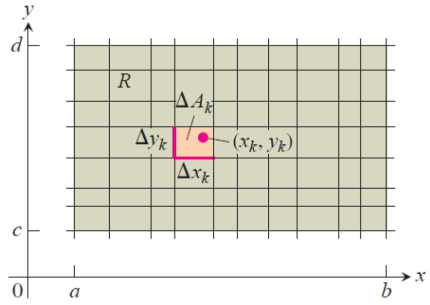
Chapter 7 Multiple Integral (Chapter 14)

Double Integral (14.1)

If $f(x,y)$ is defined on the rectangular region given by

$R: a \le x \le b, c \le y \le d$

Then we can write



$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

The double integral represents the volume under the surface $z = f(x,y)$ within the region R

Ex: Evaluate the integral

$$\iint_R f(x,y)dA$$

Where $f(x,y) = 1 - 6x^2y$ and R: $0 \le x \le 2, -1 \le y \le 1$

Sol.

$$\iint_R f(x,y)dA = \int_{-1}^{1} \int_{0}^{2} (1 - 6x^2y)dxdy = \int_{-1}^{-1} [x - 2x^3y]_0^2 dy$$

$$= \int_{-1}^{1} (2 - 16y)dy = 2y - 8y^2|_{-1}^{1} = 2 - 8 - (-2 - 8) = 4$$

This integral can be evaluated in the order of integration reversed

$$\iint_R f(x,y)dA = \int_0^2 \int_{-1}^{1} (1 - 6x^2y)dydx = \int_0^2 [y - 3x^2y^2]_{-1}^{1} dx$$

$$= \int_0^2 \left((1 - 3x^2) - (-1 - 3x^2)\right)dx = \int_0^2 (2)dx = 2x|_0^2 = 4$$

Double Integral Over Bounded Nonrectangular Regions

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

Ex: Find the volume of the prism whose base is the triangle in the *xy*-plane bounded by the *x*-axis and the lines *y*=*x* and *x*=1 and whose top lies in the plane *z*=3–*x*–*y*.

Sol.

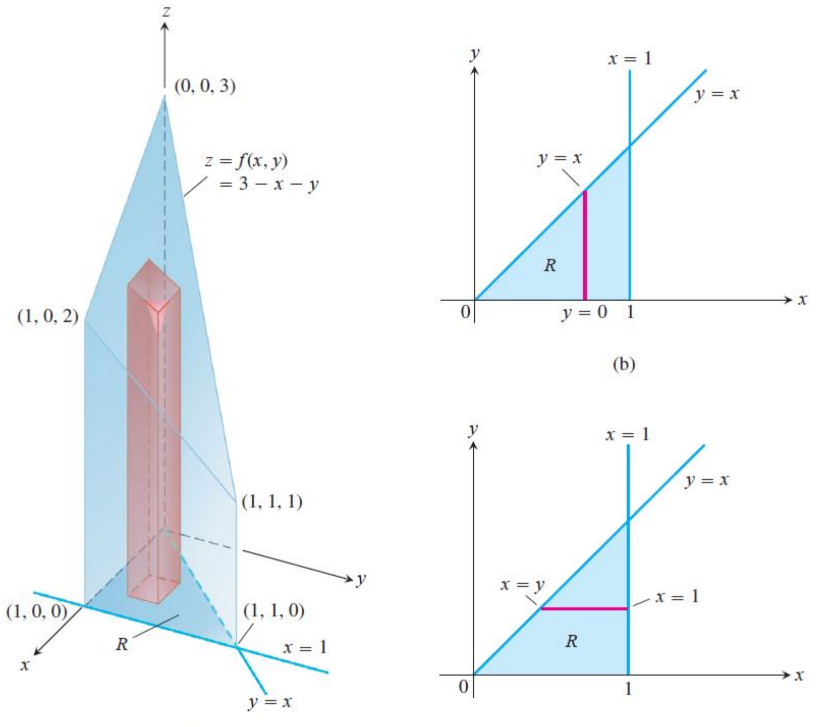
$$V=\int_0^1\int_0^x (3-x-y)dydx$$

$$=\int_0^1\left[3y-xy-\frac{y^2}{2}\right]_0^x dx$$

$$=\int_0^1\left(3x-x^2-\frac{x^2}{2}\right)dx$$

$$=\int_0^1\left(3x-\frac{3}{2}x^2\right)dx$$

$$=\left[\frac{3}{2}x-\frac{1}{2}x^3\right]_0^1=\frac{3}{2}-\frac{1}{2}=1$$



(a) (b) (c)

By reversing the order of integration

$$V=\int_0^1\int_y^1 (3-x-y)dxdy=\int_0^1\left[3x-\frac{x^2}{2}-xy\right]_y^1 dy=\int_0^1\left(\left(3-\frac{1}{2}-y\right)-\left(3y-\frac{y^2}{2}-y^2\right)\right)dy$$

$$=\int_0^1\left(\frac{5}{2}-4y+\frac{3}{2}y^2\right)dy=\left[\frac{5}{2}y-2y^2+\frac{1}{2}y^3\right]_0^1=\frac{5}{2}-2+\frac{1}{2}=1$$

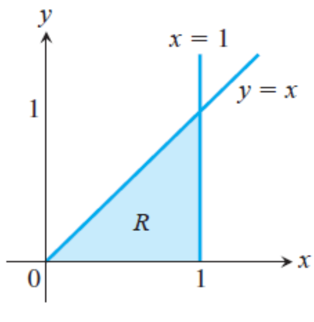
Ex: Find

$$\iint_R \frac{\sin x}{x}dA$$

Where *R* is the triangular region in the *xy*-plane bounded by the lines *y*=*x*, *x*=1 and x-axis

Sol.:

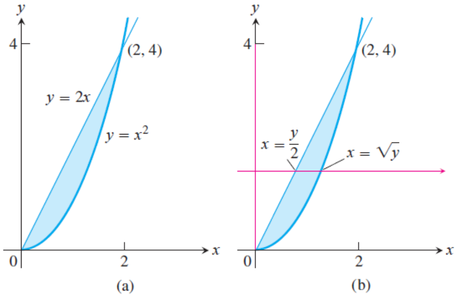
$$\int_0^1\int_0^x \frac{\sin x}{x}dydx=\int_0^1\left[y\frac{\sin x}{x}\right]_0^x=\int_0^1 \sin x=-\cos x|_0^1=1-\cos 1$$



Ex: Find the value of the integration

$$\int_0^2 \int_{x^2}^{2x} (4x+2)dydx$$

With the order of the integration reversed



Sol.:

First it is better to find the points of intersection of the curves.

$x^2 = 2x \ \Rightarrow \ x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0 \ \Rightarrow y = 0$

$x = 2 \ \Rightarrow y = 4$

$y = x^2 \ \Rightarrow \ x = \sqrt{y}$

$y = 2x \ \Rightarrow \ x = y/2$

$$\int_0^2 \int_{x^2}^{2x} (4x+2)dydx = \int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)dxdy = \int_0^4 [2x^2+2x]_{y/2}^{\sqrt{y}} = \int_0^4 \left(2y + 2\sqrt{y} - \frac{y^2}{2} - y\right)dy$$

$$= \int_0^4 \left(y + 2\sqrt{y} - \frac{y^2}{2}\right)dy = \left[\frac{y^2}{2} + \frac{4}{3}y^{3/2} - \frac{y^3}{6}\right]_0^4 = 8 + \frac{32}{3} - \frac{32}{3} - 0 = 8$$

Areas, Moments and Center of Mass

Areas of Bounded Region in the Plane

The area of a bounded region R can be expressed in terms of double integral

$$A = \iint_R dA = \iint_R dydx = \iint_R dxdy$$

Ex: Find the area of the region R bounded by the parabola $y = x^2$ and the line $y=x+2$

Sol.:

First we find the points of intersections

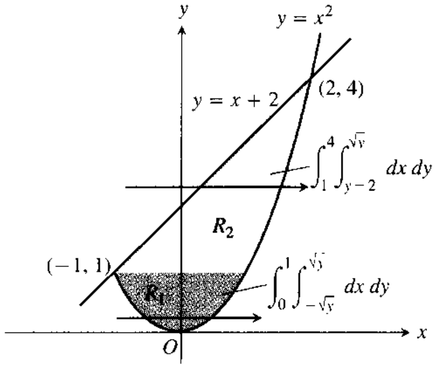
$x^2 = x + 2$

$x^2 - x - 2 = 0$

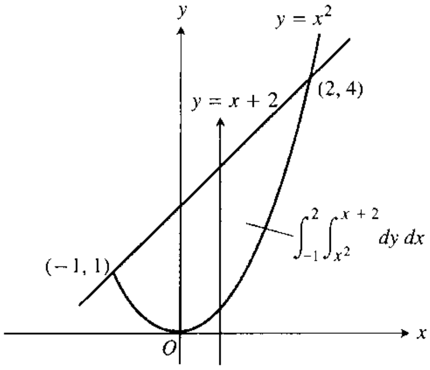
$(x+1)(x-2) = 0$

$x = -1 \Rightarrow y = 1$

$x = 2 \Rightarrow y = 4$



(a)



(b)

$$A = \int_{-1}^{2}\int_{x^2}^{x+2} dydx = \int_{-1}^{2} [y]_{x^2}^{x+2} dx = \int_{-1}^{2} (x + 2 - x^2) dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = 8 - 3 - \frac{1}{2} = \frac{9}{2}$$

First and Second Moments and Center of Mass

Density: $\delta(x, y)$

Mass: $M = \iint \delta(x, y) dA$

First Moments:

Moment about *x*-axis: $M_x = \iint y\delta(x, y) dA$

Moment about *y*-axis: $M_y = \iint x\delta(x, y) dA$

Center of Mass: $\bar{x} = \frac{M_y}{M}, \bar{y} = \frac{M_x}{M}$

Moments of Inertia (Second Moments)

About *x*-axis: $I_x = \iint y^2\delta(x, y) dA$

About *y*-axis: $I_y = \iint x^2\delta(x, y) dA$

About origin: $I_o = I_x + I_y = \iint (x^2 + y^2)\delta(x, y) dA$

Centroids of Geometric Figures

When the density of an object is constant, it cancels out of the numerator and denominator of the formulas for $\bar{x}$ and $\bar{y}$. So, we can take δ=1 in calculating $\bar{x}$ and $\bar{y}$.

Ex: Find the centroid of the region in the first quadrant that is bounded by the line *y*=*x* and the curve $y = x^2$.

Sol.:

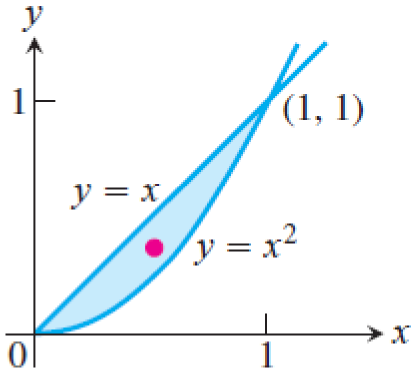
$$M = \int_{0}^{1}\int_{x^2}^{x} (1) dydx = \int_{0}^{1} [y]_{x^2}^{x} dx = \int_{0}^{1} (x - x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{0}^{1} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$M_x = \int_0^1 \int_{x^2}^{x} (1) y dy dx = \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{x} dx = \frac{1}{2}\int_0^1 (x^2 - x^4)\, dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{1}{15}$$

$$M_y = \int_0^1 \int_{x^2}^{x} (1) x dy dx = \int_0^1 [xy]_{x^2}^{x}\, dx = \int_0^1 (x^2 - x^3)\, dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{12}$$

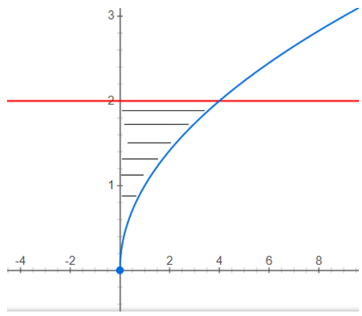
$$\bar{x} = \frac{M_y}{M} = \frac{1/12}{1/6} = \frac{1}{2}$$

$$\bar{y} = \frac{M_x}{M} = \frac{1/15}{1/6} = \frac{2}{5}$$



Ex: Evaluate the integral

$$\int_0^4 \int_{\sqrt{x}}^{2} \cos(y^3)\, dy dx$$



Sol.:

The integral cannot be solved in the given order, so, we have to reverse the order of integration. The limits of the y variable are $y = \sqrt{x}$ $(x = y^2)$ and y=2, the intersection point is (2,4), the integral in the reversed order will be

$$\int_0^2 \int_0^{y^2} \cos(y^3)\, dx dy = \int_0^2 [x\cos(y^3)]_0^{y^2} dy = \int_0^2 y^2 \cos(y^3)\, dy = \frac{1}{3}\sin(y^3)\Big|_0^2 = \frac{1}{3}\sin(8)$$

Exercises 14.2

Sketch the region bounded by the given lines and curves, then find the area by double integration.

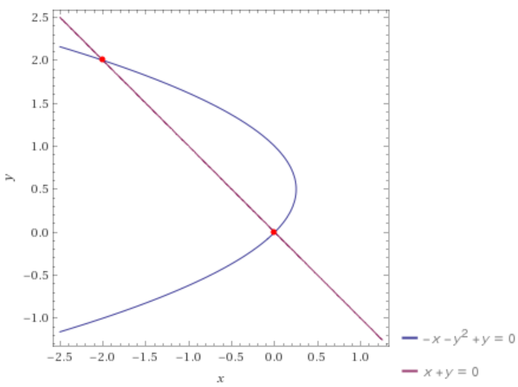
6- The parabola $x = y - y^2$ and the line $x + y = 0$

Sol.

$x + y = 0 \Rightarrow x = -y$

$-y = y - y^2$

$2y - y^2 = 0 \Rightarrow y(2 - y) = 0$



$y = 0 \Rightarrow x = 0$

$y = 2 \Rightarrow x = -2$

$$A = \iint dA = \int_0^2 \int_{-y}^{y-y^2} dxdy$$

$$A = \int_0^2 (y - y^2 - (-y))dy = \int_0^2 (2y - y^2)dy = y^2 - \frac{y^3}{3}\Big|_0^2 = 4 - \frac{8}{3} - 0 = \frac{4}{3}$$

36- Find the centroid of the region in the *xy*-plane bounded by the curves $= \frac{1}{\sqrt{1-x^2}}$, $y = \frac{-1}{\sqrt{1-x^2}}$ and the lines *x*=0 and *x*=1.

Sol.:

From symmetry $\bar{y} = 0$

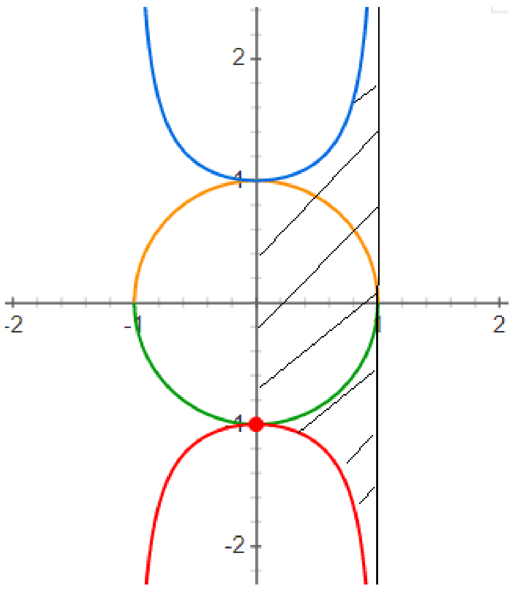
$M = \iint (1) dA$

$$M = \int_0^1 \int_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}} dydx$$

$$= \int_0^1 \left(\frac{1}{\sqrt{1-x^2}} - \frac{-1}{\sqrt{1-x^2}}\right) dx$$

$$= \int_0^1 \left(\frac{2dx}{\sqrt{1-x^2}}\right) = 2\sin^{-1}(x)|_0^1$$

$$2\left(\frac{\pi}{2} = 0\right) = \pi$$



$$M_y = \int_0^1 \int_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}} xdydx$$

$$= \int_0^1 [xy]_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}} dx$$

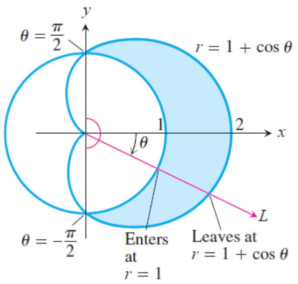
$$= \int_0^1 \left(\frac{2x}{\sqrt{1-x^2}}\right) dx = -\frac{\sqrt{1-x^2}}{1/2}\Bigg|_0^1 = -2(0-1) = 2$$

$$\bar{x} = \frac{2}{\pi}$$

Double Integral in Polar Form

To integrate a function in polar form $f(r,\theta)$ over the shaded region shown

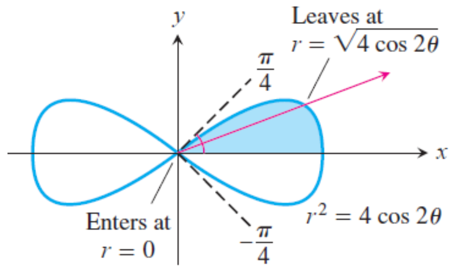
$$\iint_R f(r,\theta)dA = \int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} f(r,\theta)rdrd\theta$$



Area in Polar Coordinates

$$Area\ of\ R = \iint_R dA = \iint_R rdrd\theta$$

Ex: Find the area enclosed by the curve $r^2 = 4\cos 2\theta$



Sol.:

$$\frac{A}{4} = \int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} rdrd\theta$$

$$A = 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{\sqrt{4\cos 2\theta}} d\theta = 4\int_0^{\pi/4} 2\cos 2\theta\, d\theta = 4[\sin(2\theta)]_0^{\pi/4} = 4(1-0) = 4$$

Changing Cartesian Integration into Polar Integration

$$\iint_R f(x,y)dxdy = \iint_G f(r\cos(\theta), r\sin(\theta))rdrd\theta$$

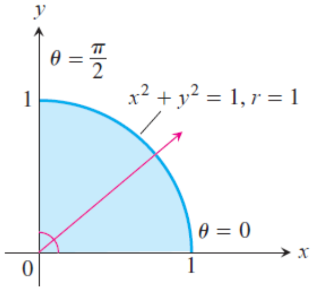
R is the region of integration described in Cartesian coordinates and G is the same region described in polar coordinates.

Ex: Find the moment of inertia about the origin of a thin plate of density $\delta(x,y)=1$ bounded by the quarter circle $x^2 + y^2 = 1$ in the first quadrant.

Sol.:

In Cartesian coordinates

$$I_o = \int_0^1\int_0^{\sqrt{1-x^2}} (x^2+y^2)dydx$$



$$\int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{\sqrt{1-x^2}} dx = \int_0^1 \left(x^2\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2}\right) dx$$

In polar coordinates

$$I_o = \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)dydx = \int_0^{\pi/2}\int_0^1 (r^2)rdrd\theta = \int_0^{\pi/2}\int_0^1 (r^3)drd\theta$$

$$= \int_0^{\pi/2} \left.\frac{r^4}{4}\right|_0^1 d\theta = \int_0^{\pi/2} \frac{1}{4}d\theta = \left.\frac{\theta}{4}\right|_0^{\pi/2} = \frac{\pi}{8}$$

Ex: Evaluate the integral (the Gaussian integral)

$$\int_{-\infty}^{\infty} e^{-x^2}dx$$

Sol.:

$$I = \int_{-\infty}^{\infty} e^{-x^2}dx$$

$$I = \int_{-\infty}^{\infty} e^{-y^2}dy$$

$$I \cdot I = \left[\int_{-\infty}^{\infty} e^{-x^2}dx\right]\left[\int_{-\infty}^{\infty} e^{-y^2}dy\right]$$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy$$

The above integral can be converted to polar coordinates. We note that the region of integration is the entire xy-plane that can be described as below

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}rdrd\theta = \int_0^{2\pi} \left.-\frac{1}{2}e^{-r^2}\right|_0^{\infty} d\theta = -\frac{1}{2}\int_0^{2\pi}(0-1)d\theta = \frac{1}{2}(2\pi) = \pi$$

$$I = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

Triple Integral in Rectangular Coordinates

The volume of a closed bounded region D in space is

$$Volume\ of\ D = \iiint_D dV = \iiint_D dzdydx$$

In triple integral we always make the order of integration starts with *dz* and the limits of the integral with respect to the variable *z* should always be clear from the problem statement. Next, the problem reduces to double integral and we follow the same rules described in earlier sections. Usually we only need to draw the region of integration with respect to *x* and *y* variable in 2-D only and no need to draw a 3-dimesional drawing for the overall region.

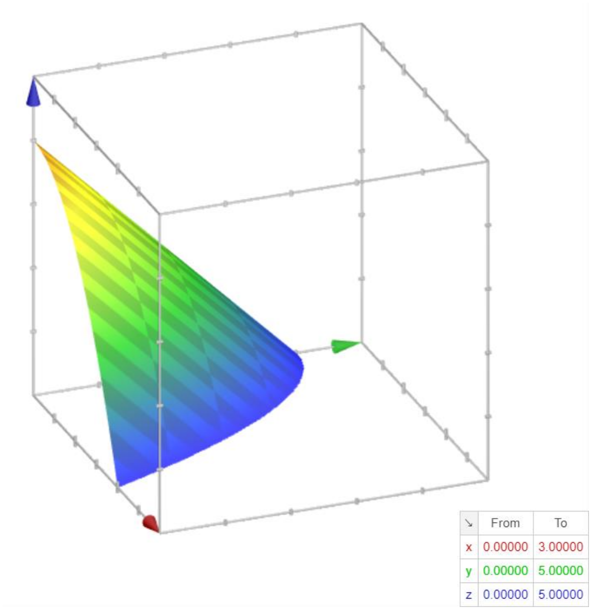
Exercises 14.4

24- Find the volume of the region in the 1st octant bounded by the surface $z = 4 - x^2 - y$

Sol.:

The first step is to find the limits of the variable *z*. Since the region is in the first octant this means that $x \geq 0$, $y \geq 0$ and $z \geq 0$. Combining this and the fact that the upper bound is $z = 4 - x^2 - y$, then we have found the limits of z variable.

The second step is to draw the region in the *xy*-plane and finding the limits of x and y variables. The limits of the x, y, z variables are

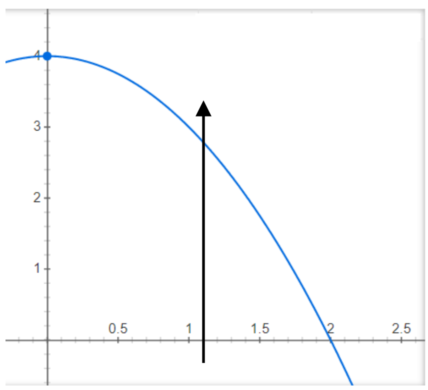


In the xy-plane z=0 $\Rightarrow y = 4 - x^2$

$0 \leq z \leq 4 - x^2 - y$

$0 \leq y \leq 4 - x^2$

$0 \leq x \leq 2$



$$V = \int_0^2 \int_0^{4-x^2} \int_0^{4-x^2-y} dzdydx$$

$$V = \int_0^2 \int_0^{4-x^2} z|_0^{4-x^2-y} dydx$$

$$V = \int_0^2 \int_0^{4-x^2} (4 - x^2 - y)dydx$$

$$V = \int_0^2 \left[(4 - x^2)y - \frac{y^2}{2}\right]_0^{4-x^2} dx = \int_0^2 \left((4 - x^2)^2 - \frac{(4 - x^2)^2}{2}\right) dx = \frac{1}{2}\int_0^2 ((4 - x^2)^2)dx$$

$$V = \frac{1}{2}\int_0^2 (16 - 8x^2 + x^4)dx = \frac{1}{2}\int_0^2 \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right] dx = \frac{1}{2}\left(32 - \frac{64}{3} + \frac{32}{5}\right) = \frac{128}{15}$$

Masses and Moments in Three Dimensions

Mass:

$$M = \iiint_R \delta dV \ \ (\delta = density)$$

First Moments:

$$M_{yz} = \iiint_R x\delta dV$$

$$M_{xz} = \iiint_R y\delta dV$$

$$M_{xy} = \iiint_R z\delta dV$$

Center of Mass

$\bar{x} = \frac{M_{yz}}{M}, \bar{y} = \frac{M_{xz}}{M}, \bar{z} = \frac{M_{xy}}{M}$

Moments of Inertia

$$I_x = \iiint_R (y^2 + z^2)\delta dV$$

$$I_y = \iiint_R (x^2 + z^2)\delta dV$$

$$I_z = \iiint_R (x^2 + y^2)\delta dV$$

Triple Integral in Cylindrical and Spherical Coordinates

Cylindrical Coordinates

To integrate a continuous function $f(r,\theta,z)$ over a region given by

$z_1(r,\theta) \le z \le z_2(r,\theta)$

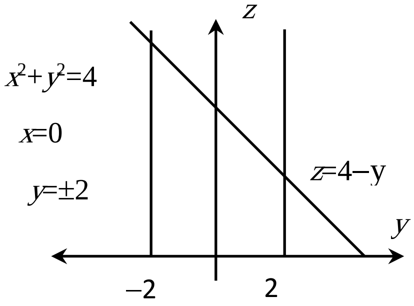
$r_1(\theta) \le r \le r_2(\theta)$

$\theta_1 \le \theta \le \theta_2$

$$\int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)}\int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r,\theta,z)dzrdrd\theta$$

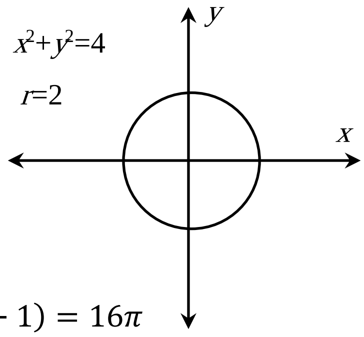
Ex: Find the volume of the solid bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$.

Sol.:



$$V=\int_0^{2\pi}\int_0^{2}\int_0^{4-r\sin\theta} rdzdrd\theta$$

$$V=\int_0^{2\pi}\int_0^{2} rz|_0^{4-r\sin\theta} rdrd\theta$$



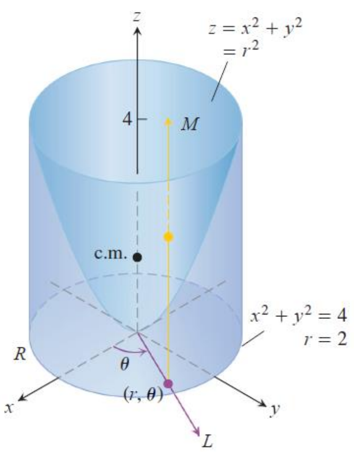
$$V=\int_0^{2\pi}\int_0^{2}(4r-r^2\sin\theta)drd\theta$$

$$V=\int_0^{2\pi}\left[2r^2-\frac{1}{3}r^3\sin\theta\right]_0^2 d\theta$$

$$V=\int_0^{2\pi}\left(8-\frac{8}{3}\sin\theta\right)d\theta = 8\theta+\frac{8}{3}\cos\theta\Big|_0^{2\pi} = 8(2\pi-1)+\frac{8}{3}(1-1)=16\pi$$

Ex: Find the centroid of the solid enclosed by the cylinder $x^2+y^2=4$, bounded above by the paraboloid $z=x^2+y^2$ and below by the xy-plane.

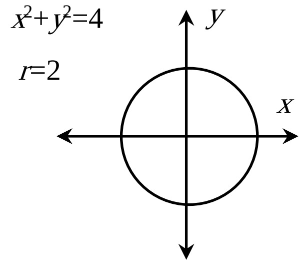
Sol.:

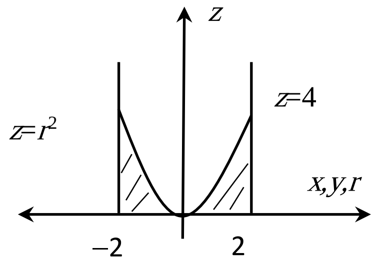


$z=x^2+y^2=r^2$

$x^2+y^2=4=r^2 \Rightarrow r=2$

$0 \le z \le r^2$





$0 \le r \le 2$

$0 \le \theta \le 2\pi$

The centroid lies on the axis of symmetry in this case the z-axis

$\bar{x} = 0, \bar{y} = 0$

$$M = \int_0^{2\pi}\int_0^{2}\int_0^{r^2} (1) r dz dr d\theta = \int_0^{2\pi}\int_0^{2} [rz]_0^{r^2} dr d\theta$$

$$= \int_0^{2\pi}\int_0^{2} r^3 dr d\theta = \left[\frac{r^4}{4}\right]_0^2 [\theta]_0^{2\pi} = (4)(2\pi) = 8\pi$$

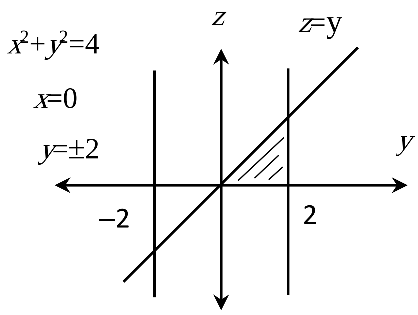
$$M_{xy} = \int_0^{2\pi}\int_0^{2}\int_0^{r^2} z(1) r dz dr d\theta = \int_0^{2\pi}\int_0^{2} \left[r\frac{z^2}{2}\right]_0^{r^2} dr d\theta$$

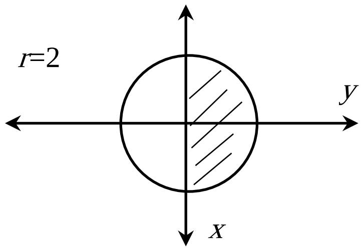
$$= \int_0^{2\pi}\int_0^{2} \frac{r^5}{2} dr d\theta = \frac{1}{2}\left[\frac{r^6}{6}\right]_0^2 [\theta]_0^{2\pi} = \frac{1}{2}\left(\frac{32}{3}\right)(2\pi) = \frac{32\pi}{3}$$

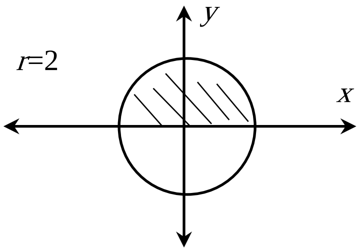
$$\bar{z} = \frac{M_{xy}}{M} = \frac{32\pi/3}{8\pi} = \frac{4}{3}$$

Ex: Find the center of mass of the solid whose density $\delta = z$ and bounded above by the plane $z = y$, below by the xy-plane and laterally by the cylinder $x^2 + y^2 = 4$.

Sol.:







$$M = \iiint \delta dV$$

$$= \int_0^{\pi}\int_0^{2}\int_0^{r\sin\theta} z r dz dr d\theta = \int_0^{\pi}\int_0^{2} \frac{z^2 r}{2}\Big|_0^{r\sin\theta} dr d\theta$$

$$= \int_0^{\pi}\int_0^{2} \frac{1}{2} r^3 \sin^2\theta \, dr d\theta = \int_0^{\pi}\int_0^{2} \frac{1}{4} r^3 (1 - \cos 2\theta) dr d\theta$$

$$= \frac{1}{4}\left[\frac{r^4}{4}\right]_0^2 \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi} = \frac{1}{4}\left(\frac{16}{4} - 0\right)(\pi - 0) = \pi$$

$$M_{xy} = \int_0^{\pi}\int_0^{2}\int_0^{r\sin\theta} z\delta r dz dr d\theta = \int_0^{\pi}\int_0^{2}\int_0^{r\sin\theta} z^2 r dz dr d\theta$$

$$= \int_0^{\pi}\int_0^{2} \frac{z^3}{3} r\Big|_0^{r\sin\theta} dr d\theta = \int_0^{\pi}\int_0^{2} \frac{1}{3} r^4 \sin^3\theta \, dr d\theta$$

$$= \int_0^{\pi}\int_0^{2} \frac{1}{3} r^4 \sin\theta\,(1-\cos^2\theta)\, dr d\theta$$

$$= \frac{1}{3}\left[\frac{r^5}{5}\right]_0^2 \left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^{\pi} = \frac{1}{3}\left(\frac{32}{5}\right)\left(1-\frac{1}{3}+1-\frac{1}{3}\right)$$

$$= \frac{32}{15}\left(\frac{4}{3}\right) = \frac{128}{45}$$

$$M_{xz} = \int_0^{\pi}\int_0^{2}\int_0^{r\sin\theta} y\delta r dz dr d\theta = \int_0^{\pi}\int_0^{2}\int_0^{r\sin\theta} z r^2 \sin\theta \, dz dr d\theta$$

$$= \int_0^{\pi}\int_0^{2} \frac{z^2}{2}\Big|_0^{r\sin\theta} r^2 \sin\theta\, dr d\theta = \int_0^{\pi}\int_0^{2} \frac{1}{2} r^4 \sin^3\theta\, dr d\theta$$

$$= \frac{64}{15}$$

$$M_{yz} = \int_0^{\pi}\int_0^{2}\int_0^{r\sin\theta} x\delta r dz dr d\theta = \int_0^{\pi}\int_0^{2}\int_0^{r\sin\theta} z r^2 \cos\theta \, dz dr d\theta$$

$$= \int_0^{\pi}\int_0^{2} \frac{z^2}{2}\Big|_0^{r\sin\theta} r^2 \cos\theta\, dr d\theta = \int_0^{\pi}\int_0^{2} \frac{1}{2} r^4 \sin^2\theta \cos\theta\, dr d\theta$$

$$= \frac{1}{2}\left[\frac{r^5}{5}\right]_0^2 \left[\frac{\sin^3\theta}{3}\right]_0^{\pi} = \frac{1}{2}\left(\frac{32}{5}\right)(0-0) = 0$$

$$\bar{x} = \frac{M_{yz}}{M} = 0$$

$$\bar{y} = \frac{M_{xz}}{M} = \frac{64/15}{\pi} = \frac{64}{15\pi}$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{128/45}{\pi} = \frac{128}{45\pi}$$

Spherical Coordinates

To integrate a continuous function $f(\rho,\phi,\theta)$ over a region given by

$\rho_1(\phi,\theta) \le \rho \le \rho_2(\phi,\theta)$

$\phi_1(\theta) \le \phi \le \phi_2(\theta)$

$\theta_1 \leq \theta \leq \theta_2$

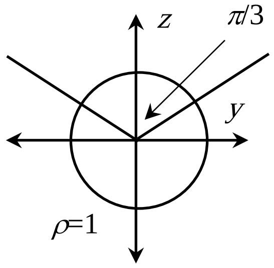
The integral will be

$$\int_{\theta_1}^{\theta_2} \int_{\phi_1(\theta)}^{\phi_2(\theta)} \int_{\rho_1(\phi,\theta)}^{\rho_2(\phi,\theta)} f(\rho,\phi,\theta)\rho^2 \sin\phi \, d\rho d\phi d\theta$$

Ex: Find the volume of the upper region cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \pi/3$.

Sol.:

Limits:



$0 \leq \rho \leq 1$

$0 \leq \phi \leq \pi/3$

$0 \leq \theta \leq 2\pi$

$$V = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \, d\rho d\phi d\theta$$

$$= \left[\frac{\rho^3}{3}\right]_0^1 [-\cos\phi]_0^{\pi/3} [\theta]_0^{2\pi} = \left(\frac{1}{3}\right)\left(-\frac{1}{2}+1\right)(2\pi) = \frac{\pi}{3}$$

Ex: find the moment of inertia about the z-axis of the region in example above.

Sol.:

$$I_z = \iiint (x^2+y^2)dV = \iiint r^2 dV = \iiint (\rho\sin\phi)^2 dV$$

$$I_z = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 (\rho\sin\phi)^2 \rho^2 \sin\phi \, d\rho d\phi d\theta$$

$$I_z = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^4 \sin^3\phi \, d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^4 (1-\cos^2\phi)\sin\phi \, d\rho d\phi d\theta$$

$$= \left[\frac{\rho^4}{5}\right]_0^1 \left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi/3} [\theta]_0^{2\pi}$$

$$= \left(\frac{1}{5}\right)\left(-\frac{1}{2}+\frac{1/8}{3}+1-\frac{1}{3}\right)(2\pi) = \left(\frac{2\pi}{5}\right)\left(\frac{1}{2}-\frac{7}{24}\right) = \left(\frac{2\pi}{5}\right)\left(\frac{5}{24}\right) = \frac{\pi}{12}$$

Exercises: (14.6)

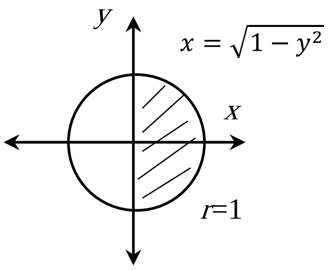
16- Convert the integral

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{x}(x^2+y^2)dzdxdy$$

into cylindrical coordinates and evaluate the result.

Sol.:

$$= \int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{0}^{r\cos\theta}(r^2)rdzdrd\theta$$



$$= \int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{0}^{r\cos\theta} r^3 dzdrd\theta$$

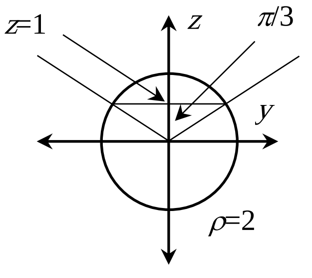
$$= \int_{-\pi/2}^{\pi/2}\int_{0}^{1}[z]_0^{r\cos\theta} r^3 drd\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_{0}^{1} r^4\cos\theta drd\theta = \left[\frac{r^5}{5}\right]_0^1 [\sin\theta]_{-\pi/2}^{\pi/2} = \left(\frac{1}{5}\right)(1-(-1)) = \frac{2}{5}$$

37- Find the volume of the smaller region cut from the sphere ρ=2 by the plane z=1.

Sol.:

Convert the plane equation z=1 to spherical coordinates



$z = 1 \Rightarrow \rho\cos\phi = 1 \Rightarrow \rho = 1/\cos\phi$

Find the limit of ϕ as the intersection of the plane z=1 and the sphere ρ=2

$$z = 1 \Rightarrow \rho\cos\phi = 1 \Rightarrow 2\cos\phi = 1 \Rightarrow \cos\phi = \frac{1}{2} \Rightarrow \phi = \pi/3$$

$$V = \int_{0}^{2\pi}\int_{0}^{\pi/3}\int_{1/\cos\phi}^{2} \rho^2\sin\phi\, d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{\rho^3}{3}\right]_{1/\cos\phi}^{2} \sin\phi \, d\phi d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/3} \left(8\sin\phi - \frac{\sin\phi}{\cos^3\phi}\right) d\phi d\theta$$

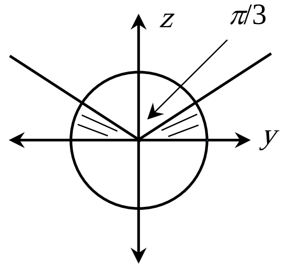
$$= \frac{1}{3}\left[-8\cos\phi - \frac{1}{2}\cos^{-2}\phi\right]_0^{\pi/3} [\theta]_0^{2\pi} = \frac{1}{3}\left(-8\left(\frac{1}{2} - 1\right) - \frac{1}{2}\left(\left(\frac{1}{2}\right)^{-2} - 1\right)\right)(2\pi)$$

$$= \frac{1}{3}\left(4 - \frac{1}{2}(4-1)\right)(2\pi) = \frac{2\pi}{3}\left(\frac{5}{2}\right) = \frac{5\pi}{3}$$

40- A conical hole is drilled inside the solid hemisphere, the cone equation is $\phi=\pi/3$. Find the center of mass, $\rho\leq 2$, $z\geq 0$

Sol.:

From symmetry $\bar{x} = 0, \bar{y} = 0$



$$M = \iiint \delta dV = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^{2} (1)\rho^2 \sin\phi \, d\rho d\phi d\theta$$

$$= \left[\frac{\rho^3}{3}\right]_0^2 [-\cos\phi]_{\pi/3}^{\pi/2} [\theta]_0^{2\pi}$$

$$= \left(\frac{8}{3}\right)\left(0 + \frac{1}{2}\right)(2\pi) = \frac{8\pi}{3}$$

$$M_{xy} = \iiint z\delta dV = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^{2\pi} (\rho\cos\phi)(1)\rho^2 \sin\phi \, d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^{2\pi} \rho^3 \sin\phi\cos\phi \, d\rho d\phi d\theta$$

$$= \left[\frac{\rho^4}{4}\right]_0^2 \left[\frac{\sin^2\phi}{2}\right]_{\pi/3}^{\pi/2} [\theta]_0^{2\pi} = \left(\frac{16}{4}\right)\left(\frac{1 - 3/4}{2}\right)(2\pi) = \pi$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{\pi}{8\pi/3} = \frac{3}{8}$$

Center of mass at (0, 0, 3/8)