, \tan^{-1}\sqrt{2}, 0)$
6	(0, 1, 1)	$(1, \pi/2, 1)$	$(\sqrt{2}, \pi/4, \pi/2)$
7	$(0, -3/2, \sqrt{3}/2)$	$(3/2, 3\pi/2, \sqrt{3}/2)$	$(\sqrt{3}, \pi/3, 3\pi/2)$
8	$(0, -2\sqrt{2}, 0)$	$(2\sqrt{2}, 3\pi/2, 0)$	$(2\sqrt{2}, \pi/2, 3\pi/2)$
9	$(0, 0, -\sqrt{2})$	$(0, \theta, -\sqrt{2})$	$(\sqrt{2}, \pi, \theta)$