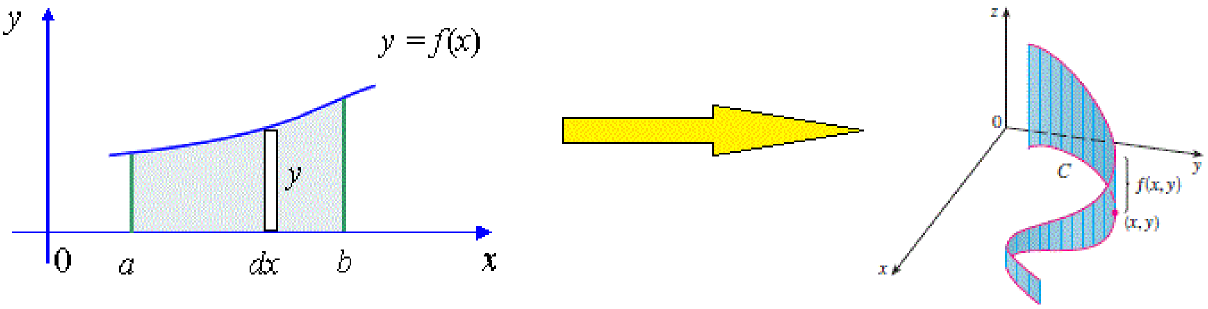
## Chapter 8

## Line Integral (15.1)

Line integrals are integrals of a function over a curve

$$\int_C f(x, y, z)ds$$




*Definite integral - Area of a flat surface*

*Line integral - Area of a curved surface*

$\int_a^b f(x)dx$ $\int_C f(x, y)ds$

To evaluate the line integral, we express *ds* and the integrand $f(x, y, z)$ in terms of the curve's parameter *t*.

$$\int_C f(x, y, z)ds = \int_a^b f(g(t), h(t)k(t))\frac{ds}{dt}dt = \int_a^b f\big(g(t), h(t)k(t)\big)|\boldsymbol{v}(t)|dt$$

$\boldsymbol{r}(t) = g(t)\boldsymbol{i} + h(t)\boldsymbol{j} + k(t)\boldsymbol{k}$ (position vector)

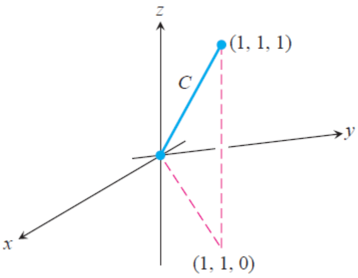
$\boldsymbol{v}(t) = \frac{dr(t)}{dt} = \dot{g}(t)\boldsymbol{i} + \dot{h}(t)\boldsymbol{j} + \dot{k}(t)\boldsymbol{k}$ (velocity vector)

**Ex:** Find the line integral of the function $f(x, y, z) = x - 3y^2 + z$ on the path joining the line segments $C_1$ and $C_2$, and over the line segment $C_3$, where:

$C_1$: from (0, 0, 0) to (1, 1, 0)

$C_2$: from (1, 1, 0) to (1, 1, 1)

$C_3$: from (0, 0, 0) to (1, 1, 1)




**Sol.:**

$C_1$: $\boldsymbol{V}_1 = \boldsymbol{i} + \boldsymbol{j}$, P(0, 0, 0)

$x(t) = t$, $y(t) = t$, $z(t) = 0$, $0 \le t \le 1$

$\boldsymbol{r}_1 = t\boldsymbol{i} + t\boldsymbol{j}$, $0 \le t \le 1 \Rightarrow \boldsymbol{v}_1 = \boldsymbol{i} + \boldsymbol{j} \Rightarrow |\boldsymbol{v}_1| = \sqrt{2}$

$C_2$: $\boldsymbol{V}_2 = \boldsymbol{k}$, P(1, 1, 0)

$x(t) = 1$, $y(t) = 1$, $z(t) = t$, $0 \le t \le 1$

$\boldsymbol{r}_2 = \boldsymbol{i} + \boldsymbol{j} + t\boldsymbol{k}$, $0 \le t \le 1 \Rightarrow \boldsymbol{v}_2 = \boldsymbol{k} \Rightarrow |\boldsymbol{v}_2| = 1$

$C_3$: $\boldsymbol{V}_3 = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{j}$, P(0, 0, 0)

$x(t) = t$, $y(t) = t$, $z(t) = t$, $0 \le t \le 1$

$\boldsymbol{r}_3 = t\boldsymbol{i} + t\boldsymbol{j} + t\boldsymbol{k}$, $0 \le t \le 1 \Rightarrow \boldsymbol{v}_3 = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k} \Rightarrow |\boldsymbol{v}_3| = \sqrt{3}$

Path1:

$$\int_{C_1 \cup C_2} f(x,y,z)ds = \int_{C_1} f(x,y,z)ds + \int_{C_2} f(x,y,z)ds = \int_0^1 f(t,t,0)\sqrt{2}dt + \int_0^1 f(1,1,t)(1)dt$$

$$= \int_0^1 (t - 3t^2 + 0)\sqrt{2}dt + \int_0^1 (1 - 3 + t)(1)dt = \sqrt{2}\left(\frac{t^2}{2} - t^3\right)\Big|_0^1 + \left(-2t + \frac{t^2}{2}\right)\Big|_0^1$$

$$= \sqrt{2}\left(\frac{1}{2} - 1\right) + \left(-2 + \frac{1}{2}\right) = -\frac{\sqrt{2} + 3}{2}$$

Path2:

$$\int_{C_3} f(x,y,z)ds = \int_0^1 f(t,t,t)\sqrt{3}dt = \int_0^1 (t - 3t^2 + t)\sqrt{3}dt = \int_0^1 (2t - 3t^2)\sqrt{3}dt$$

$$= \sqrt{3}(t^2 - t^3)\Big|_0^1 = 0$$

**Exercises 15.1**

Integrate $f(x) = \sqrt{3}/(x^2 + y^2 + z^2)$ over the path

$\boldsymbol{r}(t) = t\boldsymbol{i} + t\boldsymbol{j} + t\boldsymbol{k}$, $1 \le t \le 2$

**Sol.:**

$\boldsymbol{v}(t) = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $|\boldsymbol{v}| = \sqrt{3}$

$$\int_C f(x,y,z)ds = \int_1^2 f(t,t,t)\sqrt{3}dt = \int_1^2 \frac{\sqrt{3}}{t^2 + t^2 + t^2}\sqrt{3}dt = \int_1^2 \frac{1}{t^2}dt$$

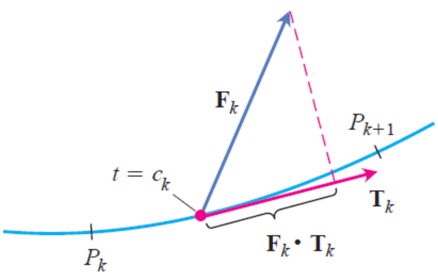
$$= \left(\frac{-1}{t}\right)\Big|_1^2 = -\left(\frac{1}{2} - 1\right) = \frac{1}{2}$$

## Vector Fields, Work and Flux (15.2)

A vector (scalar) field in a plane or space is a function that assigns a vector (value) to each point in the plane or space.

The work done by a continuous force $\boldsymbol{F} = M(x,y,z)\boldsymbol{i} + N(x,y,z)\boldsymbol{j} + P(x,y,z)\boldsymbol{k}$ over a continuous differentiable curve $\boldsymbol{r} = g(t)\boldsymbol{i} + h(t)\boldsymbol{j} + k(t)\boldsymbol{k}$ from $t$=$a$ to $t$=$b$ is the line integral

$$W = \int_{t=a}^{t=b} \boldsymbol{F}.\boldsymbol{T}ds$$




$(\boldsymbol{T}ds = \boldsymbol{dr})$ or $\left(\boldsymbol{T} = \frac{dr}{ds}\right)$

$$W = \int_{t=a}^{t=b} \boldsymbol{F}.\boldsymbol{dr} = \int_{t=a}^{t=b} \boldsymbol{F}.\frac{\boldsymbol{dr}}{dt}dt$$

$$W = \int Mdx + Ndy + Pdz$$

$$\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j} + P\boldsymbol{k}$$

**Ex:** Find the work done by the vector field

$$\boldsymbol{F} = (y - x^2)\boldsymbol{i} + (z - y^2)\boldsymbol{j} + (x - z^2)\boldsymbol{k}$$

Over the curve $\boldsymbol{r}(t) = t\boldsymbol{i} + t^2\boldsymbol{j} + t^3\boldsymbol{k}$, $0 \le t \le 1$

**Sol.:**

Step1: Evaluate $\boldsymbol{F}$ on the curve

$$\boldsymbol{F} = (t^2 - t^2)\boldsymbol{i} + (t^3 - y^4)\boldsymbol{j} + (t - t^6)\boldsymbol{k}$$

Step2: Find $\boldsymbol{v} = \frac{dr}{dt}$

$$\frac{\boldsymbol{dr}}{dt} = \boldsymbol{i} + 2t\boldsymbol{j} + 3t^2\boldsymbol{k}$$

Step3: Find $\boldsymbol{F}.\frac{dr}{dt}$

$$\boldsymbol{F}.\frac{\boldsymbol{dr}}{dt} = \left((t^3 - y^4)\boldsymbol{j} + (t - t^6)\boldsymbol{k}\right).(\boldsymbol{i} + 2t\boldsymbol{j} + 3t^2\boldsymbol{k})$$

$$= 2t^4 - 2t^5 + 3t^3 - 3t^8$$

Step4: Integrate

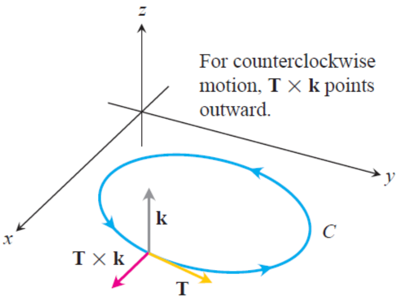
$$W = \int_0^1 (2t^4 - 2t^5 + 3t^3 - 3t^8)dt = \frac{2}{5}t^5 - \frac{2}{6}t^6 + \frac{3}{4}t^4 - \frac{3}{9}t^9\Big|_0^1$$

$$W = \frac{2}{5} - \frac{1}{3} + \frac{3}{4} - \frac{1}{3} = \frac{29}{60}$$

## Flux Across a Plane Curve

If C is a closed curve and $\boldsymbol{n}$ is the outgoing normal vector on C, the flux of the vector field

$$\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j}$$




$$Flux = \int_C \boldsymbol{F}.\boldsymbol{n}ds$$

$$\boldsymbol{n} = \boldsymbol{T} \times \boldsymbol{k}$$

This integral can be evaluated as

$$Flux = \oint \boldsymbol{F}.\boldsymbol{n}ds = \oint \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right)ds = \oint (Mdy - Ndx)$$

**Ex:** Find the flux of $\boldsymbol{F} = (x-y)\boldsymbol{i} + x\boldsymbol{j}$ across the circle $x^2 + y^2 = 1$

**Sol.:**

The parametric equations of the curve (circle) expressed

$$\boldsymbol{r} = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j}, \qquad 0 \le t \le 2\pi$$

$$M = x - y = \cos t - \sin t$$

$$N = x = \cos t$$

$$dy = d(\sin t) = \cos t\, dt$$

$$dx = d(\cos t) = -\sin t\, dt$$

$$Flux = \oint (Mdy - Ndx) = \int_0^{2\pi} (\cos^2 t - \sin t \cos t + \sin t \cos t)dt$$

$$= \int_0^{2\pi} \frac{1}{2}(1 + \cos 2t)dt = \frac{t}{2} + \frac{\sin 2t}{4}\Big|_0^{2\pi} = \pi$$

## Exercises 15.2

**7-** Find the work done by the force $\boldsymbol{F} = xy\boldsymbol{i} + y\boldsymbol{j} - yz\boldsymbol{k}$ over the curve given by $\boldsymbol{r} = t\boldsymbol{i} + t^2\boldsymbol{j} + t\boldsymbol{k}$, $0 \le t \le 1$.

**Sol.:**

$$W = \int_{t=a}^{t=b} \boldsymbol{F} \cdot \frac{d\boldsymbol{r}}{dt} dt$$

$$\boldsymbol{F} = (tt^2)\boldsymbol{i} + (t^2)\boldsymbol{j} - (t^2 t)\boldsymbol{k} = t^3\boldsymbol{i} + t^2\boldsymbol{j} - t^3\boldsymbol{k}$$

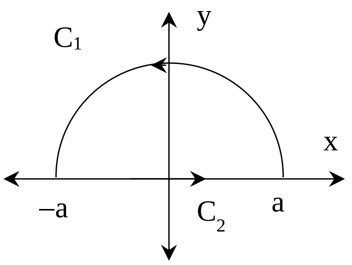
$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{i} + 2t\boldsymbol{j} + \boldsymbol{k}$$

$$\boldsymbol{F} \cdot \frac{d\boldsymbol{r}}{dt} = t^3 + 2t^3 - t^3 = 2t^3$$

$$W = \int_0^1 \boldsymbol{F}.\frac{d\boldsymbol{r}}{dt} dt = \int_0^1 2t^3 dt = \frac{2}{4} t^4 \Big|_0^1 = \frac{1}{2}$$

**17-** Find the flux of the field $\boldsymbol{F} = x\boldsymbol{i} + y\boldsymbol{j}$ across the closed semicircle path $\boldsymbol{r_1} = (a\cos t)\boldsymbol{i} + (a\sin t)\boldsymbol{j}$, $0 \le t \le \pi$ followed by the line segment $\boldsymbol{r_2} = t\boldsymbol{i}$, $-a \le t \le -a$.

**Sol.:**




$$Flux = \oint (Mdy - Ndx)$$

For $\boldsymbol{r_1} = (a\cos t)\boldsymbol{i} + (a\sin t)\boldsymbol{j}$

$$x = a\cos t$$

$$y = a\sin t$$

$$M = x = a\cos t$$

$$N = y = a\sin t$$

$$dx = d(a\cos t) = -a\sin t\, dt$$

$$dy = d(a\sin t) = a\cos t\, dt$$

$$\int (Mdy - Ndx) = \int_0^\pi [(a\cos t)(a\cos t) - (a\sin t)(-a\sin t)]\, dt$$

$$= \int_0^\pi a^2 dt = a^2\pi$$

For $\boldsymbol{r}_2 = t\boldsymbol{i}$

$x=t$, $y=0$

$dx=dt$, $dy=0$.

$$Mdy - Ndx = (t)(0) - (0)(1) = 0$$

$$\int Mdy - Ndx = 0$$

$$Flux = a^2\pi + 0$$

## Green's Theorem (15.3)

Flux-Divergence Form

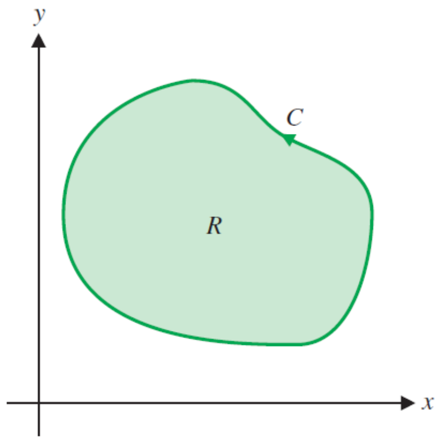
$$\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j}$$

$$\oint_C (Mdy - Ndx) = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

Outward flux — Divergence Integral

$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Counterclockwise Circulation — Curl Integral




**Ex:** Verify both forms of Green's Theorem for the field $\boldsymbol{F}(x, y) = (x - y)\boldsymbol{i} + x\boldsymbol{j}$ over the region $R$ bounded by the circle $\boldsymbol{r} = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j}$

**Sol.:**

$$M = x - y = \cos t - \sin t$$

$$N = x = \cos t$$

$$dx = d(\cos t) = -\sin t\, dt$$

$$dy = d(\sin t) = \cos t\, dt$$

$$\frac{\partial M}{\partial x} = 1, \frac{\partial M}{\partial y} = -1, \frac{\partial N}{\partial x} = 1, \frac{\partial N}{\partial y} = 0$$

$$\oint (Mdy - Ndx) = \int_0^{2\pi} [(\cos t - \sin t)\cos t - (\cos t)(-\sin t)]\, dt$$

$$= \int_0^{2\pi} \cos^2 t\, dt = \frac{1}{2}\int_0^{2\pi} (1 + \cos 2t) dt = \frac{1}{2}\left(t - \frac{\sin 2t}{2}\right)\Big|_0^{2\pi} = \pi$$

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy = \iint_R (1 + 0) dxdy = \int_0^{2\pi}\int_0^1 r dr d\theta = \left[\frac{r^2}{2}\right]_0^1 [\theta]_0^{2\pi} = \pi$$

$$\oint (Mdx + Ndy) = \int_0^{2\pi} [(\cos t - \sin t)(-\sin t) + (\cos t)(\cos t)] dt$$

$$= \int_0^{2\pi} (-\sin t \cos t + \sin^2 t + \cos^2 t) dt = \int_0^{2\pi} (-\sin t \cos t + 1) dt$$

$$= \frac{\cos^2 t}{2} + t\Big|_0^{2\pi} = \frac{1}{2} + 2\pi - \frac{1}{2} = 2\pi$$

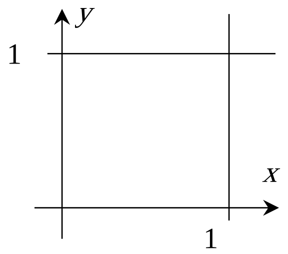
$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R (1 - (-1)) dxdy = \int_0^{2\pi}\int_0^1 2r dr d\theta = [r^2]_0^1 [\theta]_0^{2\pi} = 2\pi$$

**Ex:** Evaluate the integral

$$\oint xydy - y^2 dx$$

Around the square cut from the first quadrant by the lines $x$=1 and $y$=1.

**Sol.:**




Using the divergence integral

$$\oint (Mdy - Ndx) = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$M = xy, N = y^2$

$\frac{\partial M}{\partial x} = y, \frac{\partial N}{\partial y} = 2y$

$$\oint (xydy - y^2 dx) = \int_0^1\int_0^1 (y + 2y) dxdy = \int_0^1\int_0^1 (3y) dxdy$$

$$\left[\frac{3}{2}y^2\right]_0^1 [x]_0^1 = \frac{3}{2}$$

Using the curl integral

$$\oint (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$M = -y^2, N = xy$

$\frac{\partial N}{\partial x} = y, \frac{\partial M}{\partial y} = -2y$

$$\oint (xydy - y^2dx) = \int_0^1 \int_0^1 (y - (-2y))dxdy = \int_0^1 \int_0^1 (3y)dxdy = \frac{3}{2}$$

**Exercises 15.3**

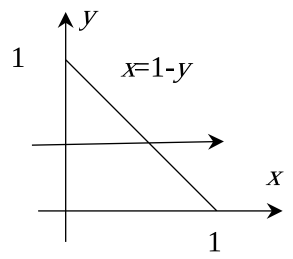
Apply Green's Theorem to evaluate the line integral

**11-** The integral

$$\oint y^2dx + x^2dy$$

Over the triangle bounded by $x$=0, $x$+$y$=1 and $y$=0.

**Sol.:**




Using Curl integral

$$\oint (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$M = y^2, N = x^2, \frac{\partial N}{\partial x} = 2x, \frac{\partial M}{\partial y} = 2y$

$$\oint y^2dx + x^2dy = \int_0^1 \int_0^{1-y} (2x - 2y)dxdy = \int_0^1 [x^2 - 2yx]_0^{1-y} dy$$

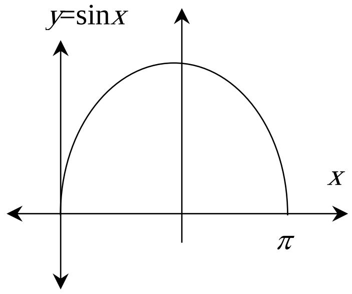
$$= \int_0^1 ((1-y)^2 - 2y(1-y))dy = \int_0^1 (1 - 2y + y^2 - 2y + 2y^2)dy$$

$$= \int_0^1 (1 - 4y + 3y^2)dy = y - 2y^2 + y^3|_0^1 = 1 - 2 + 1 = 0$$

**12-** The integral

$$\oint 3ydx + 2xdy$$

*C*: is the boundary $0\leq x\leq\pi$, $0\leq y\leq \sin x$




**Sol.:**

Using the divergence integral

$$\oint (Mdy - Ndx) = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$M = 2x,\ N = -3y,\ \frac{\partial M}{\partial x} = 2,\ \frac{\partial N}{\partial y} = -3$$

$$\oint 2xdy - (-3y)dx = \int_0^{\pi}\int_0^{\sin x} (2-3)dydx = \int_0^{\pi}\int_0^{\sin x} (-1)dydx$$

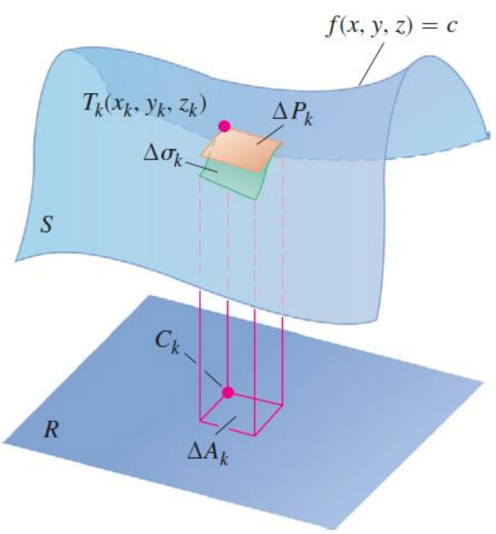
$$= \int_0^{\pi} -\sin x\, dx = \cos x|_0^{\pi} = -1 - 1 = -2$$

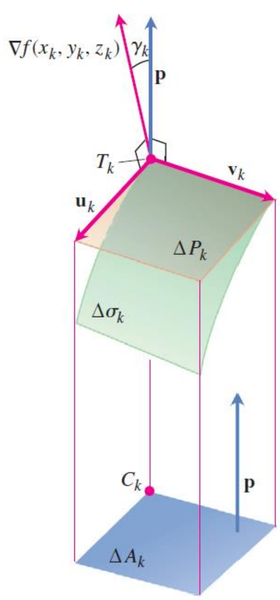
## 15.4 Surface area and Surface integral

The surface area is defined by the integral

$$\iint_R \frac{dA}{|\cos\gamma|} = \iint_R d\sigma\,,\quad d\sigma = \frac{dA}{|\cos\gamma|}$$

where *dA* is the element of area of the region *R*. *R* is the projection region of the surface *S* defined by *f*(*x*,*y*) onto an appropriate plane (usually the xy-plane). $\gamma$ is the angle between the tangent plane to the surface *S* and the plane containing the projection region *R*. Which is the same as the angle between the normal vector ***p*** to the plane of the region R and the normal vector to the tangent plane, which is the gradient $\nabla f$.



The magnitude of the dot product between $\boldsymbol{\nabla f}$ and $\boldsymbol{p}$ is

$$|\boldsymbol{\nabla f}.\boldsymbol{p}| = |\boldsymbol{\nabla f}||\boldsymbol{p}||\cos\gamma|$$

$$\frac{1}{|\cos\gamma|} = \frac{|\boldsymbol{\nabla f}|}{|\boldsymbol{\nabla f}.\boldsymbol{p}|}$$

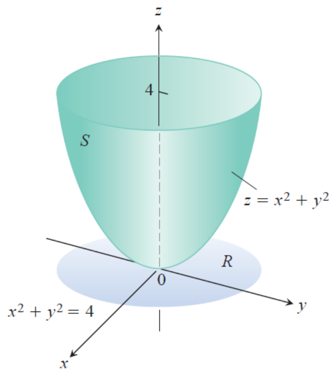
Therefore the surface area is

$$Surface\ Area = \iint\limits_R \frac{|\boldsymbol{\nabla f}|}{|\boldsymbol{\nabla f}.\boldsymbol{p}|} dA$$

The vector p is the unit vector normal to R and $\boldsymbol{\nabla f}.\boldsymbol{p} \neq 0$, usually R is the xy-plane, therefore $\boldsymbol{p}=\boldsymbol{k}$.

**Ex:** Find the area of the surface cut from the paraboloid $x^2+y^2-z=0$ by the plane $z=4$.

**Ans.**




$$f(x,y,z) = x^2 + y^2 - z = 0$$

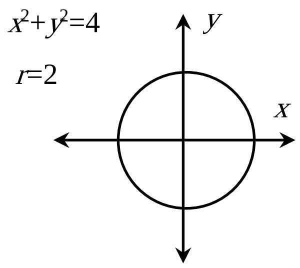
$$\boldsymbol{\nabla f} = 2x\boldsymbol{i} + 2y\boldsymbol{j} - \boldsymbol{k}$$

$$|\boldsymbol{\nabla f}| = \sqrt{4x^2 + 4y^2 + 1}$$

$$\boldsymbol{p} = \boldsymbol{k}$$

$$|\boldsymbol{\nabla f}.\boldsymbol{p}| = |\boldsymbol{\nabla f}.\boldsymbol{k}| = |-1| = 1$$

$$Surface\ Area = \iint\limits_R \frac{|\boldsymbol{\nabla f}|}{|\boldsymbol{\nabla f}.\boldsymbol{p}|} dA = \iint\limits_{x^2+y^2\le 4} \sqrt{4x^2 + 4y^2 + 1}\,dxdy$$




$$= \int_0^{2\pi}\int_0^2 \sqrt{4r^2 + 1}\,r\,dr\,d\theta = \left[\frac{1}{8}\cdot\frac{2}{3}(4r^2+1)^{3/2}\right]_0^2 [\theta]_0^{2\pi}$$

$$= \frac{1}{12}\left(17^{3/2} - 1\right)(2\pi) = \frac{\pi}{6}\left(17\sqrt{17} - 1\right)$$