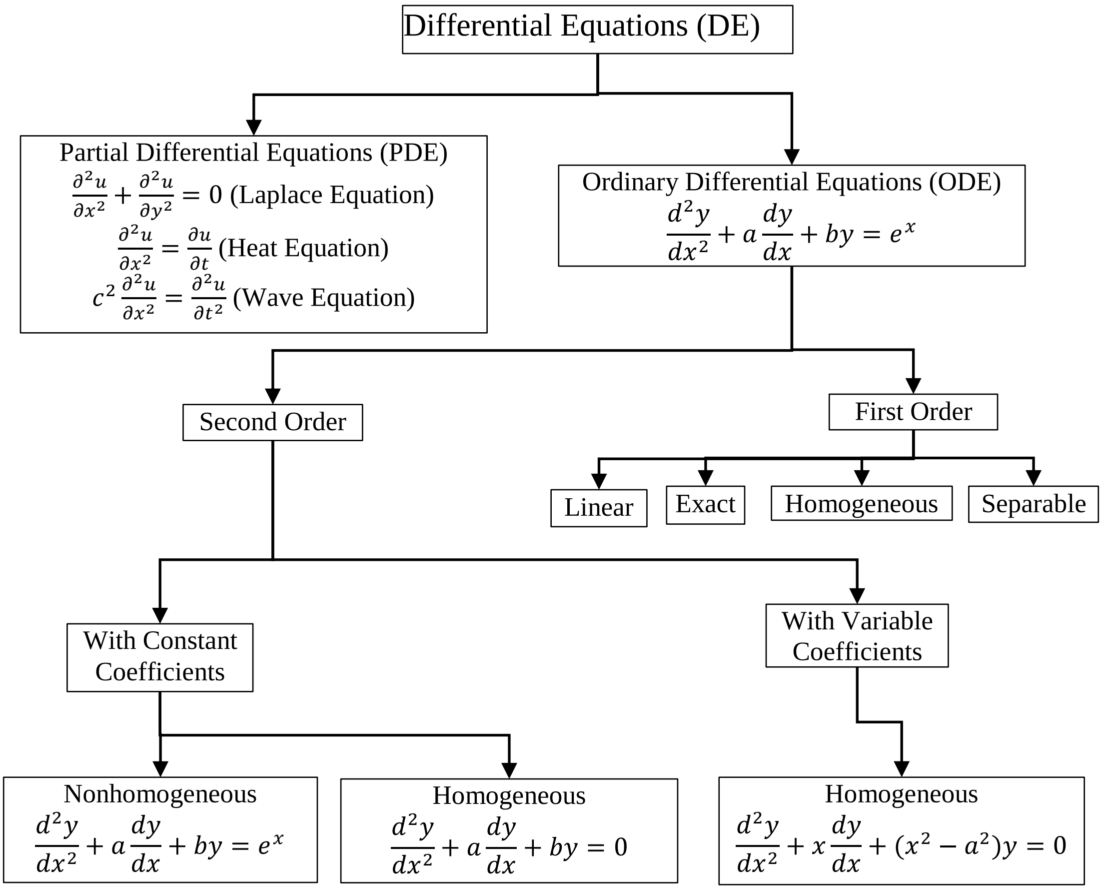
Chapter 5 Differential Equations (Chapter 16)



Note:

1- A DE equation is linear when the term y (the dependent variable) and its derivatives y' appear with power of 1 and there is no multiplication term like yy' or a function of y or y' like e^y or $\sin(y')$

2- The order of the DE is the highest order of the derivative in the DE.

Separable First Order DE (16.1)

The general form of a separable DE is

$$M(x)+N(y)\frac{dy}{dx}=0$$

Or in the differential form

$$M(x)dx+N(y)dy=0$$

Ex: Solve the DE $\frac{dy}{dx} = (1+y^2)e^x$

Sol.:

$$\frac{dy}{1+y^2} = e^x dx$$

$$\int \frac{dy}{1+y^2} = \int e^x dx$$

$$\tan^{-1} y = e^x + C \Rightarrow y = \tan(e^x + C)$$

Homogeneous First Order DE

A first order DE is homogeneous if it can be put in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

This can be changed into a separable equation by the substation

$$v = \frac{y}{x} \Rightarrow y = xv$$

By differentiating both side

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = F(v)$$

$$v - F(v) = -x\frac{dv}{dx}$$

$$-\frac{dx}{x} = \frac{dv}{v - F(v)}$$

$$\frac{dv}{v - F(v)} + \frac{dx}{x} = 0$$

Ex: Solve the DE $\frac{dy}{dx} = -\frac{x^2+y^2}{2xy}$ subjected to the condition $y(1)=1$.

$$\frac{dy}{dx} = -\frac{x^2}{2xy} - \frac{y^2}{2xy} = -\frac{1}{2}\left(\frac{x}{y} + \frac{y}{x}\right) = -\frac{1}{2}\left(\frac{1}{y/x} + \frac{y}{x}\right)$$

$$F(v) = -\frac{1}{2}\left(\frac{1}{v} + v\right)$$

$$\frac{dx}{x}+\frac{dv}{v+\frac{1}{2}\left(\frac{1}{v}+v\right)}=0$$

$$\frac{dx}{x}+\frac{dv}{v+\frac{1+v^2}{2v}}=0$$

$$\frac{dx}{x}+\frac{2vdv}{3v^2+1}=0$$

$$\ln(x)+\frac{1}{3}\ln(3v^2+1)=C$$

$$3\ln(x)+\ln(3v^2+1)=3C$$

$$\ln(x^3)+\ln(3v^2+1)=\ln\left(x^3(3v^2+1)\right)=3C$$

$$x^3(3v^2+1)=e^{3C}=C'$$

$$x^3\left(3\left(\frac{y}{x}\right)^2+1\right)=3xy^2+x^3=C'$$

$$3(1)(1)^2+(1)^3=4=C'$$

$$3xy^2+x^3=4$$

Or

$$\frac{dy}{dx}=-\frac{x^2+y^2}{2xy}$$

$$v+x\frac{dv}{dx}=-\frac{x^2+x^2v^2}{2x(xv)}=-\frac{x^2(1+v^2)}{2x^2v}=-\frac{(1+v^2)}{2v}$$

$$x\frac{dv}{dx}=-v-\frac{(1+v^2)}{2v}=-\frac{2v^2+1+v^2}{2v}=-\frac{3v^2+1}{2v}$$

$$\frac{2vdv}{3v^2+1}=-\frac{dx}{x}$$

$$\frac{dx}{x}+\frac{2vdv}{3v^2+1}=0$$

The rest is the same as before.

Exact DE (16.2)

The DE of the form

$$M(x,y)dx+N(x,y)dy=0$$

Or

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

If for some function (*x*,*y*)

$$M(x,y)dx + N(x,y)dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = df$$

The above differential form is exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Since $M = \frac{\partial f}{\partial x}$ and $N = \frac{\partial f}{\partial y}$

The solution of the exact DE

$$M(x,y)dx + N(x,y)dy = 0$$

Is $f(x,y) = C$

Ex: Solve the differential equation

$$x^2 + y^2 + (2xy + \cos y)y' = 0$$

Sol.:

In order to test it the above DE is exact we write it in the standard from

$$(x^2 + y^2)dx + (2xy + \cos y)dy = 0$$

$$M = x^2 + y^2, N = 2xy + \cos y$$

$\frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = 2y \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ The DE is exact

Since $M = \frac{\partial f}{\partial x} = x^2 + y^2$ then

$$f(x,y) = \int_{y=const} (x^2 + y^2)dx = \frac{x^3}{3} + xy^2 + k(y)$$

Note that the constant of integration is a function of *y* because we are integrating with respect to *x* and keeping *y* constant. In order to find *k*(*y*) we differentiated with respect to *y* and set the derivative equal to *N*(*x*,*y*)

$$\frac{\partial f}{\partial y} = N(x,y) = 2xy + k'(y) = 2xy + \cos y$$

$$k'(y) = \cos y \Rightarrow k(y) = \sin y \Rightarrow$$

$$f(x,y) = \frac{x^3}{3} + xy^2 + \sin y = C$$

Linear First Order DE (16.3)

Linear first order equation is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

This equation can be solved by multiplying both sides by an integrating factor $\rho(x)$ that makes the LHS of the equation

$$\rho\frac{dy}{dx} + \rho P y = \rho Q$$

into the derivative of the product (ρy)

$$\frac{d}{dx}(\rho y) = \rho Q$$

$$\rho y = \int \rho Q dx$$

$$y = \frac{1}{\rho}\int \rho Q dx$$

To find ρ

$$\frac{d}{dx}(\rho y) = \rho\frac{dy}{dx} + \rho P y$$

$$\rho\frac{dy}{dx} + y\frac{d\rho}{dx} = \rho\frac{dy}{dx} + \rho P y$$

$$y\frac{d\rho}{dx} = \rho P y$$

$$\frac{d\rho}{dx} = \rho P$$

$$\frac{d\rho}{\rho} = P dx$$

$$\ln(\rho) = \int P dx + C_1 \;\Rightarrow\; \rho = Ce^{\int P dx} = e^{\int P dx}$$

Note the constant C will be cancelled in the following equation so we can drop it

$$y(x) = \frac{1}{\rho(x)}\int \rho(x) Q(x) dx$$

Where

$$\rho = e^{\int P dx}$$

Ex: Solve the equation $x\frac{dy}{dx} - 3y = x^2$, $x>0$

Sol.:

Firs we put the DE in the standard form

$$\frac{dy}{dx} - \frac{3}{x}y = x$$

$$P(x) = \frac{-3}{x},\ Q(x) = x$$

$$\int Pdx = \int \frac{-3}{x}dx = -3\ln(x) = \ln\left(\frac{1}{x^3}\right)$$

$$\rho = e^{\int Pdx} = \frac{1}{x^3}$$

$$y = \frac{1}{\rho(x)}\int \rho(x)Q(x)dx = x^3\int \frac{x}{x^3}dx = x^3\int \frac{1}{x^2}dx = x^3\left(\frac{-1}{x} + C\right)$$

$$y = Cx^3 - x^2$$

Exercises 16.3

8- $\cosh x\frac{dy}{dx} + (\sinh x)y = e^{-x}$

$$\frac{dy}{dx} + \left(\frac{\sinh x}{\cosh x}\right)y = \frac{e^{-x}}{\cosh x}$$

$$P(x) = \frac{\sinh x}{\cosh x}$$

$$\int P(x)dx = \int \frac{\sinh x}{\cosh x}dx = \ln(\cosh x)$$

$$\rho(x) = \cosh x$$

$$y = \frac{1}{\rho(x)}\int \rho(x)Q(x)dx = \frac{1}{\cosh x}\int \cosh x\frac{e^{-x}}{\cosh x}dx$$

$$y = \frac{1}{\cosh x}\int e^{-x}dx = \frac{1}{\cosh x}(-e^{-x} + C)$$

$$y = \frac{-e^{-x} + C}{\cosh x} = \frac{-e^{-x} + C}{(e^x + e^{-x})/2} = \frac{2Ce^x - 2}{(e^{2x} + 1)}$$

Second Order Linear Homogeneous Equations with Constant Coefficients (16.4)

The equation

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = 0$$

Is called homogeneous equation, it can be written in the form

$D^2y + aDy + by = 0 \qquad (D = \frac{d}{dt})$

$(D^2 + aD + b)y = 0$

D is the linear differentiation operator $(\frac{d}{dt})$

$(D^2 + aD + b) = (D - \lambda_1)(D - \lambda_2) = 0$

Where λ_1 and λ_2 are roots of the characteristic equation

$\lambda^2 + a\lambda + b = 0$

The general solution of the homogeneous equation depends on the value of λ as follows

Roots of the characteristic equation	Solution
λ_1 and λ_2 are real and distinct	$y = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}$
λ_1 and λ_2 are real and equal	$y = C_1te^{\lambda_1 t} + C_2e^{\lambda_1 t}$
λ_1 and λ_2 are complex conjugate $=\alpha \pm j\beta$	$y = e^{\alpha t}(C_1 \cos\beta t + C_2 \sin\beta t)$

Note: The solution of a second order DE should be always a linear combination of two linearly independent solutions

Exercises 16.4

4- $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 0$

$\lambda^2 - 2\lambda - 3 = 0$

$(\lambda - 3)(\lambda + 1) = 0$

$\lambda_1 = 3, \lambda_2 = -1$

$y = C_1e^{3t} + C_2e^{-t}$

18- $4y'' - 2y' + y = 0$, $y(0)=4$, $y'(0)=2$

$4\lambda^2 - 2\lambda + 1 = 0$

$$\lambda = \frac{2 \pm \sqrt{4 - 16}}{8} = \frac{1}{4} \pm j\frac{\sqrt{3}}{4}$$

$$y(t) = e^{t/4}\left(C_1 \cos\frac{\sqrt{3}}{4}t + C_2 \sin\frac{\sqrt{3}}{4}t\right)$$

$y(0) = C_1 = 4$

$$y'(t) = \frac{1}{4}e^{\frac{t}{4}}\left(4\cos\frac{\sqrt{3}}{4}t + C_2\sin\frac{\sqrt{3}}{4}t\right) + e^{t/4}\left(-\sqrt{3}\sin\frac{\sqrt{3}}{4}t + \frac{\sqrt{3}}{4}C_2\cos\frac{\sqrt{3}}{4}t\right)$$

$$y'(0) = \frac{1}{4}(4+0) + \frac{\sqrt{3}}{4}C_2 = 2$$

$$\frac{\sqrt{3}}{4}C_2 = 1 \;\Rightarrow\; C_2 = \frac{4}{\sqrt{3}}$$

$$y(t) = 4e^{t/4}\left(\cos\frac{\sqrt{3}}{4}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{4}t\right)$$

Second Order Nonhomogeneous Linear Equations (16.5)

The nonhomogeneous equation is of the form

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = F(t)$$

First we should find the solution of the homogeneous equation (y_h)

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = 0$$

The we find the particular solution (y_p) of the nonhomogeneous equation. The complete solution will be

$$y = y_h + y_p$$

Variation of Parameters

The solution of the nonhomogeneous equation using the method of variation of parameters is of the form

$$y = v_1u_1 + v_2u_2$$

The solution of the solution of the homogeneous equation is

$$y_h = C_1u_1(t) + C_2u_2(t)$$

The method of be followed by the steps

Step1: Solve the homogeneous part

$$y'' + ay' + by = 0$$

to find the functions u_1 and u_2

Step2: Calculate D and find $v_1{}'$ and $v_2{}'$ by

$D = \begin{vmatrix} u_1 & u_2 \\ u_1{}' & u_2{}' \end{vmatrix} = u_1 u_2' - u_2 u_1'$

$v_1' = \dfrac{-u_2 F(t)}{D}$

$v_2' = \dfrac{u_1 F(t)}{D}$

Step3: Integrate $v_1{}'$ and $v_2{}'$ to find v_1 and v_2

Step4: Write the general solution as

$y = v_1 u_1 + v_2 u_2$

Exercises 16.5

1- $\frac{d^2y}{dt^2} + \frac{dy}{dt} = t$

$y'' + y' = 0$

$\lambda^2 + \lambda = 0$

$\lambda(\lambda + 1) = 0$

$\lambda = 0, \lambda = -1$

$u_1 = 1, u_2 = e^{-t}$

$D = \begin{vmatrix} u_1 & u_2 \\ u_1{}' & u_2{}' \end{vmatrix} = \begin{vmatrix} 1 & e^{-t} \\ 0 & -e^{-t} \end{vmatrix} = -e^{-t}$

$v_1' = \dfrac{-u_2 F(t)}{D} = \dfrac{-e^{-t}(t)}{-e^{-t}} = t$

$v_1 = \displaystyle\int t dt = \frac{t^2}{2} + C_1$

$v_2' = \dfrac{u_1 F(t)}{D} = \dfrac{(1)(t)}{-e^{-t}} = -te^{t}$

$v_2 = \displaystyle\int -te^{t} dt = -te^{t} + \int e^{t} dt$

$v_2 = -te^{t} + e^{t} + C_2$

$y = v_1 u_1 + v_2 u_2$

$y = (1)\left(\dfrac{t^2}{2} + C_1\right) + (e^{-t})(-te^{t} + e^{t} + C_2)$

$y = \dfrac{t^2}{2} + C_1 - t + 1 + C_2 e^{-t}$

$$y = C_3 + C_2 e^{-t} - t + \frac{t^2}{2}$$

$$C_3 = C_3 + 1$$

2- $\frac{d^2y}{dt^2} + y = \tan t$

$$y'' + y = 0$$

$$\lambda^2 + 1 = 0 \;\Rightarrow\; \lambda = \pm j$$

$$u_1 = \cos t,\, u_2 = \sin t,$$

$$D = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

$$v_1' = \frac{-u_2 F(t)}{D} = -\sin t \tan t = \frac{-\sin^2 t}{\cos t}$$

$$v_1 = \int \frac{-\sin^2 t}{\cos t} dt = \int \frac{\cos^2 t - 1}{\cos t} dt = \int (\cos t - \sec t) dt$$

$$v_1 = \sin t - \ln(\sec t + \tan t) + C_1$$

$$v_2' = \frac{u_1 F(t)}{D} = \cos t \tan t = \sin t$$

$$v_2 = \int \sin t \, dt = -\cos t + C_2$$

$$y = v_1 u_1 + v_2 u_2$$

$$y = \cos t\,(\sin t - \ln(\sec t + \tan t) + C_1) + \sin t\,(-\cos t + C_2)$$

$$y = \sin t \cos t - \cos t \ln(\sec t + \tan t) + C_1 \cos t - \sin t \cos t + C_2 \sin t$$

$$y = -\cos t \ln(\sec t + \tan t) + C_1 \cos t + C_2 \sin t$$

Undetermined Coefficients

The method of undetermined coefficients is used for special cases of F(t) to determine y_p. We choose y_p according to the table below

$F(t)$	Condition	y_p
$e^{\lambda t}$	λ is not a root of the c/c eqn.	$Ae^{\lambda t}$
	λ is single root of the c/c eqn.	$Ate^{\lambda t}$
	λ is double root of the c/c eqn.	$At^2e^{\lambda t}$
$\sin kt$, $\cos kt$	kj is not a root of the c/c eqn.	$B\cos kt + C\sin kt$
	kj is a root of the c/c eqn.	$Bt\cos kt + Ct\sin kt$
$at^2 + bt + c$	0 is not a root of the c/c eqn.	$Dt^2 + Et + F$
	0 is a single root of the c/c eqn.	$Dt^3 + Et^2 + Ft$

	0 is a double root of the c/c eqn.	$Dt^4 + Et^3 + Ft^2$

Exercises 16.5

14- $y'' - 3y' - 10y = 2t - 3$

$y'' - 3y' - 10y = 0$

$\lambda^2 - 3\lambda - 10\lambda = 0$

$(\lambda - 5)(\lambda + 2) = 0$

$\lambda_1 = 5, \lambda_2 = -2$

$y_h = C_1 e^{5t} + C_2 e^{-2t}$

$y_p = At + B$

$y_p{}' = A$

$y_p{}'' = 0$

$0 - 3A - 10(At + B) = 2t - 3$

$0 - 3A - 10At - 10B = 2t - 3$

$-10At - 3A - 10B = 2t - 3$

$-10A = 2 \Rightarrow A = -1/5$

$-3(A) - 10B = -3$

$-3\left(-\frac{1}{5}\right) - 10B = -3$

$B = 9/25$

$y = C_1 e^{5t} + C_2 e^{-2t} - \frac{t}{5} + \frac{9}{25}$

17- $y'' + y = \cos 3t$

$y'' + y = 0$

$\lambda^2 + 1 = 0$

$\lambda = \pm j$

$y_h = C_1 \cos t + C_2 \sin t$

$y_p = A \cos 3t + B \sin 3t$

$y_p' = -3A \sin 3t + 3B \cos 3t$

$y_p'' = -9A \cos 3t - 9B \sin 3t$

$-9A\cos 3t - 9B\sin 3t + A\cos 3t + B\sin 3t = \cos 3t$

$-8A\cos 3t - 8B\sin 3t = \cos 3t$

$-8A = 1 \Rightarrow A = -1/8$

$-8B = 0 \Rightarrow B = 0$

$y_p = \frac{-1}{8}\cos 3t$

$y = C_1\cos t + C_2\sin t - \frac{1}{8}\cos 3t$

21- $y'' - y = e^t + t^2$

$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$y_h = C_1e^t + C_2e^{-t}$

$y_p = Ate^t + Bt^2 + Ct + D$

$y_p' = Ae^t + Ate^t + 2Bt + C$

$y_p'' = Ae^t + Ae^t + Ate^t + 2B$

$y_p'' = 2Ae^t + Ate^t + 2B$

$2Ae^t + Ate^t + 2B - Ate^t - Bt^2 - Ct - D = e^t + t^2$

$2Ae^t - Bt^2 - Ct + 2B - D = e^t + t^2$

$2A = 1 \Rightarrow A = 1/2$

$-B = 1 \Rightarrow B = -1$

$-C = 0 \Rightarrow C = 0$

$2B - D = 0 \Rightarrow -2 - D = 0 \Rightarrow D = -2$

$y = C_1e^t + C_2e^{-t} + \frac{1}{2}te^t - t^2 - 2$

Ex: Solve the DE $\frac{dy}{dx} = \frac{x-y}{x+y}$

1- Linear DE

$(x + y)\frac{dy}{dx} = x - y$

In the LHS there is the term yy', therefore the DE is not linear

2- Exact DE

$(x+y)dy = (x-y)dx$

$(y-x)dx + (x+y)dy = 0$

$M = y - x, N = x + y,$

$\frac{\partial M}{\partial y} = 1, \frac{\partial N}{\partial x} = 1$ (Exact)

$$f = \int M dx = \int (y-x)dx = xy - \frac{x^2}{2} + k(y)$$

$$\frac{\partial f}{\partial y} = x + k'(y) = N = x + y$$

$$k'(y) = y \Rightarrow k(y) = \frac{y^2}{2}$$

$$xy - \frac{x^2}{2} + \frac{y^2}{2} = C$$

3- Separable DE

$ydx - xdx + xdy + ydy = 0$

There is no way to separate the *x* and *y* variables therefore not separable

4- First order homogeneous

$$\frac{dy}{dx} = \frac{x-y}{x+y} = \frac{1-\frac{y}{x}}{1+\frac{y}{x}} = F\left(\frac{y}{x}\right)$$

Therefore homogeneous

Let $y = xv \Rightarrow \frac{dy}{dx} = v + xv'$

$$\frac{dy}{dx} = \frac{x-y}{x+y} = v + xv' = \frac{1-v}{1+v}$$

$$(v + xv')(1+v) = 1 - v$$

$$v + v^2 + xv'(1+v) = 1 - v$$

$$xv'(1+v) = 1 - 2v - v^2$$

$$\frac{(1+v)dv}{1-2v-v^2} = \frac{dx}{x}$$

$$-\frac{1}{2}\ln(1-2v-v^2) = \ln x + C_1$$

$$\ln(1-2v-v^2) = -2\ln x - 2C_1$$

$$\ln(1-2v-v^2) = \ln\left(\frac{1}{x^2}\right) - 2C_1$$

$$1 - 2v - v^2 = \frac{C}{x^2}$$

$$x^2 - 2xy - y^2 = C$$