Chapter 4 Functions of Two or More Variables and Their Derivatives (Ch 13)

Partial Derivatives (13.3)

Let $f(x,y)$ be a function of the two variables x and y. The **partial derivative** of f with respect x means to differentiate f with respect x while holding y as a constant. Similarly, the partial derivative of f with respect to y is performed while holding x as a constant. On the other hand the derivative of a function of a single variable is called the **ordinary derivative** or the **total derivative**.

The partial derivative is denoted by

$$f_x = \frac{\partial f}{\partial x},\ f_y = \frac{\partial f}{\partial y}$$

Ex: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for

a) $f(x,y) = x^2 + y^2$
b) $f(x,y) = (y-x)e^x$

a) $\frac{\partial f}{\partial x} = 2x,\ \frac{\partial f}{\partial y} = 2y$

b) $\frac{\partial f}{\partial x} = -e^x + (y-x)e^x = (y-x-1)e^x$

$$\frac{\partial f}{\partial y} = e^x$$

Ex: Find f_x, f_y and f_z for $f(x,y,z) = x - \sqrt{y^2+z^2}$

$f_x=1$

$$f_y = \frac{-y}{\sqrt{y^2+z^2}}$$

$$f_z = \frac{-z}{\sqrt{y^2+z^2}}$$

Second Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x^2} = f_{xx},\ \frac{\partial^2 f}{\partial y^2} = f_{yy},\ \frac{\partial^2 f}{\partial y \partial x} = f_{xy},\ \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$$

$$f_{xy} = f_{yx}$$

Ex: Find the second order partial derivatives of $f(x,y) = \sin(xy)$

$$f_x = y\cos(xy) \ \Rightarrow\ f_{xx} = -y^2\sin(xy)$$

$$f_y = x\cos(xy) \ \Rightarrow\ f_{yy} = -x^2\sin(xy)$$

$$f_{xy} = \cos(xy) - xy\sin(xy)$$

$$f_{yx} = \cos(xy) - xy\sin(xy)$$

Level Curves and Level Surfaces

A function *z*=*f*(*x*,*y*) represents a surface that we can plot in a 3-D space. There are two independent variables x and y and one dependent variable *z*. If we fix the variable *z*=*c* the number of independent variables will reduce by one and we will have *c*=*f*(*x*,*y*), this function has one independent variable (*x*) and one dependent variable (*y*), so, it represents a curve and it is called a **level curve** because it is reduced from a surface (higher dimensional) function.

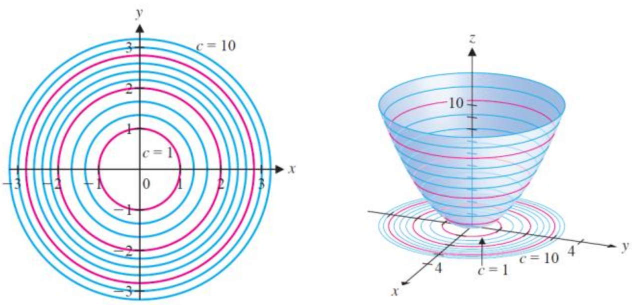
The function *w*=*f*(*x*,*y*,*z*) has three independent variable *x*, *y* and *z* and one dependent variable w, if we fix *w*=*c*, then the function *c*=*f*(*x*,*y*,*z*) will have two independent variables (*x* and *y*) and one dependent variable (*z*), so, it represents a surface and it is called a level surface because it is reduced from a higher dimensional function. For example consider the function $w = x^2 + y^2 + z^2$, when *w*=25 we get the level surface $25 = x^2 + y^2 + z^2$ which is a sphere centered a origin with radius of 5.

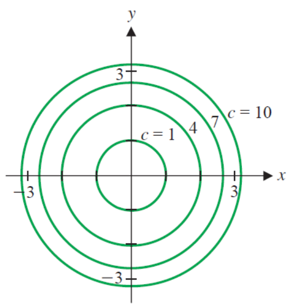
Ex: The level curves of the function $z = x^2 + y^2$ are plotted as below by assigning a certain constant to the variable *z*.

When *z*=1 we get the unit circle $x^2 + y^2 = 1$

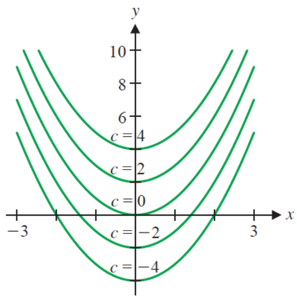
When *z*=4 we get $x^2 + y^2 = 4$ which is the circle centered at origin and radius equals 2

When *z*=10 we get $x^2 + y^2 = 10$ which is the circle centered at origin and radius equals $\sqrt{10}$

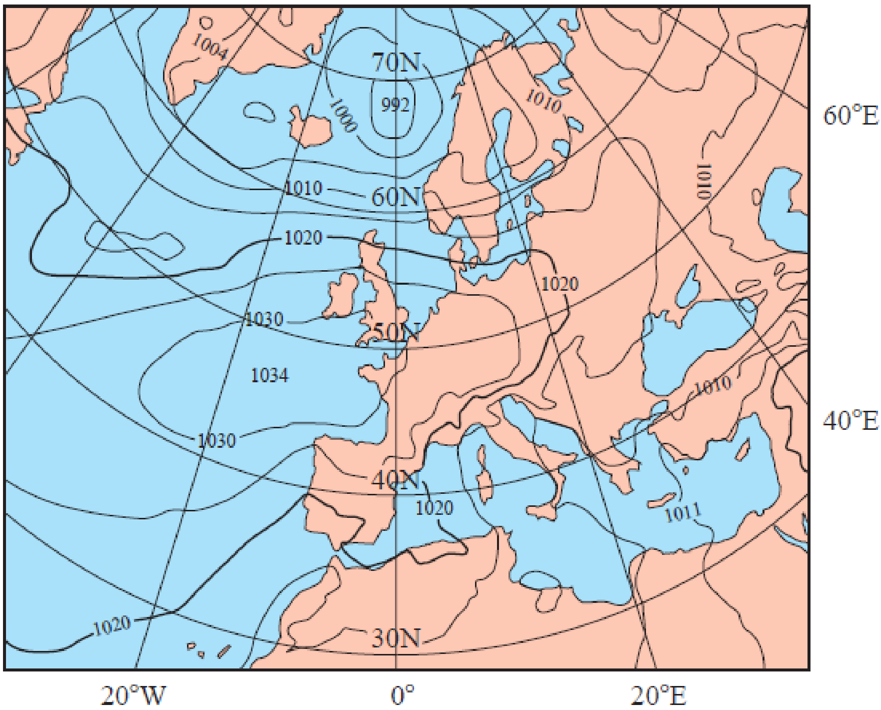




$z = x^2 + y^2$



$z = y - x^2$



The Gradient and the Directional Derivative

The gradient of the function $f(x,y,z)$ is given by the vector $\nabla \boldsymbol{f}$ or $\boldsymbol{gradf}$

$$\nabla \boldsymbol{f} = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}$$

The derivative of f at point P_o in the direction of $\boldsymbol{u}$ ($\boldsymbol{u}$ is a unit vector)

$$(\nabla \boldsymbol{f}.\boldsymbol{u})|_{P_o}$$

Ex: Find the derivative of $f = x^2 + 2y^2 - 3z^2$ at $P(1, 2, 3)$ in the direction of $\boldsymbol{A} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

Sol.:

$$f_x = 2x,\ f_y = 4y,\ f_z = -6z$$

$$\nabla \boldsymbol{f} = 2x\boldsymbol{i} + 4y\boldsymbol{j} - 6z\boldsymbol{k}$$

$$|\boldsymbol{A}| = \sqrt{4+4+1} = 3$$

$$\boldsymbol{u} = \frac{\boldsymbol{A}}{|\boldsymbol{A}|} = \frac{2}{3}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} + \frac{1}{3}\boldsymbol{k}$$

$$(\nabla \boldsymbol{f}.\boldsymbol{u})|_P = (2(1)\boldsymbol{i} + 4(2)\boldsymbol{j} - 6(3)\boldsymbol{k}).\left(\frac{2}{3}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} + \frac{1}{3}\boldsymbol{k}\right) = \frac{4}{3} + \frac{16}{3} - 6 = \frac{2}{3}$$

Ex: Find the derivative of the function $f = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of

$\boldsymbol{v} = 3\boldsymbol{i} - 4\boldsymbol{j}$

Sol.:

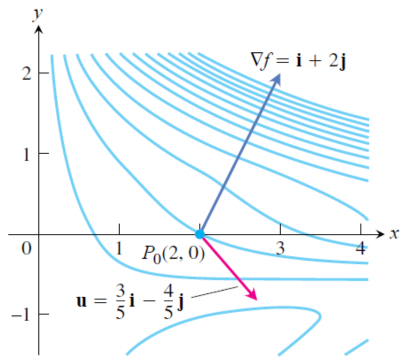
$f_x(2,0) = (e^y - y\sin(xy))_{(2,0)} = e^0 + 0 = 1$

$f_y(2,0) = (xe^y - x\sin(xy))_{(2,0)} = 2e^0 + 2(0) = 2$

$\boldsymbol{u} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{\boldsymbol{v}}{5} = \frac{3}{5}\boldsymbol{i} - \frac{4}{5}\boldsymbol{j}$

$\boldsymbol{\nabla f}_{(2,0)} = f_x(2,0)\boldsymbol{i} + f_y(2,0)\boldsymbol{j} = \boldsymbol{i} + 2\boldsymbol{j}$

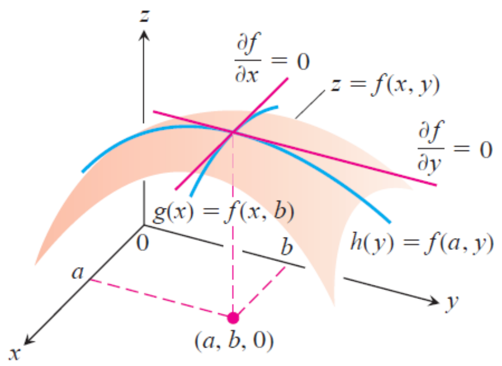
$\boldsymbol{\nabla f}_{(2,0)}.\boldsymbol{u} = (\boldsymbol{i} + 2\boldsymbol{j}).\left(\frac{3}{5}\boldsymbol{i} - \frac{4}{5}\boldsymbol{j}\right) = \frac{3}{5} - \frac{8}{5} = -1$

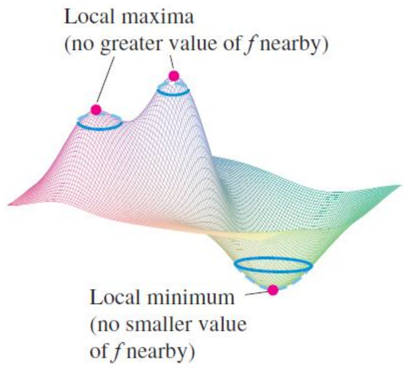


Maxima, Minima and Saddle points (Extreme Points) (13.8)

To find the maximum and minimum points of $f(x,y)$ in the region R

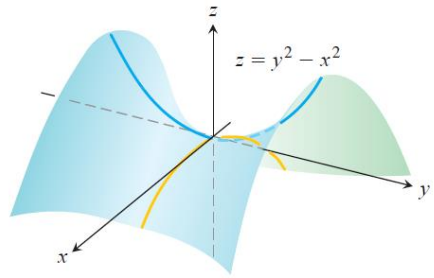
1- Find the points where f_x=0 and f_y=0 in the interior of R (local maximum and minimum).
2- Find the maximum and minimum points on the boundary of R.





Second Derivative Test

1- If $D = f_{xx}f_{yy} - (f_{xy})^2\Big|_{(a,b)} > 0$ and $f_{xx}|_{(a,b)} < 0$ then f has a local maximum at (a,b)
2- If $D = f_{xx}f_{yy} - (f_{xy})^2\Big|_{(a,b)} > 0$ and $f_{xx}|_{(a,b)} < 0$ then f has a local minimum at (a,b)
3- If $D = f_{xx}f_{yy} - (f_{xy})^2\Big|_{(a,b)} < 0$ then f has a saddle point at (a,b)
4- If $D = f_{xx}f_{yy} - (f_{xy})^2\Big|_{(a,b)} = 0$ then the test is inconclusive



Ex: Find the extreme values of $f = x^2 + y^2$

Sol.:

Note: there is no boundary for the domain

$f_x = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$

$f_y = 0 \Rightarrow 2y = 0 \Rightarrow y = 0$

$f_{xx} = 2 > 0,\ f_{yy} = 2,\ f_{xy} = 0$

$D = f_{xx}f_{yy} - (f_{xy})^2 = (2)(2) - 0 = 4 > 0$

Then (0, 0) is a local minimum

Ex: Find the extreme points of $f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$

Sol.:

$f_x = y - 2x - 2 = 0 \quad (\times 2)$

$f_y = x - 2y - 2 = 0$

$-3x - 6 = 0 \Rightarrow x = -2 \Rightarrow y = 2x + 2 = 2(-2) + 2 = -2$

Extreme point at (–2, –2)

$f_{xx} = -2 < 0,\ f_{yy} = -2,\ f_{xy} = 1$

$f_{xx}f_{yy} - (f_{xy})^2 = (-2)(-2) - 1 = 3 > 0$

Therefore $f(-2, -2) = 8$ is a local maximum

Ex: Find the extreme points of $f(x, y) = xy$

Sol.:

$f_x = y = 0$

$f_y = x = 0$

$f_{xx} = 0,\ f_{yy} = 0,\ f_{xy} = 1$

$D = f_{xx}f_{yy} - (f_{xy})^2 = (0)(0) - 1 = -1 < 0$

Therefore the point (0, 0) is a saddle point

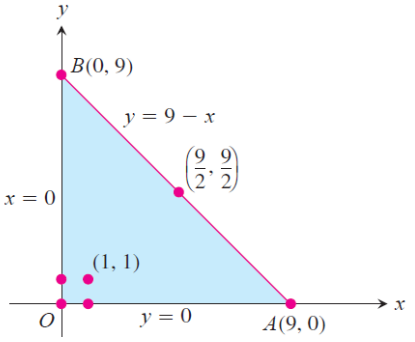
Ex: Find the absolute maximum and minimum values of $f(x, y) = 2 + 2x + 2y - x^2 - y^2$ on the triangular plate (region) in the first quadrant bounded by the lines x=0, y=0, and x+y=9

Sol.:

First we need to find the local minimum and maximum

$f_x = 2 - 2x = 0 \Rightarrow x = 1$

$f_y = 2 - 2y = 0 \Rightarrow y = 1$



$P(1, 1)$ is an interior point

$f_{xx} = -2 < 0$, $f_{yy} = -2$, $f_{xy} = 0$

$D = f_{xx}f_{yy} - \left(f_{xy}\right)^2 = (-2)(-2) - 0 = 4 > 0$

Therefore $P(1, 1)$ is a local maximum, $f(1,1)=4$

Second we find the extreme points on the boundary

On the segment OA: $y=0$

$f(x,0) = 2 + 2x - x^2$

$f'(x,0) = 2 - 2x = 0 \Rightarrow x = 1$

$f(1,0)=3$

On the segment OB: $x=0$

$f(0,y) = 2 + 2y - y^2$

$f'(0,y) = 2 - 2y \Rightarrow y = 1$

$f(0,1)=3$

On the segment AB: $y=9-x$

$f(x, 9-x) = 2 + 2x + 2(9-x) - x^2 - (9-x)^2$

$f'^{(x,9-x)} = 2 - 2 - -2x - 2(9-x)(-) = -4x + 18 = 0 \Rightarrow x = \frac{9}{2} \Rightarrow y = 9 - \frac{9}{2} = \frac{9}{2}$

$f\left(\frac{9}{2}, \frac{9}{2}\right) = \frac{-41}{2}$

The value of f at the boundary points A, B and O

$f(9, 0)=-61$, $f(0,9)=-61$ and $f(0,0)=2$

In summary, the extreme values of f are: 4, 3, –41/2, –61

The absolute maximum is 4 at (1,1) and the absolute minimum is –61 at (0,9) and (9,0)

Exercises 13.8

Find the maxima, minima and saddle points

24- $f(x,y) = 4xy - x^4 - y^4$

$f_x = 4y - 4x^3 = 0 \Rightarrow y = x^3$

$f_y = 4x - 4y^3 = 0$

$4x - 4x^9 = 0$

$$x(1-x^8) = x(1-x^4)(1+x^4) = x(1-x^2)(1+x^2)(1+x^4)$$
$$= x(1-x)(1+x)(1+x^2)(1+x^4) = 0$$

$x = 0 \Rightarrow y = 0$

$x = 1 \Rightarrow y = 1$

$x = -1 \Rightarrow y = -1$

$f_{xx} = -12x^2$

$f_{yy} = -12y^2$

$f_{xy} = 4$

$D = f_{xx}f_{yy} - (f_{xy})^2 = (-12x^2)(-12y^2) - 4^2 = 144x^2y^2 - 16$

At (0, 0): $f(0,0)=0$, $D=-16<0$ Saddle point

At (1, 1): $f(1,1)=2$, $D=128>0$, $f_{xx} = -12 < 0$ local minimum

At (–1, –1): $f(-1,-1)=2$, $D=128>0$, $f_{xx} = -12 < 0$ local minimum

Lagrange Multipliers (13.8)

To find the maxima and minima of a function $f(x, y, z)$ with the constraint $g(x, y, z)=0$ we need to solve the equations

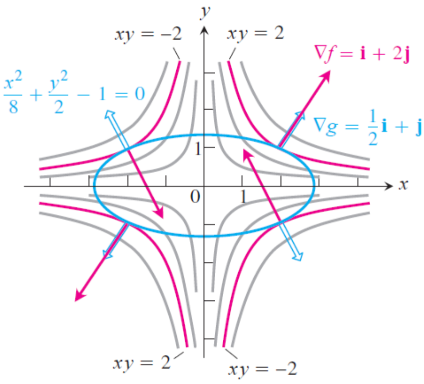
$\nabla \boldsymbol{f} = \lambda \nabla \boldsymbol{g}$ and $g(x, y, z) = 0$ (the parameter λ is called Lagrange Multiplier)

$f_x\boldsymbol{i} + f_y\boldsymbol{j} + f_z\boldsymbol{k} = \lambda(g_x\boldsymbol{i} + g_y\boldsymbol{j} + g_z\boldsymbol{k})$ and $g(x, y, z) = 0$

Ex: Find the greatest and smallest values of the function $f(x, y)=xy$ that takes on the ellipse

$\frac{x^2}{8} + \frac{y^2}{2} = 1$

Sol.:



The function $f(x, y) = xy$

The constraint $g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$

$\nabla \boldsymbol{f} = y\boldsymbol{i} + x\boldsymbol{j}$

$\nabla \boldsymbol{g} = \frac{x}{4}\boldsymbol{i} + y\boldsymbol{j}$

$\nabla \boldsymbol{f} = \lambda \nabla \boldsymbol{g}$

$y\boldsymbol{i} + x\boldsymbol{j} = \lambda\left(\frac{x}{4}\boldsymbol{i} + y\boldsymbol{j}\right)$

$y = \frac{\lambda}{4}x$ (1)

$x = \lambda y$ (2)

$\frac{x^2}{8} + \frac{y^2}{2} = 1$ (3)

Substitute (2) in (1)

$y = \frac{\lambda}{4}(\lambda y)$

$y = \frac{\lambda}{4}(\lambda y) = \frac{\lambda^2}{4}y$

$y - \frac{\lambda^2}{4}y = 0$

$y\left(1 - \frac{\lambda^2}{4}\right) = 0$

$y = 0$ or

$\left(1 - \frac{\lambda^2}{4}\right) = 0 \Rightarrow \lambda = \pm 2$

Case 1: If $y=0 \Rightarrow x=0$ but the point (0, 0) is not on the ellipse (dose not satisfy the constraint equation ($g(x,y)=0$) then $y \neq 0$

Case 2: $y \neq 0$, $\lambda = \pm 2 \Rightarrow x = \pm 2y$ by substituting in (3) ($g(x,y)=0$)

$\frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1$

$y^2 + y^2 = 2 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

The extreme points on the ellipse are (2, 1), (–2, 1), (2, –1) and (–2, –1)

The extreme values of the function $f(x,y)=xy$ on the ellipse

$f(2,1) = f(-2,-1) = 2$ maximum point ($\lambda=2$)

$f(-2,1) = f(2,-1) = -2$ minimum point ($\lambda=-2$)

Ex: Find the extreme points of the function $f(x,y) = \frac{x^2}{\sqrt{2}} + y$ on the circle $x^2 + y^2 = 1$

Sol.:

$\nabla \boldsymbol{f} = \sqrt{2}x\boldsymbol{i} + \boldsymbol{j}$

$\nabla \boldsymbol{g} = 2x\boldsymbol{i} + 2y\boldsymbol{j}$

$\nabla \boldsymbol{f} = \lambda \nabla \boldsymbol{g}$

$\sqrt{2}x\boldsymbol{i} + \boldsymbol{j} = \lambda(2x\boldsymbol{i} + 2y\boldsymbol{j})$

$\sqrt{2}x = 2\lambda x$ (1)

$1 = 2\lambda y$ (2)

$x^2 + y^2 = 1$ (3)

From (1)

$\sqrt{2}x\left(1 - \sqrt{2}\lambda\right) = 0$

Case1: x=0 substitute in (3) ⇒ y=±1

Case2: $\left(1 - \sqrt{2}\lambda\right) \Rightarrow \lambda = \frac{1}{\sqrt{2}}$

From (2)

$y = \frac{1}{2\lambda} = \frac{1}{\sqrt{2}}$ substitute in (3) ⇒ $x = \pm\frac{1}{\sqrt{2}}$

The extreme points are

$f(0,1) = 1$, $f(0,-1) = -1$ (minimum)

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{3}{2\sqrt{2}}$ (maximum)

Exercises 13.9

5- Find the point on the curve $x^2y = 2$ nearest to origin

Sol.

The problem is asking for the nearest point, this means the minimum distance. Therefore, the function that we need to find its minimum is the distance between a general point (x, y) and the origin (0, 0)

$d = \sqrt{x^2 + y^2}$

When finding the minimum of the distance, it is better to work on the square of the distance to avoid working with square root, this does not affect the answer. Therefore

$d^2 = f(x, y) = x^2 + y^2$

The problem is asking for the point to be on the curve $x^2y = 2$ this means that this equation is the constraint equation

$g(x, y) = x^2y - 2 = 0$

$\boldsymbol{\nabla f} = 2x\boldsymbol{i} + 2y\boldsymbol{j}$

$\boldsymbol{\nabla g} = 2xy\boldsymbol{i} + x^2\boldsymbol{j}$

$\boldsymbol{\nabla f} = \lambda\boldsymbol{\nabla g} \Rightarrow 2x\boldsymbol{i} + 2y\boldsymbol{j} = \lambda(2xy\boldsymbol{i} + x^2\boldsymbol{j}) = 2\lambda xy\boldsymbol{i} + \lambda x^2\boldsymbol{j}$

$2x = 2\lambda xy$

$x = \lambda xy$ (1)

$2y = \lambda x^2$ (2)

$x^2 y = 2$ (3)

From (1)

$x(1 - \lambda y) = 0$

Due to the constraint *x* cannot equal to zero, $x \neq 0$, therefore

$1 - \lambda y = 0 \Rightarrow \lambda = \frac{1}{y}$ substitute in (2)

$2y = \frac{x^2}{y} \Rightarrow 2y^2 = x^2$ substitute in (3)

$2y^3 = 2 \Rightarrow y^3 = 1 \Rightarrow y = 1$

$x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

The points on $x^2 y = 2$ nearest to origin are $(\sqrt{2}, 1)$ and $(-\sqrt{2}, 1)$

$f(\pm\sqrt{2}, 1) = (\pm\sqrt{2})^2 + (1)^2 = 3$

$d = \sqrt{3}$

8- Find the points on the curve $x^2 + xy + y^2 = 1$ in the *xy*-plane that are nearest and farthest to the origin.

Sol.

Similar to the previous exercise, since it is requires the minimum and maximum distance we can set the function as

$f(x, y) = x^2 + y^2$

The constraint is

$g(x, y) = x^2 + xy + y^2 - 1$

$\nabla \boldsymbol{f} = 2x\boldsymbol{i} + 2y\boldsymbol{j}$

$\nabla \boldsymbol{g} = (2x + y)\boldsymbol{i} + (x + 2y)\boldsymbol{j}$

$\nabla \boldsymbol{f} = \lambda \nabla \boldsymbol{g} \Rightarrow 2x\boldsymbol{i} + 2y\boldsymbol{j} = \lambda\big((2x + y)\boldsymbol{i} + (x + 2y)\boldsymbol{j}\big) = \lambda(2x + y)\boldsymbol{i} + \lambda(x + 2y)\boldsymbol{j}$

$2x = \lambda(2x + y)$ (1)

$2y = \lambda(x + 2y)$ (2)

$x^2 + xy + y^2 = 1$ (3)

From (2) $y = 2x(1-\lambda)/\lambda$ (4)

From (3) $x = 2y(1-\lambda)/\lambda$ (5)

Substitute (4) into (5)

$$x = 4x\frac{(1-\lambda)^2}{\lambda^2}$$

From the constraint, the point (0, 0) is not on the curve $x^2 + xy + y^2 = 1$, therefore $x \neq 0$ and we can cancel x from above

$$\frac{4(1-\lambda)^2}{\lambda^2} = 1$$

$$\frac{2(1-\lambda)}{\lambda} = \pm 1$$

Case1: $\frac{2(1-\lambda)}{\lambda} = 1 \Rightarrow$

$2(1-\lambda) = \lambda \Rightarrow 3\lambda = 2 \Rightarrow \lambda = 2/3$

Substitute in (4)

$y = \frac{2x\left(1-\frac{2}{3}\right)}{\frac{2}{3}} \Rightarrow y = x$ substitute in (3)

$x^2 + x^2 + x^2 = 1 \Rightarrow 3x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{3}} \Rightarrow y = \pm\frac{1}{\sqrt{3}}$

$f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) = f\left(\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right) = \frac{1}{3}+\frac{1}{3} = \frac{2}{3} \Rightarrow d = \sqrt{\frac{2}{3}}$ (minimum values of nearest distance)

Case2: $\frac{2(1-\lambda)}{\lambda} = -1 \Rightarrow$

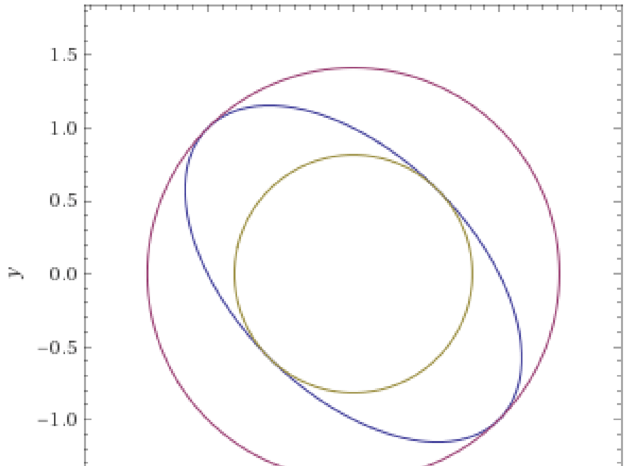
$2(1-\lambda) = -\lambda \Rightarrow \lambda = 2$

Substitute in (4)

$y = \frac{2x(1-2)}{2} \Rightarrow y = -x$ substitute in (3)

$x^2 - x^2 + x^2 = 1 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \Rightarrow y = \mp 1$

$f(1,-1) = f(-1,1) = 1 + 1 = 2 \Rightarrow d = \sqrt{2}$ (maximum values of farthest distance)



Assignment Chapter 4

1- Find the minimum and maximum values of the function $f(x, y) = 2x^2 + 3xy + 2y^2 + x - y$ inside and on the triangular region bounded by the lines $y=0$, $y=3-x$ and $y=3+x$.

2- Find the extreme values of the function $f(x, y) = x^2 y$ on the ellipse $\frac{x^2}{2} + y^2 = 1$