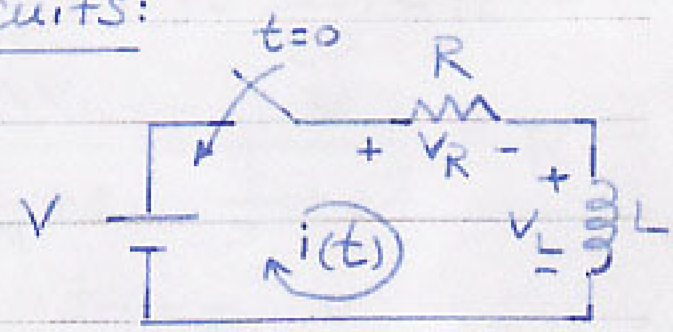
Transient Response of RLC circuits:

When a network is switched from one state to another by a change in the applied voltage or by a change in one of the circuit elements then branch currents and voltages change to new values. The response of the output of a network during transition period is called "Transient Response".

RLC circuits
- series
  - natural response
  - step response
- parallel
  - natural response
  - step response

Transient Response of RL and RC circuits:

1. Series RL circuit:



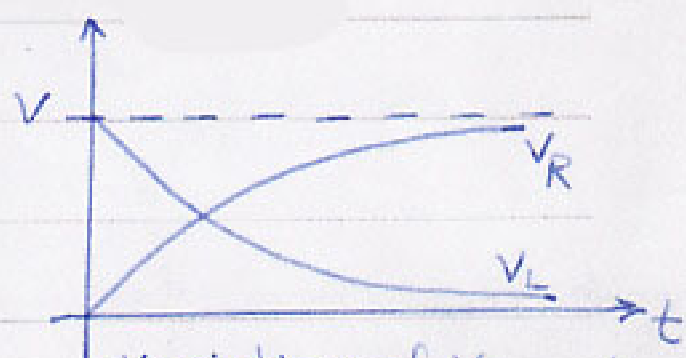
Applying KVL: $V - V_R - V_L = 0$

$$V - R\,i(t) - L\frac{di(t)}{dt} = 0$$

or: $$L\frac{di(t)}{dt} + R\,i(t) = V$$

This is a homogeneous D.E. with a solution of:

$$i(t) = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right) \quad \text{amp}$$

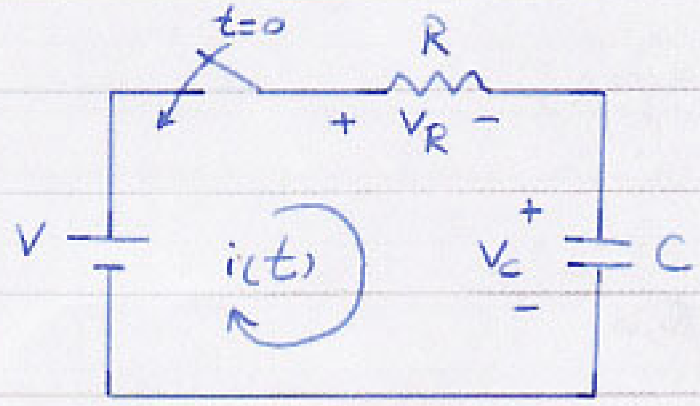


Variation of V_R and V_L with time

$$V_R = R\,i(t) = V\left(1 - e^{-\frac{R}{L}t}\right) \quad \text{volt}$$

$$V_L = L\frac{di(t)}{dt} = L\frac{d}{dt}\left[\frac{V}{R} - \frac{V}{R}e^{-\frac{R}{L}t}\right] = V e^{-\frac{R}{L}t} \quad \text{volt}$$

2. Series RC circuit:



Applying KVL:

$V - V_R - V_c = 0$

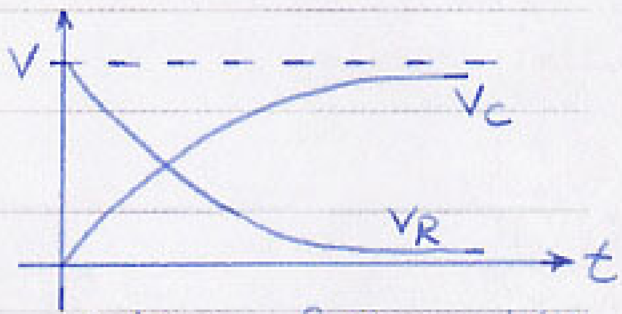
$V - Ri(t) - \frac{1}{c}\int i(t)dt = 0$

Differentiate the above eq. with respect to t, then:

$0 - R\frac{di(t)}{dt} - \frac{1}{c}i(t) = 0 \Rightarrow R\frac{di(t)}{dt} + \frac{1}{c}i(t) = 0$

The solution for the above eq. is:

$i(t) = \frac{V}{R}e^{-\frac{1}{RC}t}$ amp



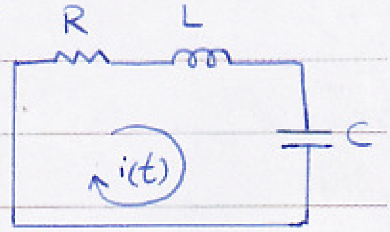
variation of V_R and V_c with time

$V_R = Ri(t) = Ve^{-\frac{1}{RC}t}$ volt

$V_c = V - V_R = V - Ve^{-\frac{1}{RC}t}$ volt

Natural Response of series RLC circuits:

Consider the cct. shown in figure.
The cct. is excited by the energy stored in both L and C.



Applying KVL, then:

$$V_R + V_L + V_C = 0$$

$$R\,i(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt = 0 \quad \text{—— (1)}$$

Differentiate eq. (1) with respect to t, then:

$$R\frac{di(t)}{dt} + L\frac{d^2i(t)}{dt^2} + \frac{1}{C}i(t) = 0$$

Rearranging: $$\frac{d^2i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = 0 \quad \text{—— (2)}$$

Eq. (2) is a second order D.E. The solution of such eq. can be obtained by replacing $\frac{d}{dt}$ with s and $\frac{d^2}{dt^2}$ with s^2, then:

$$s^2 i(t) + \frac{R}{L}s\,i(t) + \frac{1}{LC}i(t) = 0 \quad \text{—— (3)}$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \quad \text{—— (4)}$$

Eq. (4) is called the "Characteristic Equation".
The response of the cct. depends on the roots of eq. (4) which are:

$$s_1 = \frac{-\frac{R}{L} + \sqrt{(\frac{R}{L})^2 - \frac{4}{LC}}}{2} = -\frac{R}{2L} + \sqrt{(\frac{R}{2L})^2 + (\frac{1}{\sqrt{LC}})^2} = -\alpha + \sqrt{\alpha^2 - \omega_n^2}$$

$$s_2 = \frac{-\frac{R}{L} - \sqrt{(\frac{R}{L})^2 - \frac{4}{LC}}}{2} = -\frac{R}{2L} - \sqrt{(\frac{R}{2L})^2 + (\frac{1}{\sqrt{LC}})^2} = -\alpha - \sqrt{\alpha^2 - \omega_n^2}$$

$\alpha = \frac{R}{2L}$ rad/sec. neper frequency

$\omega_n = \frac{1}{\sqrt{LC}}$ rad/sec. natural (resonant) frequency

The complete solution of eq.② represents the natural response of the series RLC cct., which is:

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where A_1 and A_2 are constants determined from the initial values $i(0)$ and $\frac{di(0)}{dt}$.

From the values of s_1 and s_2, there are 3 types of solutions (responses):

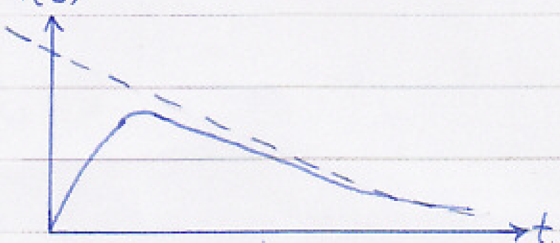
1. Overdamped case: $(\alpha > \omega_n)$

If $\alpha > \omega_n$ $\left(C > \frac{4L}{R^2}\right)$ then the cct. is overdamped, and the response is:

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where s_1 and s_2 are real, negative and unequall roots.

In such case, the response is exponential and slow.



overdamped response

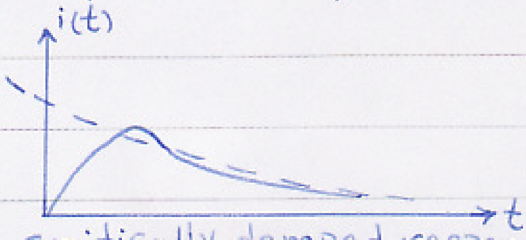
2. Critically damped case: $(\alpha = \omega_n)$

If $\alpha = \omega_n$ $\left(C = \frac{4L}{R^2}\right)$ then the cct. is critically damped and the response is:

$$i(t) = A_1 e^{s_1 t} + A_2 t\, e^{s_2 t} = (A_1 + A_2 t)\, e^{-\alpha t}$$

where $s_1 = s_2 = -\alpha$ (real, negative and equal)

In such case, the response is exponential but faster.



critically damped response

3. Underdamped case: ($\alpha < \omega_n$)

If $\alpha < \omega_n$ ($C < \frac{4L}{R^2}$) then the cct. is underdamped and the response is:

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = A_1 e^{(-\alpha + j\omega d)t} + A_2 e^{(-\alpha - j\omega d)t}$$

where s_1 and s_2 are complex conjugates

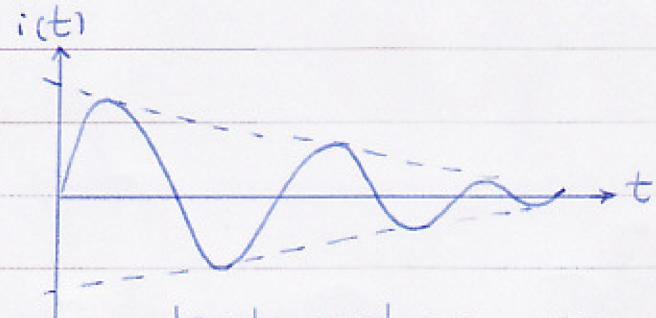
$$\omega d = \sqrt{\omega_n^2 - \alpha^2} \text{ rad/sec.} \quad \text{damping frequency}$$

$$i(t) = e^{-\alpha t}(A_1 e^{j\omega d t} + A_2 e^{-j\omega d t})$$

Knowing that: $e^{\pm jt} = \cos t \pm j \sin t$

$$\therefore \quad i(t) = e^{-\alpha t}(B_1 \cos \omega d t + B_2 \sin \omega d t)$$

In such case, the response is exponentially oscillating (due to the sine term).

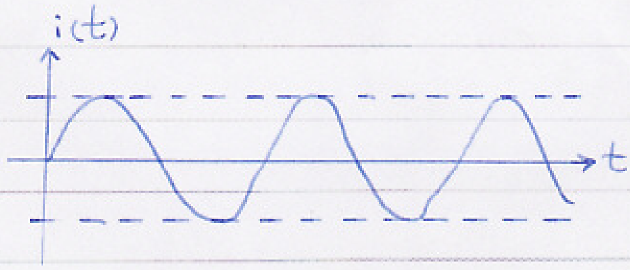


underdamped response

Note that if $R = 0$ then $\alpha = \frac{R}{2L} = 0$ and $s_1 = s_2 = j\omega d$

$$\therefore i(t) = B_1 \cos \omega d t + B_2 \sin \omega d t$$

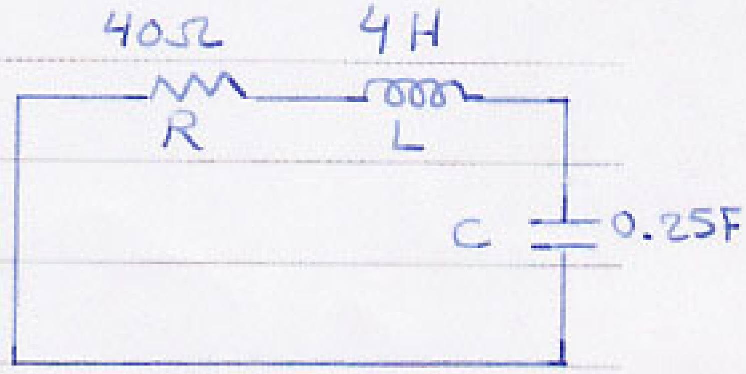
In this case, the cct. is said to be "damped" and will have oscillation.



The oscillatory response of RLC networks is due to the presence of both L and C that allows the flow of energy back and forth between them.

The behaviour of such networks is based on the idea of damping, which is the gradual loss of the initially-stored energy.

ex. For the cct. shown, calculate the c/cs roots of the cct. and define its response.



Sol. $\alpha = \frac{R}{2L} = \frac{40}{2(4)} = 5$ rad/sec.

$\omega_n = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{4 * 0.25}} = 1$ rad/sec.

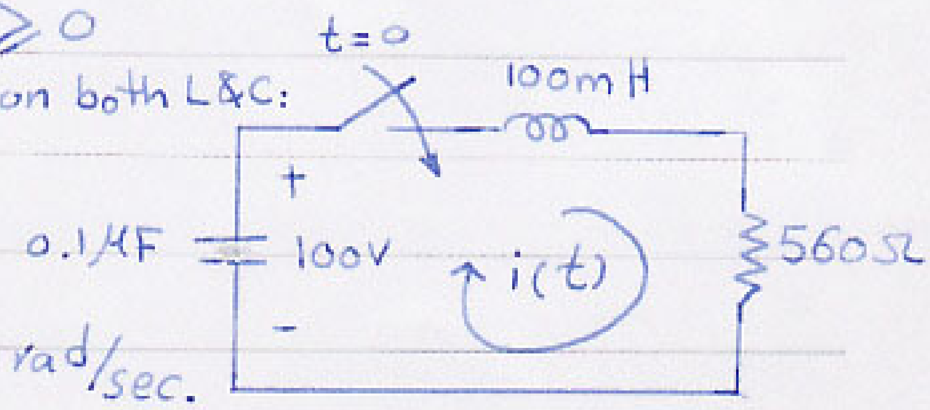
$\because \alpha > \omega_n \Rightarrow \therefore$ The response is overdamped

$S_1 = -\alpha + \sqrt{\alpha^2 - \omega_n^2} = -5 + \sqrt{25-1} = -0.101$ rad/sec.

$S_2 = -\alpha - \sqrt{\alpha^2 - \omega_n^2} = -5 - \sqrt{25-1} = -9.899$ rad/sec.

ex. The 0.1 μF capacitor in the cct. shown is charged to 100V. At t=0, the capacitor is discharged through a series combination of L=100mH and R=560 Ω. Find:

a) $i(t)$; b) $V_c(t)$ for $t \geq 0$



Sol. First we should find the initial charge on both L&C: $i_L(0) = 0$; $V_c(0) = 100V$

a) $\alpha = \frac{R}{2L} = \frac{560}{2(100 * 10^{-3})} = 2800$ rad/sec.

$\omega_n = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(100 * 10^{-3})(0.1 * 10^{-6})}} = 10\,000$ rad/sec.

$\because \alpha < \omega_n \Rightarrow \therefore$ The response is underdamped.

$i(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t$

$\omega_d = \sqrt{\omega_n^2 - \alpha^2} = \sqrt{(10000)^2 - (2800)^2} = 9600$ rad/sec

B_1 and B_2 are calculated from the capacitor's initial charge $i(0)$ and $di(0)/dt$.

$i(0) = 0 = B_1$

$\therefore\ i(t) = B_2 e^{-2800t} \sin 9600t$

$$\frac{di(t)}{dt} = B_2 e^{-2800t}(9600 \cos 9600t) + B_2(-2800 e^{-2800t}) \sin 9600t$$

$$\frac{di(0)}{dt} = B_2 e^{0}(9600 \cos 9600t) + B_2(-2800 e^{0}) \sin 0$$

$$= B_2(1)(9600) + B_2(-2800)(0) = 9600 B_2$$

Also, applying KVL:

$$V_R + V_L + V_C = 0$$

$$R i(t) + L\frac{di(t)}{dt} - v_c(t) = 0$$

$$R i(0) + L\frac{di(0)}{dt} - v_c(0) = 0$$

$$0 + (100 * 10^{-3})\frac{di(0)}{dt} - 100 = 0 \Rightarrow \frac{di(0)}{dt} = \frac{100}{100 * 10^{-3}} = 1000\ A/sec$$

$\therefore\ 1000 = 9600 B_2 \Rightarrow B_2 = 0.104$ A

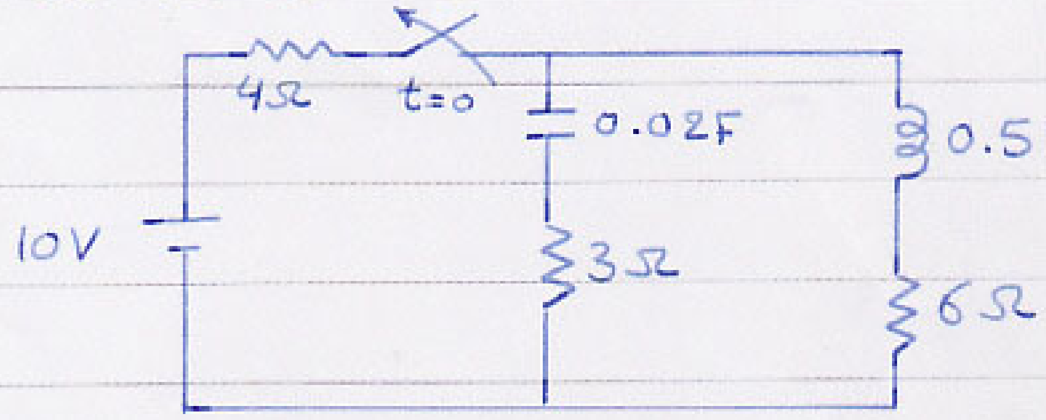
$\therefore\ i(t) = 0.104 e^{-2800t} \sin 9600t$ amp

b) $v_c(t) = R i(t) + L\frac{di(t)}{dt}$

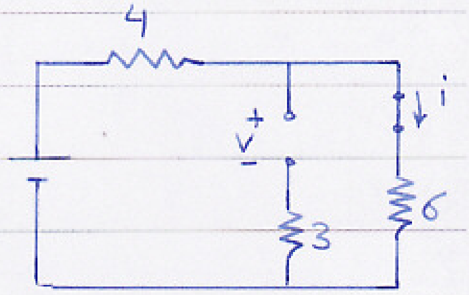
$$= (560)(0.104 e^{-2800t} \sin 9600t) + 0.1(0.104 e^{-2800t}(9600 \cos 9600t) + 0.104(-2800 e^{-2800t}) \sin 9600t)$$

$$= 100 e^{-2800t} \cos 9600t + 29.166 e^{-2800t} \sin 9600t \text{ volt}$$

ex. Find $i(t)$ in the cct. shown assuming that the cct. has reached to steady-state at $t = 0^-$.



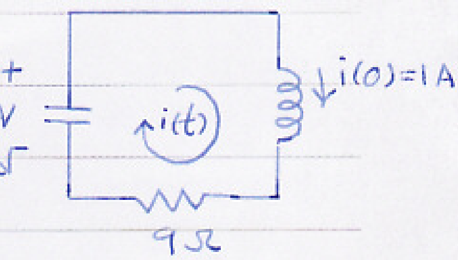
Sol. For $t<0$, the switch is closed.
At steady state, the capacitor acts as open cct. and the inductor acts as short cct. Then the equivalent cct. will be as shown: →



$i(0) = \frac{10}{4+6} = 1\,A$ initial current through L.

$V(0) = 10 * \frac{6}{4+6} = 6\,V$ initial voltage across C.

For $t>0$, the switch is open and the cct. will be as shown:



$\alpha = \frac{R}{2L} = \frac{9}{2(0.5)} = 9$ rad/sec.

$\omega_n = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(0.5)(0.02)}} = 10$ rad/sec.

$\because \alpha < \omega_n \Rightarrow \therefore$ The response is underdamped

$i(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$

$\omega_d = \sqrt{\omega_n^2 - \alpha^2} = \sqrt{100-81} = 4.358$ rad/sec.

$\therefore i(t) = e^{-9t}(B_1 \cos 4.358t + B_2 \sin 4.358t)$

$i(0) = 1 = B_1$

$\therefore i(t) = e^{-9t}(\cos 4.358t + B_2 \sin 4.358t)$

$$\frac{di(t)}{dt} = e^{-9t}(-4.358 \sin 4.358t) + (-9e^{-9t})\cos 4.358t + B_2 e^{-9t}(4.358 \cos 4.358t) + B_2(-9e^{-9t})\sin 4.358t$$

$$\frac{di(0)}{dt} = -9 + 4.358 B_2$$

Also, Applying KVL: $V_R + V_L + V_C = 0 \Rightarrow Ri(t) + L\frac{di(t)}{dt} + v_c(t) = 0$

$Ri(0) + L\frac{di(0)}{dt} - v_c(0) = 0$

$(9)(1) + (0.5)\frac{di(0)}{dt} - 6 = 0 \Rightarrow \frac{di(0)}{dt} = \frac{-9+6}{0.5} = -6$ A/sec.

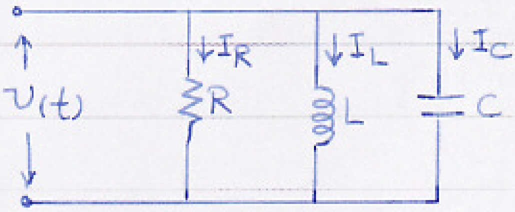
$\therefore -6 = -9 + 4.358 B_2 \Rightarrow B_2 = 0.688$

$\therefore i(t) = e^{-9t}(\cos 4.358t + 0.688 \sin 4.358t)$ amp

Natural Response of Parallel RLC circuits:

Parallel RLC cct. applications are notably in communication networks and filter design.

Consider the cct. shown in fig.



Applying KCL yields:

$$I_R + I_L + I_C = 0$$

$$\frac{v(t)}{R} + \frac{1}{L}\int v(t)\,dt + C\frac{dv(t)}{dt} = 0 \quad \text{—— ①}$$

differentiate eq. ① with respect to t, then:

$$\frac{1}{R}\frac{dv(t)}{dt} + \frac{1}{L}v(t) + C\frac{d^2v(t)}{dt^2} = 0$$

rearranging: $\frac{d^2v(t)}{dt^2} + \frac{1}{RC}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = 0$ —— ②

To find c/cs eq.:

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \quad \text{—— ③}$$ characteristic equation

$$s_1 = \frac{-\frac{1}{RC} + \sqrt{(\frac{1}{RC})^2 - \frac{4}{LC}}}{2} = \frac{-1}{2RC} + \sqrt{(\frac{1}{2RC})^2 - (\frac{1}{\sqrt{LC}})^2} = -\alpha + \sqrt{\alpha^2 - \omega_n^2}$$

$$s_2 = \frac{-\frac{1}{RC} - \sqrt{(\frac{1}{RC})^2 - \frac{4}{LC}}}{2} = \frac{-1}{2RC} - \sqrt{(\frac{1}{2RC})^2 - (\frac{1}{\sqrt{LC}})^2} = -\alpha - \sqrt{\alpha^2 - \omega_n^2}$$

where: $\alpha = \frac{1}{2RC}$; $\omega_n = \frac{1}{\sqrt{LC}}$

A complete solution for eq. ② is:

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where A_1 and A_2 are constants determined from $v(0)$ and $dv(0)/dt$.

The cct. will have 3 responses:

1. Overdamped: $(\alpha > \omega_n)$

If $\alpha > \omega_n$ $(L > 4R^2C)$ then the cct. is overdamped and the response is:

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

2. Critically damped: ($\alpha = \omega_n$)

If $\alpha = \omega_n$ ($L = 4R^2C$) then the cct. is critically damped and the response is:

$$v(t) = A_1 e^{s_1 t} + A_2 t\, e^{s_2 t} = (A_1 + A_2 t)\, e^{-\alpha t}$$

3. Underdamped: ($\alpha < \omega_n$)

If $\alpha < \omega_n$ ($L < 4R^2C$) then the cct. is underdamped and the response is:

$$v(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)$$

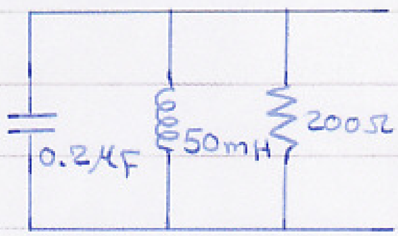
where $\omega_d = \sqrt{\omega_n^2 - \alpha^2}$

ex. For the cct. shown:

a) Find the roots of the c/cs eq.

b) Specify the response of the cct.

c) Determine R that causes the response to be critically damped.



sol.

a) $\alpha = \dfrac{1}{2RC} = \dfrac{1}{2(200)(0.2*10^{-6})} = 12500$ rad/sec

$\omega_n = \dfrac{1}{\sqrt{LC}} = \dfrac{1}{\sqrt{(50*10^{-3})(0.2*10^{-6})}} = 10\,000$ rad/sec

$S_1 = -\alpha + \sqrt{\alpha^2 - \omega_n^2} = -5000$ rad/sec

$S_2 = -\alpha - \sqrt{\alpha^2 - \omega_n^2} = -20\,000$ rad/sec

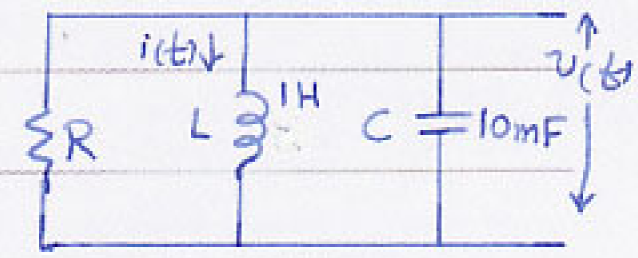
b) $\because \alpha > \omega_n \Rightarrow \therefore$ the response is overdamped

c) For the response to be critically damped, then:

$\alpha = \omega_n \Rightarrow L = 4R^2C$

$R = \dfrac{1}{2}\sqrt{\dfrac{L}{C}} = \dfrac{1}{2}\sqrt{\dfrac{50*10^{-3}}{0.2*10^{-6}}} = 250\,\Omega$

ex. For the cct. shown, assuming $v(0)=5V$; $i(0)=0$. Find $v(t)$ for $R=1.923\ \Omega$; $R=5\Omega$ and $R=6.25\Omega$.



Sol.

- For $R = 1.923\Omega$

$\alpha = \frac{1}{2RC} = 26$ rad/sec. ; $\omega_n = \frac{1}{\sqrt{LC}} = 10$ rad/sec.

$\because \alpha > \omega_n \Rightarrow \therefore$ The response is overdamped

$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_n^2} = -2$ rad/sec. ; $S_2 = -\alpha - \sqrt{\alpha^2 - \omega_n^2} = -50$ rad/sec.

$\therefore v(t) = A_1 e^{-2t} + A_2 e^{-50t}$

$v(0) = A_1 + A_2 = 5$ —— ①

$\frac{dv(t)}{dt} = -2A_1 e^{-2t} + (-50)A_2 e^{-50t} \Rightarrow \frac{dv(0)}{dt} = -2A_1 - 50A_2$

also, applying KCL: $I_R + I_L - I_C = 0$

$\frac{v(t)}{R} + i_L(t) - C\frac{dv(t)}{dt} = 0$

$\frac{v(0)}{R} + i_L(0) - C\frac{dv(0)}{dt} = 0$

$\therefore \frac{dv(0)}{dt} = \frac{v(0)}{RC} = \frac{5}{RC} = 260$ v/sec.

$\therefore 260 = -2A_1 - 50A_2$ —— ②

Solving eqs. ① and ② to get: $A_1 = 10.625$; $A_2 = -5.625$

$\therefore v(t) = 10.625\, e^{-2t} - 5.625\, e^{-50t}$ volt

- For $R = 5\Omega$:

$\alpha = 10$ rad/sec. ; $\omega_n = 10$ rad/sec.

$\because \alpha = \omega_n \Rightarrow \therefore$ the response is critically damped

$v(t) = (A_1 + A_2 t)\, e^{-\alpha t} = (A_1 + A_2 t)\, e^{-10t}$

$v(0) = 5 = A_1$

$$\frac{dv(t)}{dt} = -10A_1e^{-10t} + A_2t(-10e^{-10t}) + A_2e^{-10t}$$

$$\frac{dv(0)}{dt} = -10A_1 + A_2 = -50 + A_2$$

also: $I_R + I_L - I_C = 0$

$$\frac{v(t)}{R} + i_L(t) - C\frac{dv(t)}{dt} = 0$$

$$\frac{v(0)}{R} + i_L(0) - C\frac{dv(0)}{dt} = 0 \Rightarrow \frac{dv(0)}{dt} = \frac{v(0)}{RC} = \frac{5}{RC} = 100 \text{ V/sec}$$

$\therefore 100 = -50 + A_2 \Rightarrow A_2 = 150$

$\therefore v(t) = (5 + 150t)e^{-10t}$ volt

- For $R = 6.25\,\Omega$:

$\alpha = 8$ rad/sec. ; $\omega_n = 10$ rad/sec.

$\because \alpha < \omega_n \Rightarrow \therefore$ The response is underdamped

$$v(t) = e^{-\alpha t}(A_1\cos\omega_d t + A_2\sin\omega_d t)$$

$\omega_d = \sqrt{\omega_n^2 - \alpha^2} = 6$ rad/sec.

$\therefore v(t) = e^{-8t}(A_1\cos 6t + A_2\sin 6t)$

$v(0) = 5 = A_1$

$$\frac{dv(t)}{dt} = A_1e^{-8t}(-6\sin 6t) + A_1(-8e^{-8t}\cos 6t) + A_2e^{-8t}(6\cos 6t) + A_2(-8e^{-8t}\sin 6t)$$

$$\frac{dv(0)}{dt} = -8A_1 + 6A_2 = -40 + 6A_2$$

also: $\frac{dv(0)}{dt} = \frac{5}{RC} = 80$ V/sec.

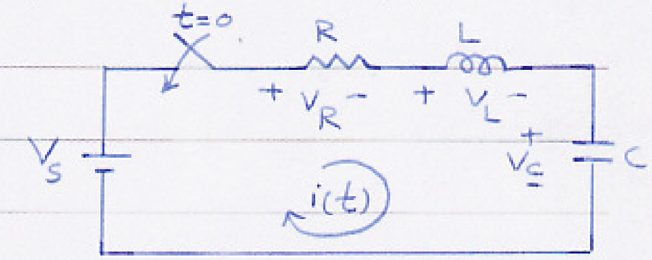
$\therefore 80 = -40 + 6A_2 \Rightarrow A_2 = 20$

$\therefore v(t) = e^{-8t}(5\cos 6t + 20\sin 6t)$ volt

Step (Forced) Response of Series RLC circuits:

The step response is obtained by the sudden application of a dc source.

Consider the cct. shown in fig.



Applying KVL for $t>0$, then:

$$V_R + V_L + V_C = V_s$$

$$R\,i(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt = V_s \quad \text{—— (1)}$$

Differentiate eq. (1) with respect to t, then:

$$R\frac{di(t)}{dt} + L\frac{d^2 i(t)}{dt^2} + \frac{1}{C}\,i(t) = 0$$

rearranging: $$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}\,i(t) = 0 \quad \text{—— (2)}$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \quad \text{—— (3)} \quad \text{c/cs eq.}$$

Note that the c/cs eq. is not affected by the presence of the dc source.

$$S_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_n^2} \quad ; \alpha = \frac{R}{2L} \quad ; \omega_n = \frac{1}{\sqrt{LC}}$$

The cct. will have 3 responses:

1. Overdamped : $(\alpha > \omega_n)$

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

2. Critically damped : $(\alpha = \omega_n)$

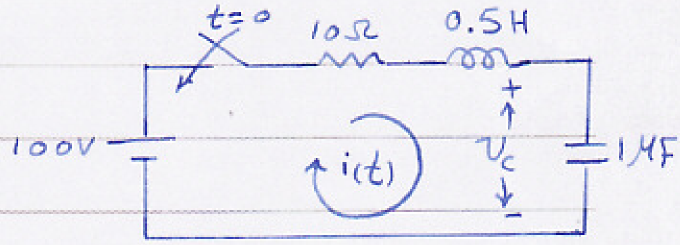
$$i(t) = (A_1 + A_2 t)\, e^{-\alpha t}$$

3. Underdamped : $(\alpha < \omega_n)$

$$i(t) = e^{-\alpha t}(A_1 \cos\omega_d t + A_2 \sin\omega_d t)$$

$$\omega_d = \sqrt{\omega_n^2 - \alpha^2}$$

ex. For the cct. shown, find $i(t)$ and $v_c(t)$ for $t \geq 0$.



sol. To find the initial conditions

at $t<0$:

$i_L(0)=0$; $v_c(0)=0$

$$\alpha = \frac{R}{2L} = \frac{10}{2(0.5)} = 10 \text{ rad/sec.}$$

$$\omega_n = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(0.5)(1*10^{-6})}} = 1414.21 \text{ rad/sec}$$

$\because \alpha < \omega_n \Rightarrow \therefore$ the response is underdamped

$$i(t) = (A_1 \cos \omega_d t + A_2 \sin \omega_d t)\, e^{-\alpha t}$$

$$\omega_d = \sqrt{\omega_n^2 - \alpha^2} = 1414.17 \text{ rad/sec}$$

$$i(t) = e^{-10t}(A_1 \cos 1414.17t + A_2 \sin 1414.17t)$$

$$i(0) = 0 = A_1$$

$$\therefore i(t) = A_2\, e^{-10t} \sin 1414.17t$$

$$\frac{di(t)}{dt} = A_2 e^{-10t}(1414.17 \cos 1414.17t) + A_2(-10e^{-10t}) \sin 1414.17t$$

$$\frac{di(0)}{dt} = 1414.17 A_2$$

For $t \geq 0$, the switch is closed. Applying KVL yields:

$$V_R + V_L + V_C = 100$$

$$Ri(t) + L\frac{di(t)}{dt} + v_c(t) = 100$$

$$Ri(0) + L\frac{di(0)}{dt} + v_c(0) = 100$$

$$\frac{di(0)}{dt} = \frac{100}{L} = \frac{100}{0.5} = 200 \text{ amp/sec.}$$

$$\therefore 200 = 1414.17 A_2 \Rightarrow A_2 = 0.1414$$

$$\therefore i(t) = 0.1414\, e^{-10t} \sin 1414.17t \quad \text{amp}$$

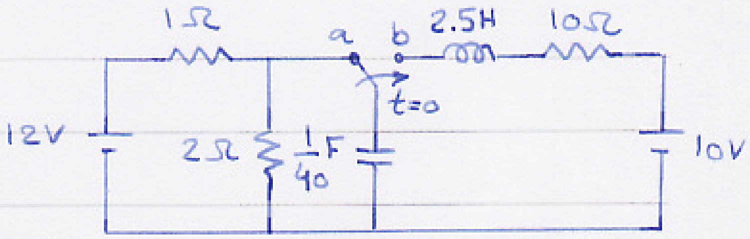
$$R\,i(t) + L\frac{di(t)}{dt} + v_c(t) = 100$$

$$v_c(t) = 100 - R\,i(t) - L\frac{di(t)}{dt}$$

$$= 100 - 10(0.1414\,e^{-10t}\sin 1414.17t) - 0.5\left(0.1414\,e^{-10t}(1414.17\cos 1414.17t)\right.$$
$$\left. + 0.1414(-10\,e^{-10t})\sin 1414.17t\right)$$

$$= 100 - 0.707\,e^{-10t}\sin 1414.17t - 100\,e^{-10t}\cos 1414.17t \text{ volts}$$

ex. For the cct. shown, the switch have been in position (a) for a long time. Find $v_c(t)$ and $v_R(t)$ at $t>0$.

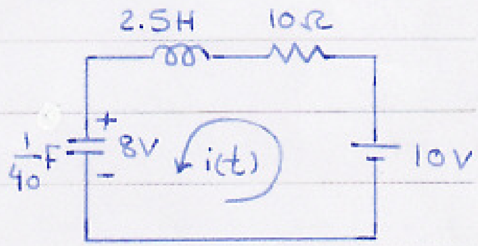


Sol.

- When the switch is at position a (i.e. $t<0$):

$v_c(0) = 12 * \frac{2}{1+2} = 8V$ initial voltage on C when the switch moves to b

- When the switch at position (b):



$v_c(0) = 8V$; $i_L(0) = 0$

$\alpha = \frac{R}{2L} = 2$ rad/sec.

$\omega_n = \frac{1}{\sqrt{LC}} = 4$ rad/sec.

$\because \alpha < \omega_n \Rightarrow \therefore$ the response is underdamped

$i(t) = e^{-\alpha t}(A_1\cos\omega_d t + A_2\sin\omega_d t)$

$\omega_d = \sqrt{\omega_n^2 - \alpha^2} = 3.464$ rad/sec.

$\therefore i(t) = e^{-2t}(A_1\cos 3.464t + A_2\sin 3.464t)$

$i(0) = 0 = A_1$

$\therefore i(t) = A_2\,e^{-2t}\sin 3.464t$

$\frac{di(t)}{dt} = A_2\,e^{-2t}(3.464\cos 3.464t) + A_2(-2e^{-2t})\sin 3.464t$

$$\frac{di(0)}{dt} = 3.464\,A_2$$

also: $V_R + V_L + V_c = 10 \Rightarrow Ri(t) + L\frac{di(t)}{dt} + v_c(t) = 10$

$$Ri(0) + L\frac{di(0)}{dt} + v_c(0) = 10 \Rightarrow \frac{di(0)}{dt} = \frac{10-8}{2.5} = 0.8 \text{ amp/sec.}$$

$\therefore 0.8 = 3.464\,A_2 \Rightarrow A_2 = 0.23$

$\therefore i(t) = 0.23\,e^{-2t}\sin 3.464t$ amp

$$v_c(t) = 10 - Ri(t) - L\frac{di(t)}{dt}$$

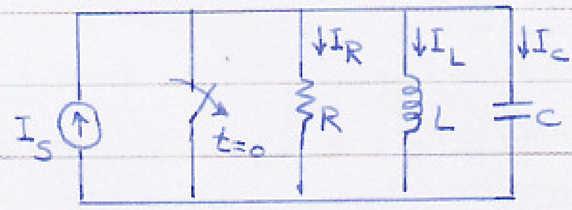
$$= 10 - 10(0.23e^{-2t}\sin 3.464t) - 2.5\left(0.23e^{-2t}(3.469\cos 3.464t) + 0.23(-2e^{-2t})\sin 3.464t\right)$$

$$= 10 - 1.145e^{-2t}\sin 3.464t - 2e^{-2t}\cos 3.464t \text{ volt}$$

$v_R(t) = Ri(t) = 2.3\,e^{-2t}\sin 3.464t$ volt

Step (Forced) Response of Parallel RLC circuits:

Consider the cct. shown in fig.



Applying KCL yields:

$$I_R + I_L + I_C = I_s$$

$$\frac{v(t)}{R} + \frac{1}{L}\int v(t)\,dt + C\frac{dv(t)}{dt} = I_s \quad \text{——} ①$$

differentiate eq. ① with respect to t:

$$\frac{1}{R}\frac{dv(t)}{dt} + \frac{1}{L}v(t) + C\frac{d^2v(t)}{dt} = 0$$

rearranging: $$\frac{d^2v(t)}{dt} + \frac{1}{RC}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = 0 \quad \text{——} ②$$

To find c/cs eq.:

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \quad \text{——} ③$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_n^2} \quad ; \alpha = \frac{1}{2RC} \quad ; \omega_n = \frac{1}{\sqrt{LC}}$$

The cct. will have 3 responses:

1. Overdamped: $(\alpha > \omega_n)$

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

2. Critically damped: $(\alpha = \omega_n)$

$$v(t) = (A_1 + A_2 t)e^{-\alpha t}$$

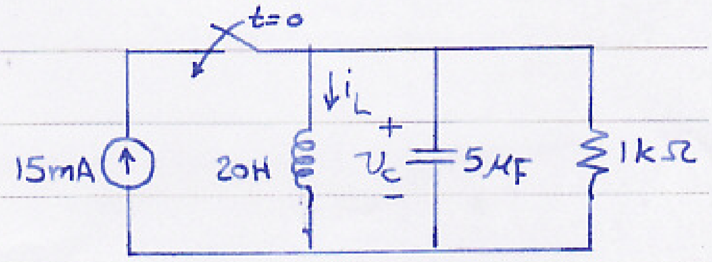
3. Underdamped: $(\alpha < \omega_n)$

$$v(t) = e^{-\alpha t}(A_1 \cos\omega_d t + A_2 \sin\omega_d t)$$

$$\omega_d = \sqrt{\omega_n^2 - \alpha^2}$$

ex. Assume that the initial current through the inductor is $-30\,mA$, and the initial voltage across the capacitor is $60\,V$. Find an expression for $i_L(t)$ for $t \geq 0$.

Sol. $v_c(0) = 60\,V$



$i_L(0) = -30\,mA$

$\alpha = \dfrac{1}{2RC} = 100$ rad/sec.

$\omega_n = \dfrac{1}{\sqrt{LC}} = 100$ rad/sec.

$\because \alpha = \omega_n \Rightarrow \therefore$ the response is critically damped.

$v(t) = (A_1 + A_2 t)e^{-\alpha t} = (A_1 + A_2 t)e^{-100t}$

$v(0) = 60 = A_1 \Rightarrow v(t) = (60 + A_2 t)e^{-100t}$

$\dfrac{dv(t)}{dt} = 60(-100e^{-100t}) + A_2 t(-100e^{-100t}) + A_2 e^{-100t}$

$\dfrac{dv(0)}{dt} = -6000 + A_2$

Also: $I_R + I_L + I_c = I_s \Rightarrow \dfrac{v(t)}{R} + i_L(t) + C\dfrac{dv(t)}{dt} = 15\,mA$

$\dfrac{v(0)}{R} + i_L(0) + C\dfrac{dv(0)}{dt} = 15\,mA$

$\dfrac{dv(0)}{dt} = \dfrac{15 * 10^{-3} - 60 * 10^{-3} - (-30 * 10^{-3})}{5 * 10^{-6}} = -3000$ v/sec

$\therefore -3000 = -6000 + A_2 \Rightarrow A_2 = 3000$

$\therefore v(t) = (60 + 3000t)e^{-100t}$ volt

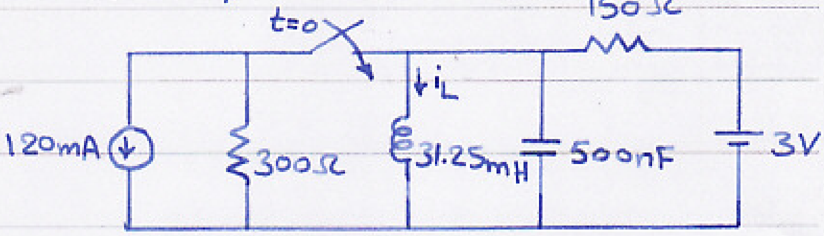
$i_L(t) = 15\,mA - \dfrac{v(t)}{R} - C\dfrac{dv(t)}{dt}$

$= 15\,mA - \dfrac{1}{1000}(60 + 3000t)e^{-100t} - (5*10^{-6})\left(60(-100e^{-100t}) + 3000t(-100e^{-100t}) + 3000e^{-100t}\right)$

$= 15 - 45e^{-100t} - 1500t\,e^{-100t}$ mamp

H.W. Solve the previous ex. for $R = 800\,\Omega$ and $R = 1250\,\Omega$

ex. The switch in the cct. shown has been open for a long time before closing at $t=0$. Find $i_L(t)$ for $t \geqslant 0$.

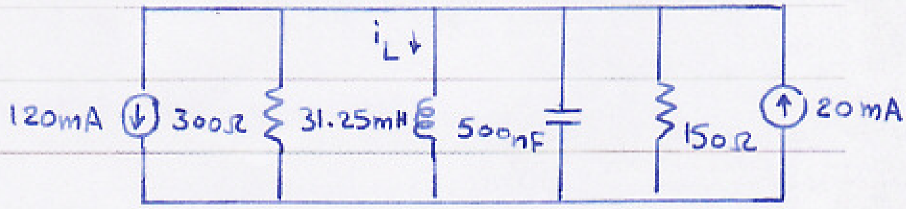


Sol. _For $t<0$:

$$i_L(0) = \frac{3}{150} = 20mA$$

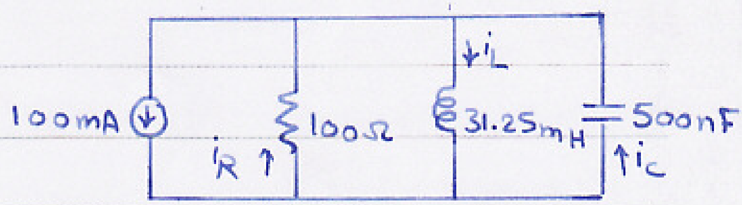
$$v_c(0) = 0$$

_For $t \geqslant 0$:



$$\alpha = \frac{1}{2RC} = 10\,000 \text{ rad/sec}$$

$$\omega_n = \frac{1}{\sqrt{LC}} = 8000 \text{ rad/sec.}$$



$\because \alpha > \omega_n$

$\therefore$ The response is overdamped

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_n^2} = -4000 \text{ rad/sec}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_n^2} = -16\,000 \text{ rad/sec}$$

$$\therefore v(t) = A_1 e^{-4000t} + A_2 e^{-16000t}$$

$$v(0) = 0 = A_1 + A_2 \quad \text{—} (1)$$

$$\frac{dv(t)}{dt} = -4000 A_1 e^{-4000t} - 16000 A_2 e^{-16000t}$$

$$\frac{dv(0)}{dt} = -4000 A_1 - 16000 A_2$$

Also: $I_R + I_L + I_C = I_S \Rightarrow \frac{v(t)}{R} - i_L(t) + C\frac{dv(t)}{dt} = 100\,mA$

$$\frac{v(0)}{R} - i_L(0) + C\frac{dv(0)}{dt} = 100 * 10^{-3}$$

$$\therefore \frac{dv(0)}{dt} = \frac{100 * 10^{-3} + 20 * 10^{-3}}{500 * 10^{-9}} = 240000 \text{ V/sec}$$

$\therefore\ 240\,000 = -4000\,A_1 - 16\,000\,A_2$ —— ②

Solving eqs. ① and ② to get: $A_1 = 20$; $A_2 = -20$

$$\therefore\ v(t) = 20\,e^{-4000t} - 20\,e^{-16000t} \quad \text{volt}$$

$$\frac{v(t)}{R} - i_L(t) + C\frac{dv(t)}{dt} = 100\,mA$$

$$i_L(t) = -100 *10^{-3} + \frac{v(t)}{R} + C\frac{dv(t)}{R}$$

$$= -100 * 10^{-3} + \frac{1}{100}\left(20\,e^{-4000t} - 20\,e^{-16000t}\right) + (500 * 10^{-9})\Big(20\left(-4000\,e^{-4000t}\right)$$

$$- 20\left(-16000\,e^{-16000t}\right)\Big)$$

$$= -100 + 160\,e^{-4000t} - 40\,e^{-16000t} \quad \text{mamp.}$$

Resonance:

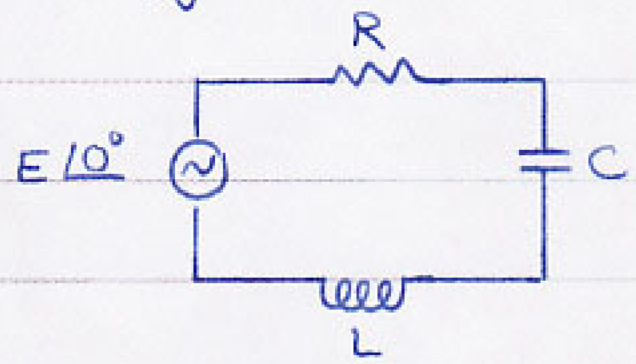
Resonance is a condition in RLC circuits in which the capacitive and inductive reactances are equal in magnitude, thus they cancel each other and result in a purely resistive impedance. Resonant circuits are useful for constructing filters.

Series Resonance:

A simple series resonant circuit is constructed by combining an ac source with RLC, as shown in figure.

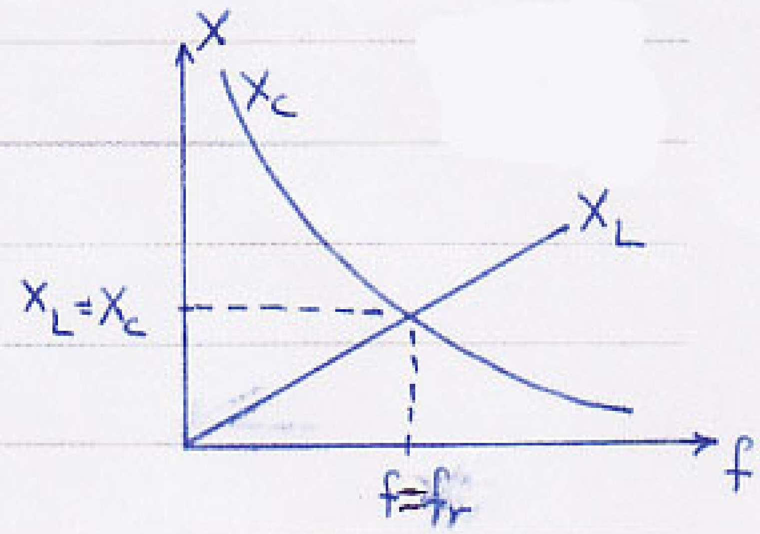
$X_L = \omega L = 2\pi f L$

$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$



$Z = R + j(X_L - X_C) = R + j(\omega L - \frac{1}{\omega C})$

The figure shows the variation of X_L and X_C with frequency. If the frequency of the source is changed from $0 \longrightarrow \infty$, then there would be a certain frequency which makes $X_L = X_C$ in magnitude. In this case, the circuit is said to be in electrical resonance, resulting imaginary part of Z is zero.



$\therefore \; X_L - X_C = 0 \implies Z = R$

To find the resonant frequency (f_r):

$X_L = X_C \implies 2\pi f_r L = \frac{1}{2\pi f_r C}$

$\therefore \; f_r = \frac{1}{2\pi\sqrt{LC}} \; Hz$

Note that at resonance:

1. The impedance is purely resistive ($Z = R$), then LC acts like a short circuit, and the entire voltage is across R.

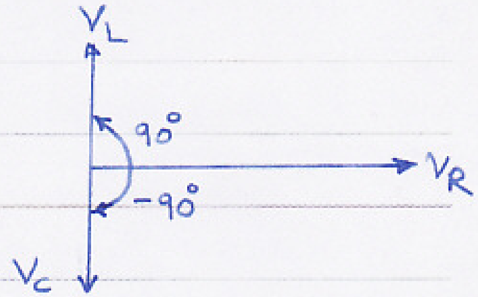
2. The input voltage and current are in phase.

$$I = \frac{E\angle 0^\circ}{Z} = \frac{E\angle 0^\circ}{R\angle 0^\circ} = \frac{E}{R}\angle 0^\circ$$

3. $V_R = IR\angle 0^\circ$

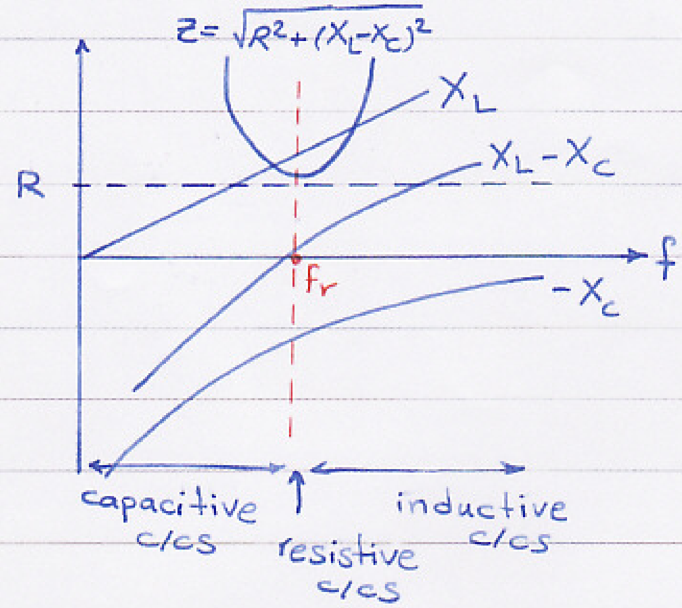
$V_L = IX_L\angle 90^\circ$

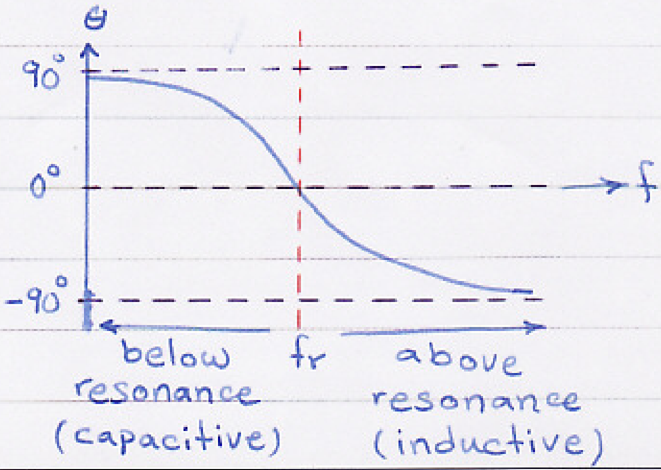
$V_C = IX_C\angle -90^\circ$



4. The current passing through the circuit is maximum at resonance.

Characteristics of series resonance:

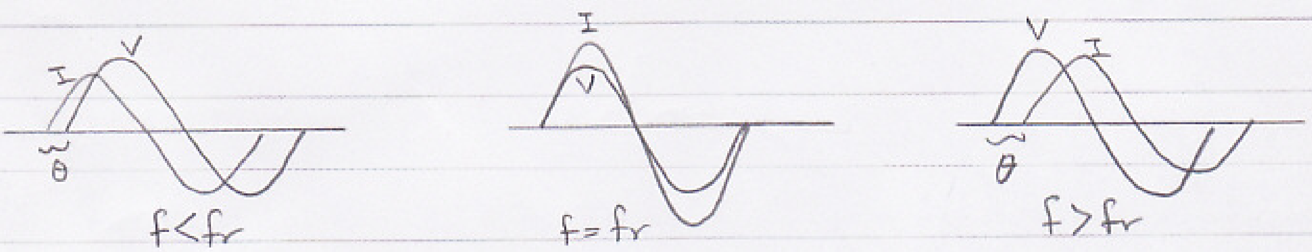




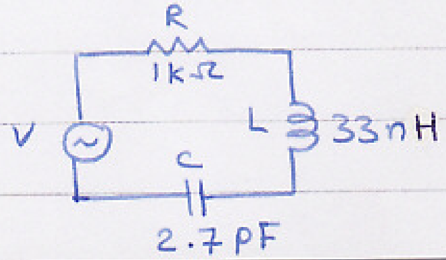
These c/cs can be divided into curves of:

1. Resistance (R): independent of f, hence it is represented by a straight line.
2. Inductive reactance (X_L): given by $X_L = 2\pi f L$. It is directly proportional with f, hence it is a straight line passing through the origin.
3. Capacitive reactance (X_C): given by $X_C = \frac{1}{2\pi f C}$. It is inversely proportional with f, hence it is represented by a hyperbola.
4. Net reactance (X): is given by $X = X_L - X_C$. It is represented by a hyperbola that crosses the x-axis at $f = f_r$.
5. Impedance (Z): is given by $Z = \sqrt{R^2 + (X_L - X_C)^2}$

- If $f < f_r$, then $X_C > X_L$ and the circuit will have capacitive c/cs (I leads V by θ).
- If $f = f_r$, then $X_L = X_C$ and the circuit will have pure resistive c/cs (I and V are in phase).
- If $f > f_r$, then $X_L > X_C$ and the circuit will have inductive c/cs (I lags V by θ).



ex. Calculate the resonant freq. for the circuit shown.

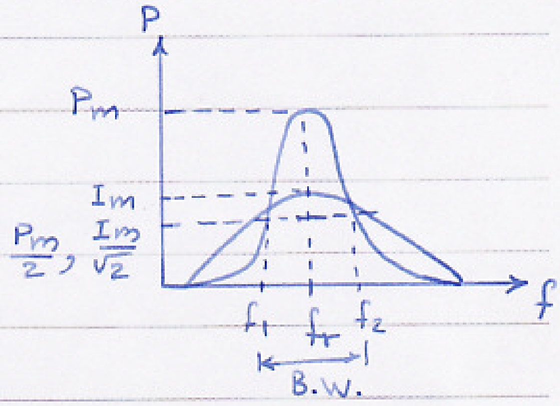


sol. $f_r = \frac{1}{2\pi\sqrt{LC}}$

$f_r = \frac{1}{2\pi\sqrt{(33*10^{-9})(2.7*10^{-12})}} = 533\,MHz$

Bandwidth of series resonance:

At resonance, I is maximum and Z is minimum. Hence, $P = I^2 Z = I_m^2 R$ is also maximum as shown in figure. It can be observed that at two different frequencies f_1 and f_2 the power is half of its maximum value. These frequencies are called "Half-power frequencies" or "cutoff frequencies".



The difference between f_1 and f_2 is called "Bandwidth".

$\therefore$ B.W. $= \Delta f = f_2 - f_1$ (Hz)

The frequency f_1 is called "lower cut-off frequency" and f_2 is called "upper cut-off frequency".

Selectivity:

It is defined as the ratio of the resonant frequency to the bandwidth.

$$\text{Selectivity} = \frac{f_r}{B.W.} = \frac{f_r}{f_2 - f_1}$$

Expressions for lower and upper cut-off frequencies:

For a series RLC circuit:

$$I = \frac{V}{Z} \quad ; \quad Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

$$\therefore I = \frac{V}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$|I| = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \quad \text{——— (1)}$$

At resonance : $I = I_m = \frac{V}{R}$

and $P_m = I_m^2 R$

At half-power points: $P = \frac{P_m}{2} = \frac{I_m^2 R}{2} = \left(\frac{I_m}{\sqrt{2}}\right)^2 R$

$\therefore\ I = \frac{I_m}{\sqrt{2}} = \frac{V/R}{\sqrt{2}}$ —— ② at half-power frequencies

Equating equations ① and ② yields:

$$\frac{V/R}{\sqrt{2}} = \frac{V}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} \Rightarrow \sqrt{2} R = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}$$

$$2R^2 = R^2 + (\omega L - \frac{1}{\omega C})^2 \Rightarrow R^2 = (\omega L - \frac{1}{\omega C})^2$$

$$\mp R = \omega L - \frac{1}{\omega C}$$

Solving for ω, then:

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \quad ; \quad \omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$B.W. = f_2 - f_1 = \frac{\omega_2 - \omega_1}{2\pi}$$

$$\omega_2 - \omega_1 = \frac{R}{L} \Rightarrow B.W. = \frac{R}{2\pi L} \quad Hz$$

also $\omega_1 \omega_2 = \frac{1}{LC}$

but $\omega_r = \frac{1}{\sqrt{LC}} \Rightarrow \therefore\ \omega_r^2 = \omega_1 \omega_2$ or $f_r = \sqrt{f_1 f_2}$

Quality Factor (Q):

It is the ratio of the resonant frequency to the band width.

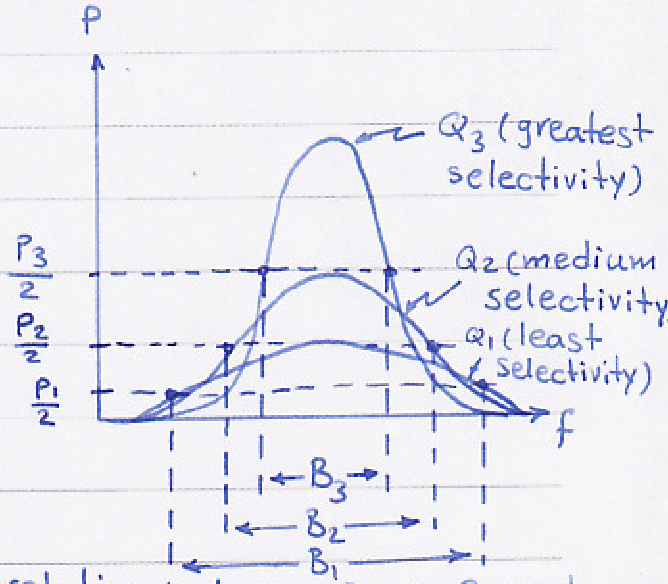
$$Q = \frac{f_r}{B.W.} = \frac{\frac{1}{2\pi\sqrt{LC}}}{\frac{R}{2\pi L}} = \frac{1}{R}\sqrt{\frac{L}{C}} \quad \text{(unitless)} \quad ; \quad Q \geqslant 1$$

Another definition for Q is the ratio of the reactive power to the average power at resonance.

$$Q = \frac{\text{reactive power}}{\text{average power}} = \frac{\text{energy stored}}{\text{energy lost}} = \frac{\text{Power of L or C}}{\text{power in R}}$$

$$= \frac{I^2 X_L}{I^2 R} = \frac{X_L}{R} = \frac{2\pi f_r L}{R} = \frac{2\pi \frac{1}{2\pi\sqrt{LC}} L}{R}$$

$$\therefore\ Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$



relation between B.W., Q and selectivity

The significance of quality factor is that it gives an indication of the selectivity of series RLC circuits which are used in radio circuits.
As illustrated in the fig, the higher the value of Q, the more selective the circuit is but the smaller the bandwidth.
High quality factor circuits are used in communication networks.

ex. A series RLC circuit with $R = 10\Omega$, $L = 0.2H$ and $C = 40\mu F$ is supplied with a 100V supply at variable frequency.
Find: a) resonant frequency ; b) current at resonance c) power at resonance ; d) voltage across R, L and C at resonance ; e) quality factor ; f) half-power frequencies g) bandwidth ; h) phasor diagram.

Sol.

a) $f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(0.2)(40*10^{-6})}} = 56.269 \text{ Hz}$

b) I (at resonance) $= I_m = \frac{V}{R} = \frac{100}{10} = 10 \text{ A}$

c) P at resonance $= P_m = I_m^2 R = (10)^2 10 = 1000 \text{ W}$

d) $V_R = I_m R = 10*10 = 100V$

$V_L = I_m X_L = I_m(2\pi f_r L) = 10(2\pi*56.269*0.2) = 707.106V$

$V_C = I_m X_C = I_m\left(\frac{1}{2\pi f_r C}\right) = \frac{10}{2\pi(56.269)(40*10^{-6})} = 707.106V$

$\therefore V_L = V_C$ at resonance

ex. continued:

e) $Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{10}\sqrt{\frac{0.2}{40*10^{-6}}} = 7.071$

f) $\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} = -\frac{10}{0.4} + \sqrt{\left(\frac{10}{0.4}\right)^2 + \frac{1}{(0.2)(40*10^{-6})}}$

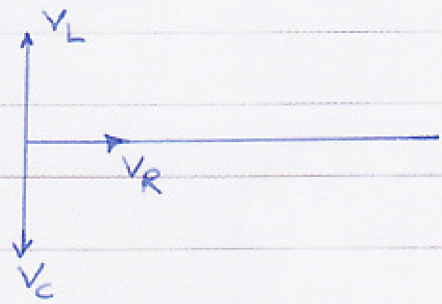
$= 329.43$ rad / sec.

$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} = \frac{10}{0.4} + \sqrt{\left(\frac{10}{0.4}\right)^2 + \frac{1}{(0.2)(40*10^{-6})}} = 379.43 \ \frac{rad}{sec}$

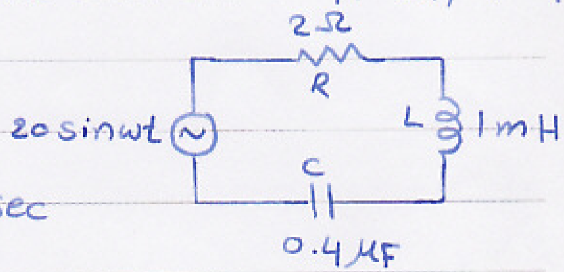
$\therefore f_1 = \frac{\omega_1}{2\pi} = 52.431$ Hz ; $f_2 = \frac{\omega_2}{2\pi} = 60.38$ Hz

g) B.W. $= \Delta f = f_2 - f_1 = 7.95$ Hz

h)



ex. For the circuit shown, Find: a) f_r, f_1 and f_2
b) Q and B.W., c) amplitude of the current at f_r, f_1 and f_2.



sol. a) $f_r = \frac{1}{2\pi\sqrt{LC}} = 7.95$ KHz

$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} = 49$ K rad/sec

$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} = 51$ K rad/sec

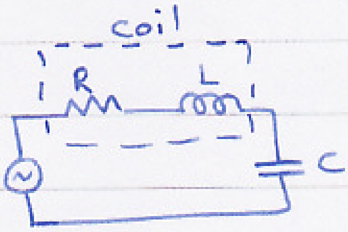
$\therefore f_1 = 7.8$ KHz ; $f_2 = 8.11$ KHz

b) $Q = \frac{1}{R}\sqrt{\frac{L}{C}} = 25$

B.W. $= f_2 - f_1 = 318.49$ Hz

c) At resonance: $I = I_m = \frac{V}{R} = \frac{20}{2} = 10$ A

At half-power points : $I = \frac{I_m}{\sqrt{2}} = \frac{10}{\sqrt{2}} = 7.071$ A ex. A coil having a resistance of 25Ω and a quality factor of 10 at a resonant freq. of 10 KHz is fed with a 100V supply. Calculate: a) Value of series capacitance required to produce resonance with the coil ; b) the inductance of the coil ; c) voltage across the capacitor ; d) voltage across the coil.



sol.

a) $Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{R}\sqrt{\frac{LC}{C^2}} = \frac{1}{R}\frac{\sqrt{LC}}{C}$

$= \frac{1}{R} \cdot \frac{1}{C} \cdot \frac{1}{2\pi f_r}$

$\therefore C = \frac{1}{Q} \cdot \frac{1}{R} \cdot \frac{1}{2\pi f_r} = \frac{1}{10} \cdot \frac{1}{25} \cdot \frac{1}{2\pi(10*10^3)} = 63.66 \text{ nF}$

b) $f_r = \frac{1}{2\pi\sqrt{LC}} \Rightarrow L = \frac{1}{(2\pi f_r)^2 C} = \frac{1}{(2\pi*10*10^3)^2(63.66*10^{-9})} = 3.97 \text{ mH}$

c) at resonance:

$V_C = I_m X_C = \frac{V}{R} \cdot \frac{1}{2\pi f_r C} = \frac{100}{25}\left(\frac{1}{2\pi(10*10^3)(63.66*10^{-9})}\right) = 1000V$

d) $V_{coil} = V_R + V_L$

$V_R = I_m R = 100V$

$V_L = I_m X_L = 100(2\pi(10*10^3)(3.97*10^{-3})) = 1000V$

$\therefore V_{coil} = 100 + 1000 = 1100V$

Parallel Resonance:

Similar to series ac circuits, there can be a resonance in parallel ac circuits. When the voltage and total current are in phase at a particular frequency (f_r) then the circuit is said to be at resonance.

Consider the circuit shown in fig.:

$$Y = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

From the above equation, the impedance $\left(\frac{1}{Y}\right)$ is maximum at resonance. Hence the current drawn is minimum, so the parallel circuit at resonance is called "rejector circuit". This indicates that it rejects the unwanted frequencies and hence it is used as filter in radio receivers.

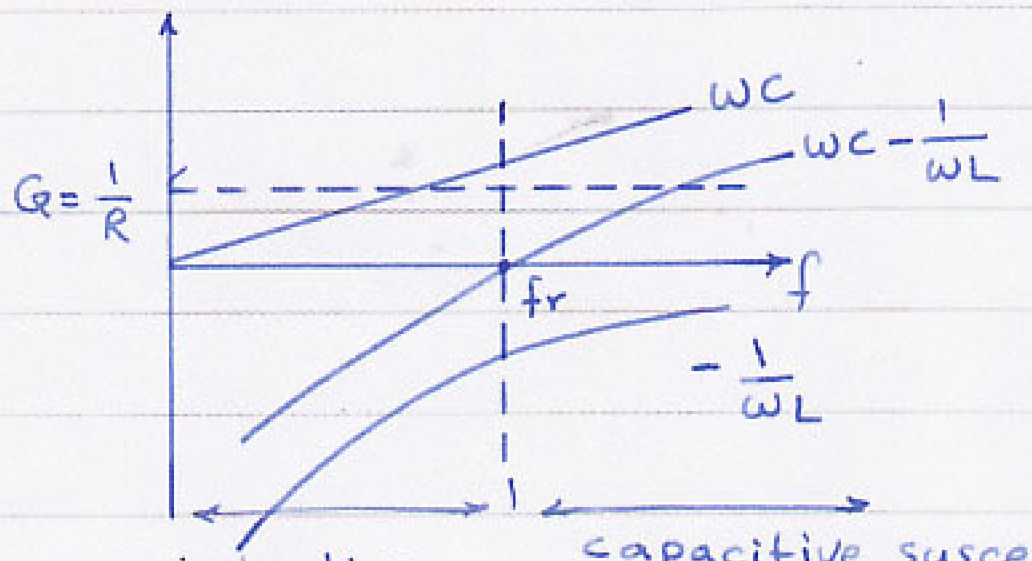
The resonance occurs when imaginary part $= 0$;

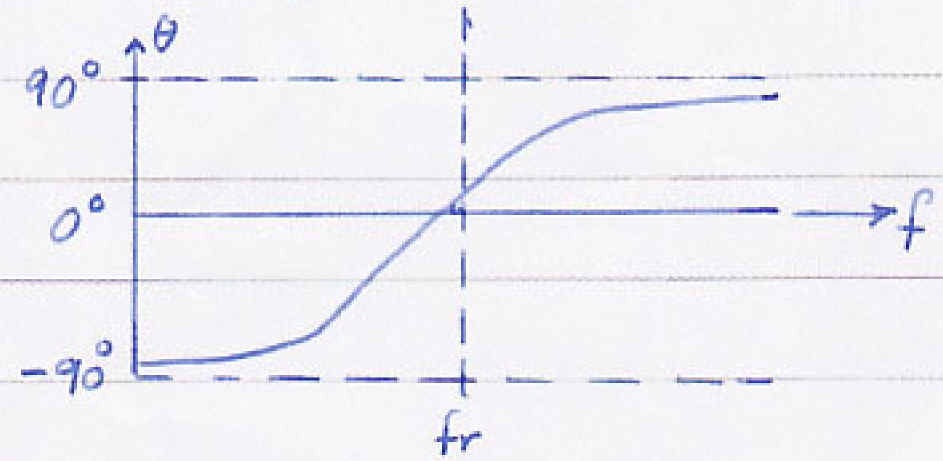
i.e. $\omega_r C - \frac{1}{\omega_r L} = 0 \Rightarrow \omega_r C = \frac{1}{\omega_r L}$

$$2\pi f_r C = \frac{1}{2\pi f_r L} \Rightarrow f_r = \frac{1}{2\pi\sqrt{LC}} \quad Hz$$

and $Y = \frac{1}{R}$

Characteristics of parallel resonance:





Bandwidth, cut-off frequencies and quality factor:

For a parallel resonance circuit:

$$\frac{1}{R^2} = \left(\omega C - \frac{1}{\omega L}\right)^2 \Rightarrow \pm\frac{1}{R} = \omega C - \frac{1}{\omega L}$$

Solving for ω, then:

$$\omega_1 = \frac{-1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \quad ; \omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$B.W. = \omega_2 - \omega_1 = \frac{1}{RC} \quad rad/sec.$$

$$\text{or } B.W. = \frac{1}{2\pi RC} \quad Hz$$

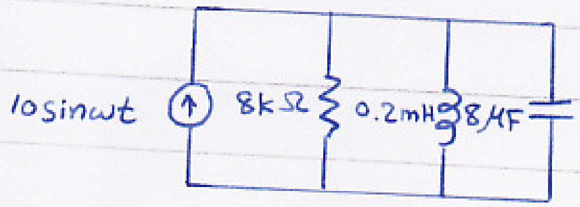
$$\omega_1 \omega_2 = \frac{1}{LC} \quad ; \text{ but } \omega_r = \frac{1}{\sqrt{LC}} \Rightarrow \omega_1 \omega_2 = \omega_r^2 \Rightarrow f_r = \sqrt{f_1 f_2}$$

$$Q = \frac{f_r}{B.W.} = \frac{\frac{1}{2\pi\sqrt{LC}}}{\frac{1}{2\pi RC}} = R\sqrt{\frac{C}{L}}$$

ex. For the parallel RLC circuit shown, calculate:

a) ω_r, Q and B.W. ; b) ω_1 and ω_2

sol.



a) $\omega_r = \dfrac{1}{\sqrt{LC}} = \dfrac{1}{\sqrt{(0.2*10^{-3})(8*10^{-6})}}$

$= 25$ K rad/sec

$Q = R\sqrt{\dfrac{C}{L}} = 8000\sqrt{\dfrac{8*10^{-6}}{0.2*10^{-3}}} = 1600$

$B.W. = \dfrac{1}{RC} = \dfrac{1}{(8000)(8*10^{-6})} = 15.625$ rad/sec

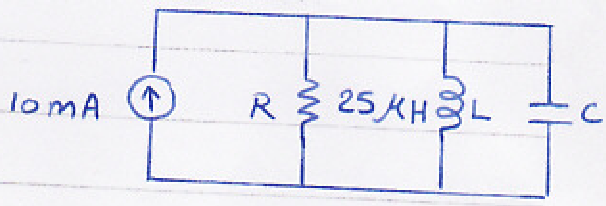
b) $\omega_1 = \dfrac{-1}{2RC} + \sqrt{\left(\dfrac{1}{2RC}\right)^2 + \dfrac{1}{LC}} = 24.992$ K rad/sec

$\omega_2 = \dfrac{1}{2RC} + \sqrt{\left(\dfrac{1}{2RC}\right)^2 + \dfrac{1}{LC}} = 25.007$ K rad/sec

ex. For the cct. shown, if $f_r = 2$ MHz and $Q = 10$,

calculate: C, f_1, f_2, B.W., V_r, I_L and I_C at resonance.

sol.



$f_r = \dfrac{1}{2\pi\sqrt{LC}} \Rightarrow 2*10^{6} = \dfrac{1}{2\pi\sqrt{(25*10^{-6})C}}$

$\therefore C = \dfrac{1}{(2\pi f_r)^2 L} = 0.253$ nF

$Q = R\sqrt{\dfrac{C}{L}} \Rightarrow R = Q\sqrt{\dfrac{L}{C}} = 10\sqrt{\dfrac{25*10^{-6}}{0.253*10^{-9}}} = 3.141$ kΩ

$f_1 = \dfrac{1}{2\pi}\left(\dfrac{-1}{2RC} + \sqrt{\left(\dfrac{1}{2RC}\right)^2 + \dfrac{1}{LC}}\right) = 1.903$ MHz

$f_2 = \dfrac{1}{2\pi}\left(\dfrac{1}{2RC} + \sqrt{\left(\dfrac{1}{2RC}\right)^2 + \dfrac{1}{LC}}\right) = 2.103$ MHz

$B.W. = f_2 - f_1 = 0.2$ MHz

ex. continued:

$$V \text{ at resonance} = V_r = I * \frac{1}{Y} = I * Z = IR$$

$$= (10*10^{-3})(3141) = 31.415\,V$$

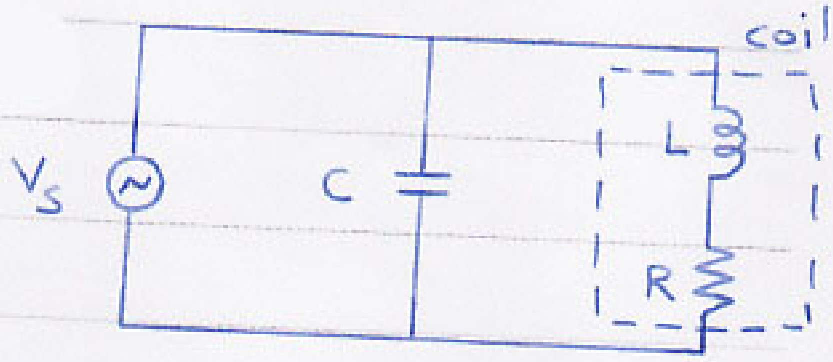
$$I_C = \frac{V_r}{X_C} = V_r(2\pi f_r C) = (31.415)(2\pi * 2*10^6 * 0.253*10^{-9})$$
$$= 0.1\,A$$

$$I_L = \frac{V_r}{X_L} = \frac{V_r}{2\pi f_r L} = \frac{31.415}{2\pi(2*10^6)(25*10^{-6})} = 0.1\,A$$

$\therefore I_C = I_L$ at resonance

Practical Resonance Circuits:

In a practical resonant circuit, it is assumed that the capacitor has zero losses and the inductor contains a resistance, as shown in fig.



$$Y = \frac{R}{R^2+\omega^2L^2} + j\left(\omega C - \frac{\omega L}{R^2+\omega^2L^2}\right)$$

at resonance, imaginary part = 0

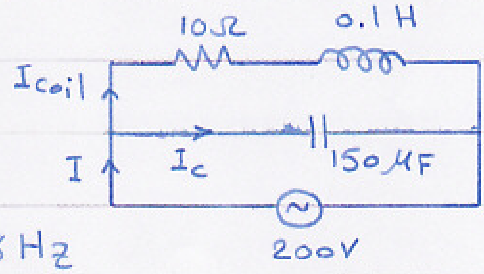
$$\therefore \omega_r C = \frac{\omega_r L}{R^2+\omega_r^2L^2} \Rightarrow C = \frac{L}{R^2+\omega_r^2L^2} \Rightarrow \omega_r^2 = \frac{1}{LC} - \frac{R^2}{L^2}$$

$$\therefore \omega_r = 2\pi f_r = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

$$B.W. = \frac{R}{L}$$

$$Q = \frac{f_r}{B.W.}$$

Ex. A coil of resistance 10 Ω and inductance 0.1 H is connected in parallel with a 150 μF capacitor to 200 V variable freq. supply. Find the resonant freq., total current, branch currents at resonance.



Sol.

$$f_r = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{1}{(0.1)(150*10^{-6})} - \frac{100}{0.01}} = 37.88\ Hz$$

$$Z_{coil} = R + j\omega L = 10 + j(2\pi * 37.88 * 0.1) = 10 + j\,23.8 = 25.82\angle 67.21^\circ\ \Omega$$

$$I_{coil} = \frac{V}{Z_{coil}} = \frac{200\angle 0^\circ}{25.82\angle 67.21^\circ} = 7.745\angle -67.21^\circ\ A$$

$$X_C = \frac{1}{2\pi f C} = \frac{1}{2\pi * 37.88 * 150 * 10^{-6}} = 28\angle -90^\circ\ \Omega$$

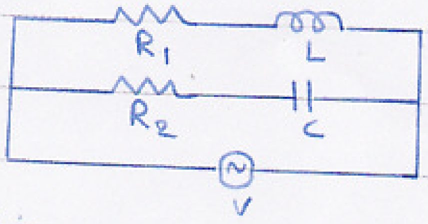
$$I_C = \frac{V}{X_C} = \frac{200\angle 0^\circ}{28\angle -90^\circ} = 7.141\angle 90^\circ\ A$$

$$Z_T = Z_{coil} // X_C = \frac{Z_{coil}\,X_C}{Z_{coil} + X_C} = \frac{(25.82\angle 67.21^\circ)(28\angle -90^\circ)}{(10 + j\,23.8) + (0 - j28)} = \frac{722.96\angle -22.79^\circ}{10.844\angle -22.79^\circ}$$

$$= 66.67\ \Omega \quad \text{pure resistive at resonance}$$

$$\therefore I = \frac{V}{Z_T} = \frac{200}{66.67} = 3\ A$$

ex. Derive an expression for the resonant freq. for the cct shown.



sol. $$Y = \frac{1}{R_1 + j\omega L} + \frac{1}{R_2 - \frac{j}{\omega C}}$$

$$= \frac{1}{R_1 + j\omega L} \cdot \frac{R_1 - j\omega L}{R_1 - j\omega L} + \frac{1}{R_2 - \frac{j}{\omega C}} \cdot \frac{R_2 + \frac{j}{\omega C}}{R_2 + \frac{j}{\omega C}}$$

$$= \frac{R_1 - j\omega L}{R_1^2 + \omega^2 L^2} + \frac{R_2 + \frac{j}{\omega C}}{R_2^2 + \frac{1}{\omega^2 C^2}}$$

$$= \frac{R_1}{R_1^2 + \omega^2 L^2} + \frac{R_2}{R_2^2 + \frac{1}{\omega^2 C^2}} + j\left(\frac{\frac{1}{\omega C}}{R_2^2 + \frac{1}{\omega^2 C^2}} - \frac{\omega L}{R_1^2 + \omega^2 L^2}\right)$$

At resonance, imaginary part = 0 :

$$\frac{\frac{1}{\omega_r C}}{R_2^2 + \frac{1}{\omega_r^2 C^2}} - \frac{\omega_r L}{R_1^2 + \omega_r^2 L^2} = 0 \Rightarrow \frac{\frac{1}{\omega_r C}}{R_2^2 + \frac{1}{\omega_r^2 C^2}} = \frac{\omega_r L}{R_1^2 + \omega_r^2 L^2}$$

$$\frac{\omega_r C}{\omega_r^2 R_2^2 C^2 + 1} = \frac{\omega_r L}{R_1^2 + \omega_r^2 L^2} \Rightarrow C(R_1^2 + \omega_r^2 L^2) = L(\omega_r^2 R_2^2 C^2 + 1)$$

$$R_1^2 C + \omega_r^2 L^2 C = \omega_r^2 R_2^2 C^2 L + L \Rightarrow \omega_r^2 (L^2 C - C^2 R_2^2 L) = L - R_1^2 C$$

$$\therefore \omega_r = \sqrt{\frac{L - R_1^2 C}{L^2 C - C^2 R_2^2 L}} = \sqrt{\frac{L - CR_1^2}{LC(L - CR_2^2)}} = \frac{1}{\sqrt{LC}} \sqrt{\frac{L - CR_1^2}{L - CR_2^2}} \ \frac{\text{rad}}{\text{sec.}}$$

Filters (Frequency Selective Networks):

A filter is a network designed to pass signals having certain frequencies (called passbands) with a little or no attenuation, but greatly attenuates signals within other frequencies (called stopbands).

Filters can be classified into passive filters and active filters.

i/p → Filter → o/p

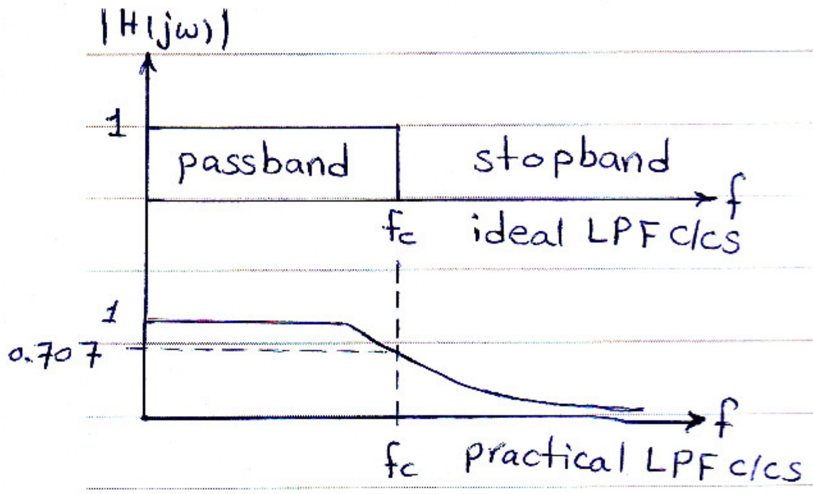
An active filter consists of active elements such as transistors, op-amps as well as resistors, capacitors and inductors. The main advantage (along with filtering) is the gain of the o/p signal (voltage, current and power gain) but its disadvantage is that they require power supplies to operate. In passive filters, only passive elements (resistors, capacitors and inductors) are used. They do not require power supplies to operate but it may become costly if inductors are used.

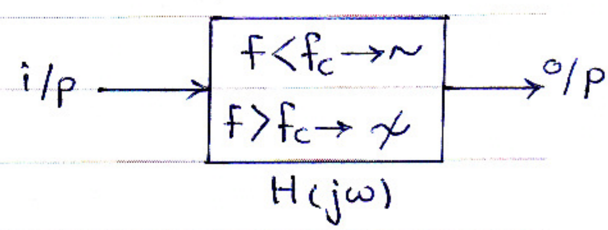
Definitions:

1. Passband: is the range of frequencies over which attenuation by filter is zero.
2. Stopband (attenuation band): is the range of frequencies over which attenuation is infinite.
3. Attenuation: is the reduction of the o/p signal compared to the i/p signal.
4. Cutoff Frequency (f_c): is the frequency which separate the passband from stopband.

Basic Filter Networks:

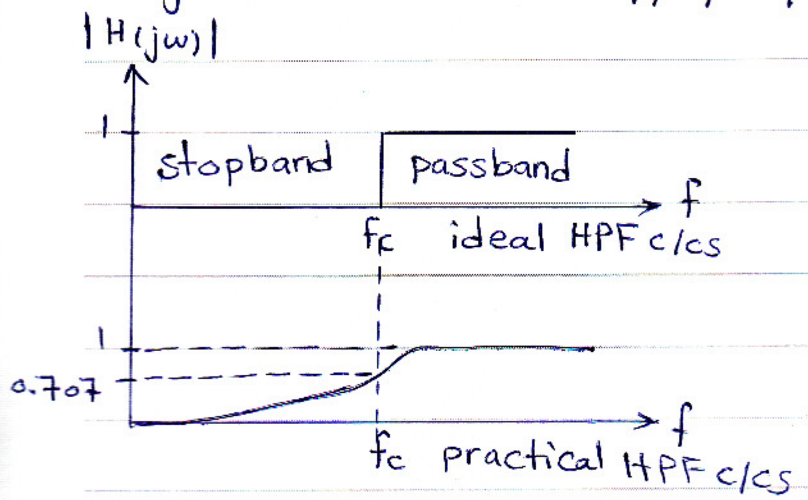
1. Low pass filter (LPF): passes signal at frequencies lower than the cutoff frequency from the i/p to the o/p.

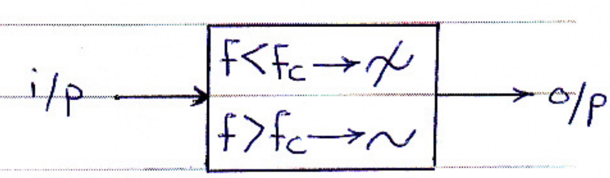




An ideal filter would have zero attenuation in the passband and infinite attenuation in the stopband. Practically, the attenuation cannot change from zero to infinite instantaneously but it gradually changes.

2. High pass filter (HPF): passes signals at frequencies higher than the cutoff frequency from the i/p to the o/p.

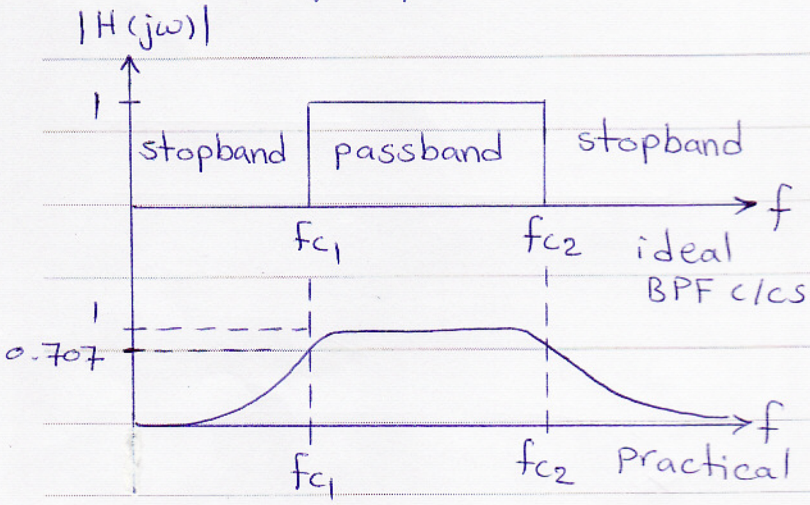




Any simple type of filter has only one passband, one stopband and a single cutoff frequency.

One can design a filter with two cutoff frequencies to get two filter c/cs, as the following types:

3. Band pass filter (BPF): passes all the frequencies between the two cutoff frequencies and attenuates all other frequencies.

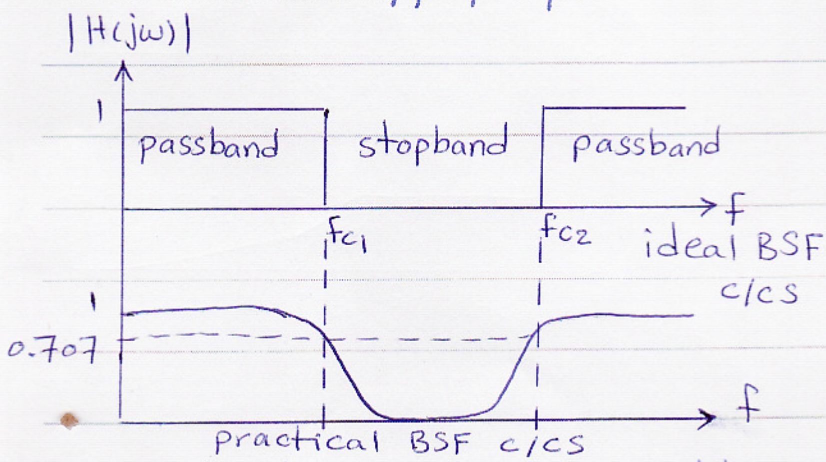


i/p $\longrightarrow$ $f<f_{c_1} \rightarrow \nsim$, $f_{c_1}<f<f_{c_2} \sim$, $f>f_{c2} \rightarrow \nsim$ $\longrightarrow$ o/p

Such a filter may be formed by cascading a HPF with cutoff frequency f_{cH} and a LPF with cutoff frequency f_{cL} provided that $f_{cL} > f_{cH}$

i/p $\longrightarrow$ HPF @ f_{cH} $\longrightarrow$ LPF @ f_{cL} $\longrightarrow$ o/p

4. Band stop filter (BSF): also called "Band reject filter" It attenuates (or rejects) all the frequencies between the two cutoff frequencies and passes all other frequencies.



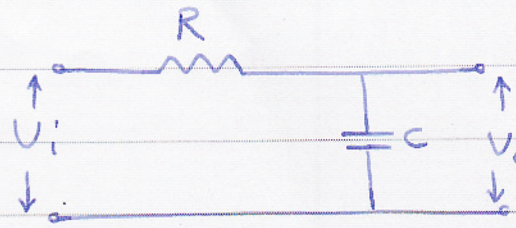
i/p $\longrightarrow$ $f<f_{c1} \rightarrow \sim$, $f_{c_1}<f<f_{c_2} \rightarrow \nsim$, $f>f_{c2} \rightarrow \sim$ $\longrightarrow$ o/p

Such a filter may be formed by cascading a LPF with $f_{c_1} = f_{cL}$ and a HPF with $f_{c2} = f_{cH}$ provided that $f_{cH} > f_{cL}$.

i/p $\longrightarrow$ LPF @ f_{cL} $\longrightarrow$ HPF @ f_{cH} $\longrightarrow$ o/p

Low Pass Filter:

1. The RC LPF:



At low frequencies, the capacitor has large reactance, resulting open circuit across o/p terminals and V_o equals V_i.

At high frequencies, the capacitor has small reactance, resulting short circuit across o/p terminals and V_o approaches to zero.

$$H(\omega) = \frac{V_o}{V_i} = \frac{X_c}{R + X_c} \quad ; \quad X_c = \frac{1}{j\omega C}$$

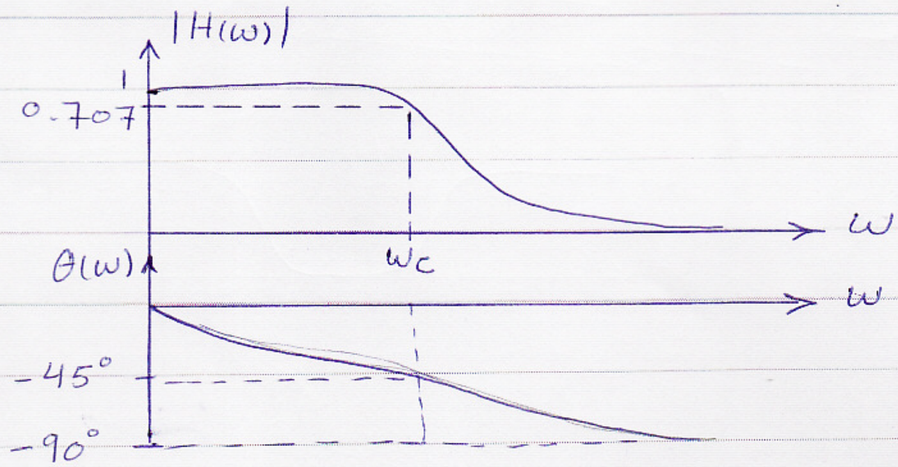
$$\therefore \; H(\omega) = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC}$$

$$|H(\omega)| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} \quad ; \quad \theta(\omega) = -\tan^{-1}(\omega RC)$$

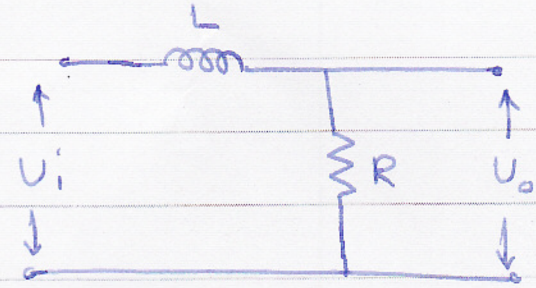
The cutoff frequency occurs when the o/p voltage has an amplitude which is 0.707 (or $\frac{1}{\sqrt{2}}$) of the i/p voltage.

$\therefore$ At cutoff : $|H(\omega)| = \frac{1}{\sqrt{2}}$

$$\frac{1}{\sqrt{1 + \omega_c^2 R^2 C^2}} = \frac{1}{\sqrt{2}} \Rightarrow \boxed{\omega_c = \frac{1}{RC}}$$



2. The RL LPF :



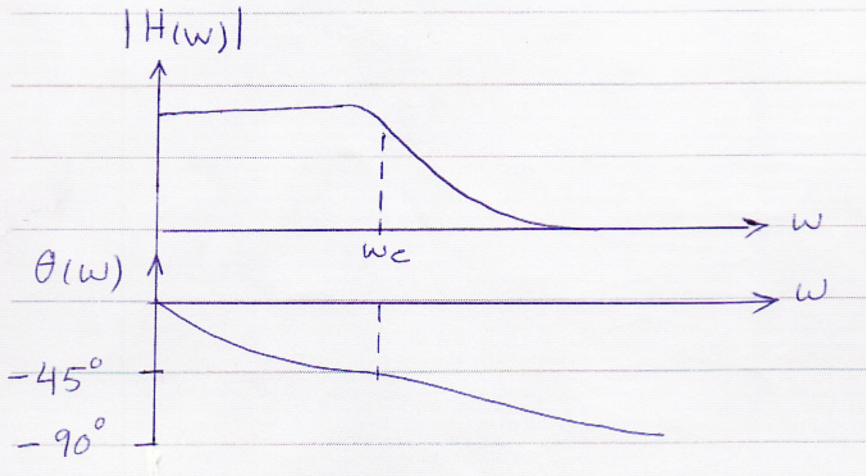
At low frequencies, the inductor has small reactance, resulting short circuit and $V_o = V_i$.
As the frequency increases, the reactance increases too until it acts as an open circuit and $V_o = 0$.

$$H(\omega) = \frac{V_o}{V_i} = \frac{R}{R + X_L} = \frac{R}{R + j\omega L} = \frac{1}{1 + j\omega \frac{L}{R}}$$

$$|H(\omega)| = \frac{1}{\sqrt{1 + \omega^2 \frac{L^2}{R^2}}} \quad ; \quad \theta(\omega) = -\tan^{-1} \frac{\omega L}{R}$$

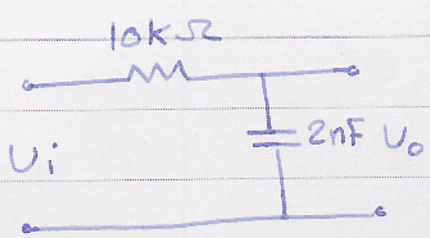
At cutoff : $|H(\omega)| = \frac{1}{\sqrt{2}} \Rightarrow \frac{1}{\sqrt{1 + \omega^2 \frac{L^2}{R^2}}} = \frac{1}{\sqrt{2}}$

$$\therefore \boxed{\omega_c = \frac{R}{L}}$$



Ex. Calculate the cut off frequency for the circuit shown.

Sol. This is an RC LPF circuit



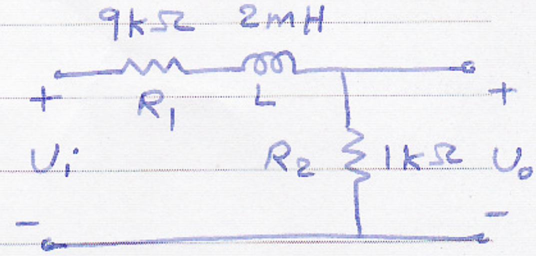
$$f_c = \frac{1}{2\pi RC} = \frac{1}{2\pi (10 * 10^3)(2 * 10^{-9})}$$

$$\therefore f_c = 7.95 \text{ KHz}$$

Ex. Consider the LPF shown:

a) Derive the transfer function $H(\omega)$

b) Calculate the cutoff frequency.



Sol. a) $V_o = V_i \dfrac{R_2}{(R_1 + X_L) + R_2}$

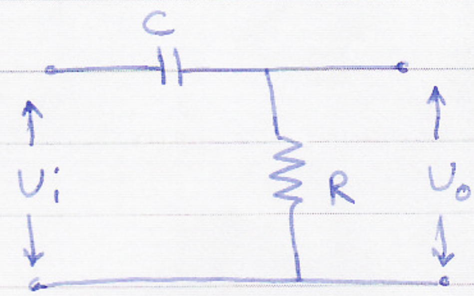
$$\therefore H(\omega) = \frac{V_o}{V_i} = \frac{R_2}{R_1 + R_2 + X_L} = \frac{R_2}{R_1 + R_2 + j\omega L} = \frac{R_2}{R_1 + R_2}\left(\frac{1}{1 + j\omega \frac{L}{R_1 + R_2}}\right)$$

b) At cutoff: $|H(\omega)| = \dfrac{1}{\sqrt{2}}$

$$\therefore 1 + \omega_c^2 \frac{L^2}{(R_1 + R_2)^2} = 2 \Rightarrow \omega_c = \frac{R_1 + R_2}{L} = \frac{10\,000}{2 * 10^{-3}} = 5\ M\ rad/sec$$

High Pass Filter:

1. The RC HPF:

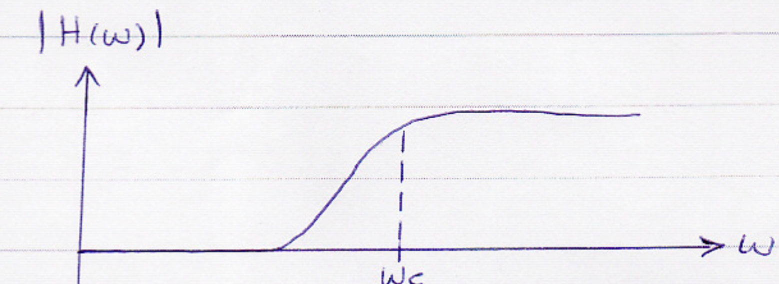


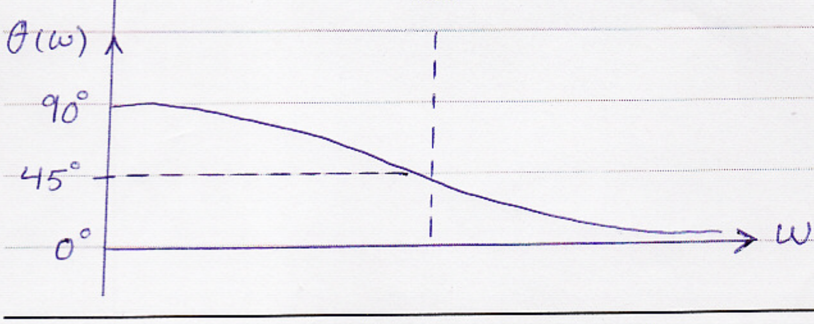
$$\frac{V_o}{V_i} = \frac{R}{R + X_C} = \frac{R}{R + 1/j\omega C}$$

$$\therefore H(\omega) = \frac{j\omega RC}{1 + j\omega RC}$$

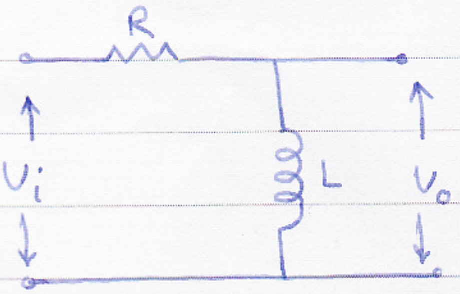
$$|H(\omega)| = \frac{\omega RC}{\sqrt{1 + \omega^2 R^2 C^2}} \quad ; \quad \theta(\omega) = 90 - \tan^{-1} \omega RC$$

At cutoff: $|H(\omega)| = \dfrac{1}{\sqrt{2}} \Rightarrow 1 + \omega_c^2 R^2 C^2 = 2 \Rightarrow \boxed{\omega_c = \dfrac{1}{RC}}$





2. The RL HPF:



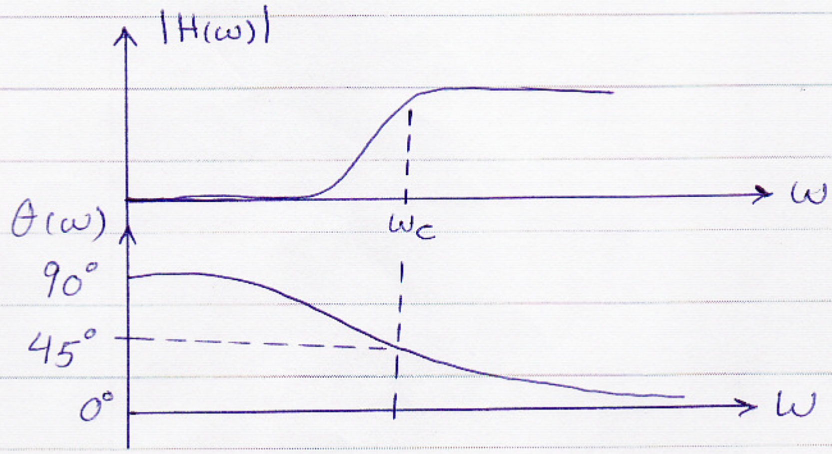
$$\frac{V_o}{V_i} = \frac{X_L}{R+X_L} = \frac{j\omega L}{R+j\omega L}$$

$$H(\omega) = \frac{j\omega \frac{L}{R}}{1+j\omega \frac{L}{R}}$$

$$|H(\omega)| = \frac{\omega \frac{L}{R}}{\sqrt{1+\omega^2 \frac{L^2}{R^2}}} \quad ; \quad \theta(\omega) = 90 - \tan^{-1} \omega \frac{L}{R}$$

At cutoff: $|H(\omega)| = \frac{1}{\sqrt{2}} \Rightarrow 1 + \omega_c^2 \frac{L^2}{R^2} = 2$

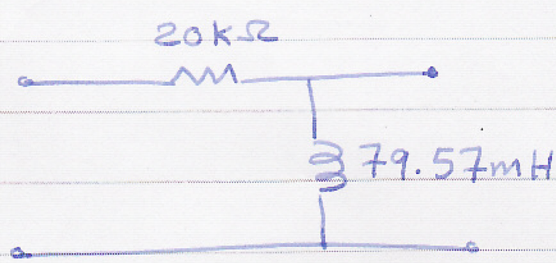
$$\therefore \boxed{\omega_c = \frac{R}{L}}$$



EX. Design an RL HPF circuit with cutoff frequency of 40 KHz.

Sol. $f_c = \frac{1}{2\pi} \frac{R}{L} \Rightarrow 40*10^3 = \frac{1}{2\pi} \frac{R}{L}$

Let $R = 20\,k\Omega \Rightarrow L = \frac{1}{2\pi} \frac{20*10^3}{40*10^3} = 79.57\,mH$

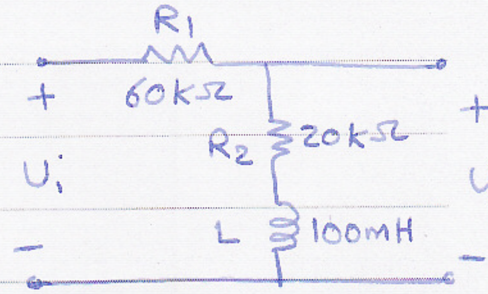


RL HPF

Ex. Consider the HPF circuit shown:

a) Derive the transfer function $H(\omega)$.

b) Calculate ω_c.



Sol. a) $V_o = V_i * \dfrac{R_2 + X_L}{R_1 + (R_2 + X_L)}$

$$\therefore H(\omega) = \frac{V_o}{V_i} = \frac{R_2 + j\omega L}{R_1 + R_2 + j\omega L} = \frac{R_2 + j\omega L}{R_1 + R_2} \left(\frac{1}{1 + j\omega \frac{L}{R_1 + R_2}} \right)$$

b)

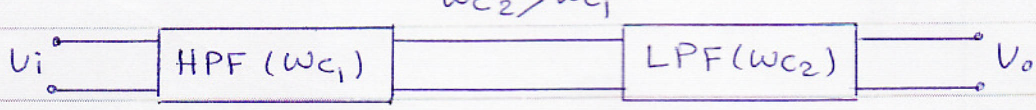
At cutoff: $|H(\omega)| = \dfrac{1}{\sqrt{2}} \Rightarrow \dfrac{1}{\sqrt{1 + \omega_c^2 \frac{L^2}{(R_1+R_2)^2}}} = \dfrac{1}{\sqrt{2}}$

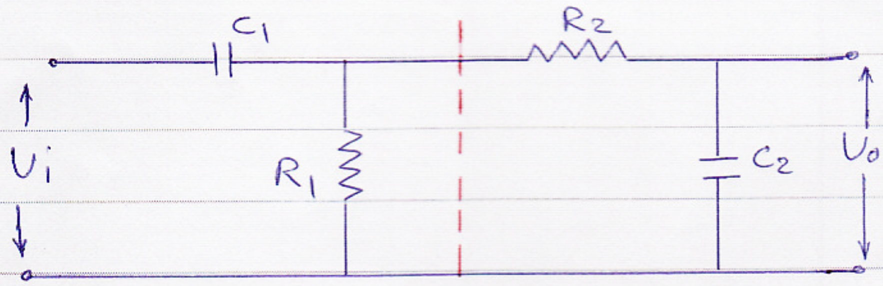
$$\omega_c^2 \frac{L^2}{(R_1+R_2)^2} = 1 \Rightarrow \omega_c = \frac{R_1 + R_2}{L} = \frac{60000 + 20000}{100 * 10^{-3}} = 800 \text{ k rad/sec}$$

Band Pass Filter:

The BPF is constructed by using a HPF cascaded with a LPF as shown.

$\omega_{c_2} > \omega_{c_1}$





RC HPF | RC LPF

- For the HPF section: $\omega_{c_1} = \dfrac{1}{R_1 C_1}$
- For the LPF section: $\omega_{c_2} = \dfrac{1}{R_2 C_2}$

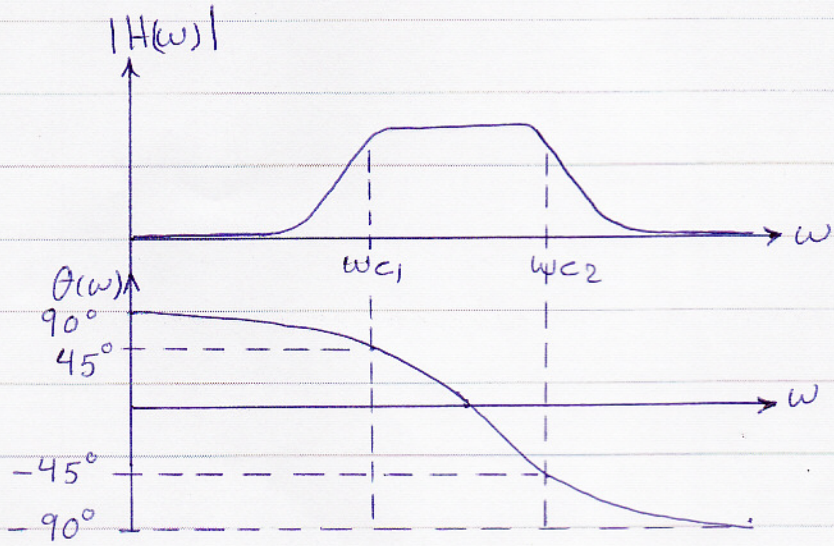
$$B.W. = \omega_{c_2} - \omega_{c_1}$$

$$H(\omega) = H_1(\omega)\, H_2(\omega)$$

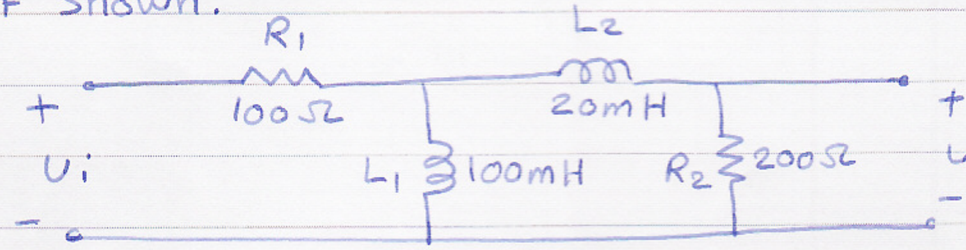
where: $H_1(\omega) = \dfrac{j\omega R_1 C_1}{1 + j\omega R_1 C_1}$ Transfer function of HPF

$H_2(\omega) = \dfrac{1}{1 + j\omega R_2 C_2}$ Transfer function of LPF

$$\therefore H(\omega) = \frac{j\omega R_1 C_1}{1 + j\omega R_1 C_1} \cdot \frac{1}{1 + j\omega R_2 C_2}$$



Ex. Calculate the cutoff frequency and B.W. of the BPF shown.



sol.

$$\omega_{c1} = \frac{R_1}{L_1} = 1\text{K rad/sec}$$

$$\omega_{c2} = \frac{R_2}{L_2} = 10\text{ K rad/sec}$$

$$B.W. = \omega_{c2} - \omega_{c1} = 10\text{ K} - 1\text{K} = 9\text{k rad/sec}$$

Ex. Given a 0.1 μF and 0.04 μF capacitors, design a BPF with Bandwidth of 30 Krad/sec and lower cutoff frequency of 5 Krad/sec.

Sol. Let's design an RC BPF ← لا تحتاج لذكره لان المعطيات هي مكثفات لذا فأكيد RC

$$\omega_{c1} = \frac{1}{R_1 C_1} \quad ; \quad \omega_{c2} = \frac{1}{R_2 C_2}$$

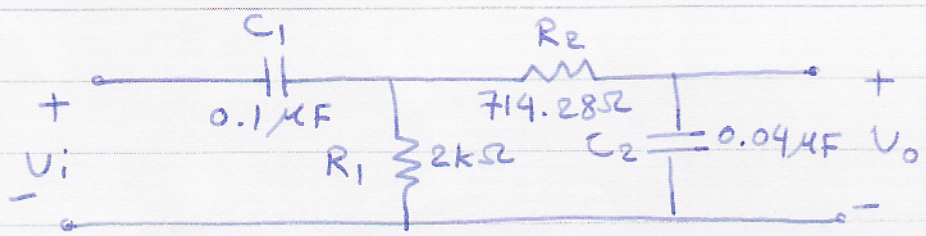
$$\because \omega_{c2} > \omega_{c1} \Rightarrow C_2 < C_1$$

$$\therefore C_2 = 0.04\ \mu F \quad \text{and} \quad C_1 = 0.1\ \mu F$$

$$5000 = \frac{1}{R_1 (0.1 * 10^{-6})} \Rightarrow R_1 = 2\,k\Omega$$

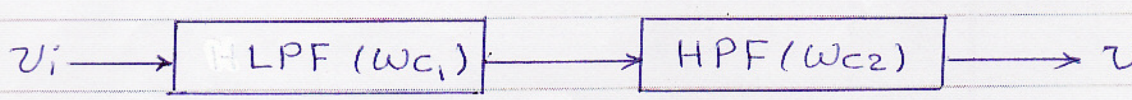
$$B.W. = \omega_{c2} - \omega_{c1} \Rightarrow 30\,000 = \omega_{c2} - 5000 \Rightarrow \omega_{c2} = 35\ \text{Krad/sec}$$

$$35000 = \frac{1}{R_2 (0.04 * 10^{-6})} \Rightarrow R_2 = 714.28\,\Omega$$

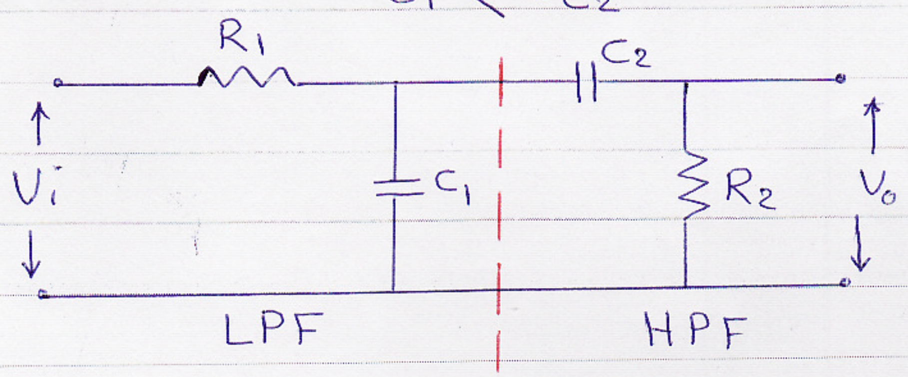


Band Stop Filter:

A BSF is constructed from a LPF and HPF as shown:



$$\omega_{c1} < \omega_{c2}$$



- For the LPF section: $\omega_{c_1} = \dfrac{1}{R_1 C_1}$

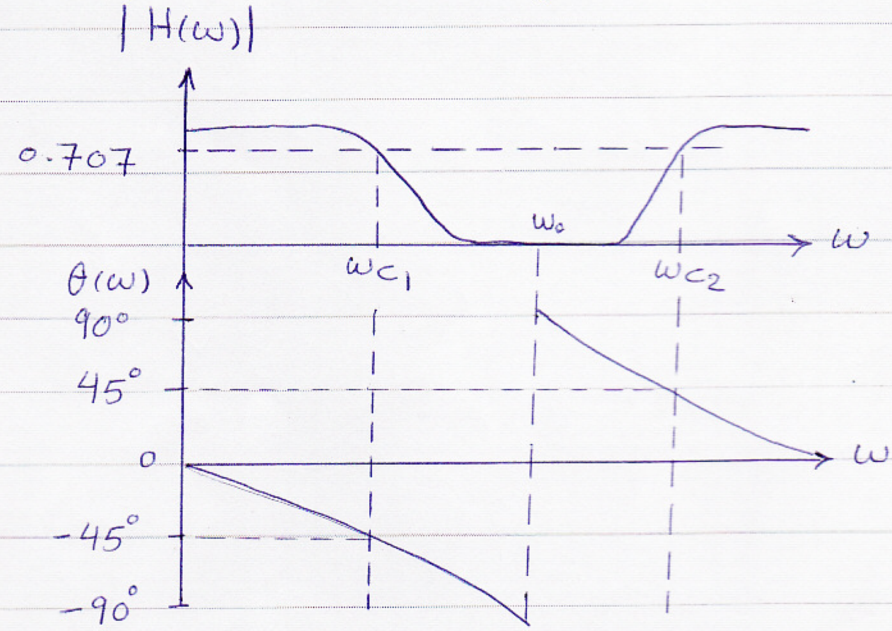
- For the HPF section: $\omega_{c_2} = \dfrac{1}{R_2 C_2}$

B.W. $= \omega_{c_2} - \omega_{c_1}$

$H(\omega) = H_1(\omega) \cdot H_2(\omega)$

where: $H_1(\omega) = \dfrac{1}{1 + j\omega R_1 C_1}$ Transfer function of LPF

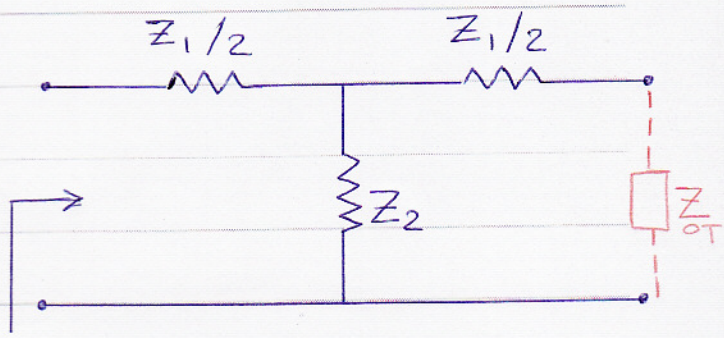
$H_2(\omega) = \dfrac{j\omega R_2 C_2}{1 + j\omega R_2 C_2}$ Transfer function of HPF



Prototype Filter Sections: (Constant-K Sections)

There are two prototype sections to realize a filter, T and π sections.

A T-section is shown in fig. It consists of two series arms (the two $\frac{Z_1}{2}$ impedance) and parallel arm (Z_2 impedance)



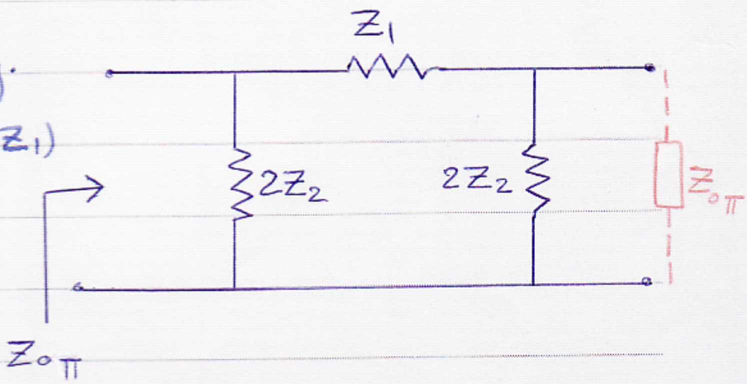
Symmetrical T-section

The characteristic impedance Z_{oT} is given by:

$$Z_{oT} = \frac{\left(\frac{Z_1}{2} + Z_{oT}\right) Z_2}{\frac{Z_1}{2} + Z_{oT} + Z_2} + \frac{Z_1}{2}$$

$$\therefore \boxed{Z_{oT} = \frac{Z_1^2}{4} + Z_1 Z_2}$$

A π-section is shown in fig. It consists of a series arm (Z_1) and two shunt arms ($2Z_2$).



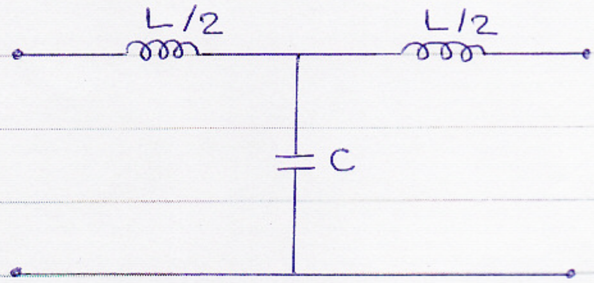
$$Z_{o\pi} = \frac{\left(\left(\frac{Z_{o\pi} \cdot 2Z_2}{Z_{o\pi} + 2Z_2}\right) + Z_1\right) \cdot 2Z_2}{\frac{Z_{o\pi} \cdot 2Z_2}{Z_{o\pi} + 2Z_2} + Z_1 + 2Z_2}$$

$$\therefore \boxed{Z_{o\pi} = \frac{Z_1 Z_2}{\sqrt{\frac{Z_1^2}{4} + Z_1 Z_2}} = \frac{Z_1 Z_2}{Z_{oT}}}$$

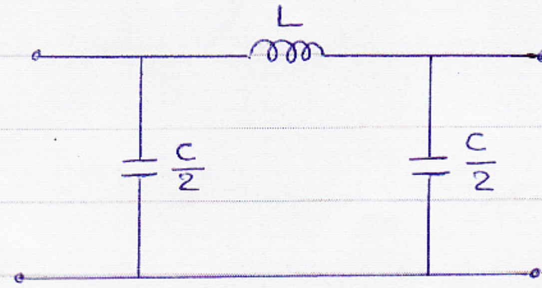
Constant_K Filters:

1. Lowpass Filter:

A prototype (Constant-K) LPF consists of inductive series arm and capacitive shunt arm, as shown.



T-section LPF



Π-section LPF

a) Design impedance (R_o):

$$R_o = \sqrt{Z_1 Z_2} \quad ; \quad Z_1 = j\omega L \quad ; \quad Z_2 = \frac{1}{j\omega C}$$

$$\therefore R_o = \sqrt{j\omega L\left(\frac{1}{j\omega C}\right)} \Rightarrow \boxed{R_o = \sqrt{\frac{L}{C}}}$$

b) Cutoff frequency:

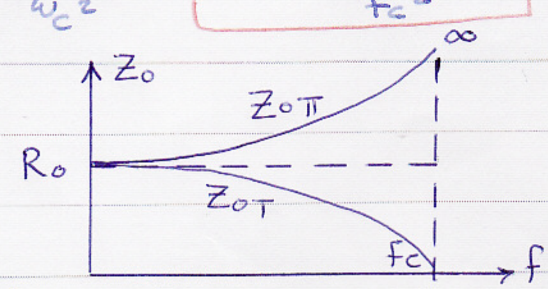
The prototype LPF passes all frequencies below $\omega = \frac{2}{\sqrt{LC}}$ while attenuates all frequencies above this value.

$$\therefore \omega_c = \frac{2}{\sqrt{LC}} \quad \text{or} \quad \boxed{f_c = \frac{1}{\pi\sqrt{LC}}}$$

c) Variation of Z_{oT} and $Z_{o\pi}$ with frequency:

$$Z_{oT} = R_o\sqrt{\left(1-\frac{\omega^2}{\omega_c^2}\right)} = \boxed{R_o\sqrt{\left(1-\frac{f^2}{f_c^2}\right)}}$$

$$Z_{o\pi} = \frac{R_o}{\sqrt{1-\frac{\omega^2}{\omega_c^2}}} = \boxed{\frac{R_o}{\sqrt{1-\frac{f^2}{f_c^2}}}}$$



Variation of c/cs impedance with frequency

d) Variation of attenuation constant (α) and phase shift (β) with frequency:

The ability of any filter network to perform as required is decided by the propagation constant (γ) of the filter.

$$\gamma = \alpha + j\beta$$

where : α = attenuation constant (neper)

β = phase shift

Generally:

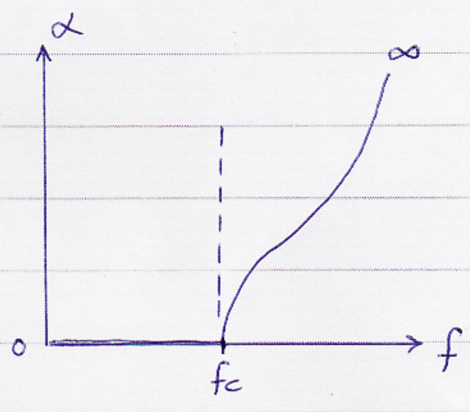
$$\boxed{\begin{aligned} \alpha &= 2\cosh^{-1}\sqrt{\frac{Z_1}{4Z_2}} \\ \beta &= 2\sin^{-1}\sqrt{\frac{Z_1}{4Z_2}} \end{aligned}}$$

neper

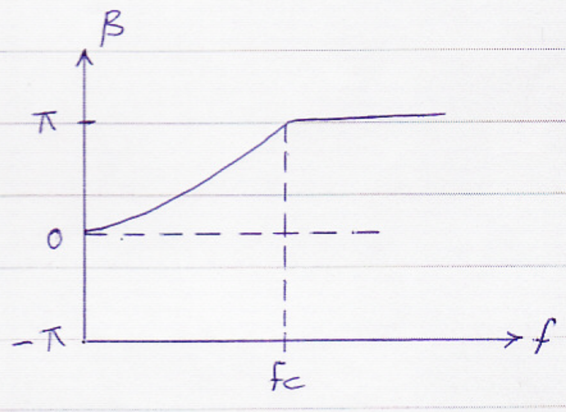
degree

For a LPF:

$$\boxed{\begin{aligned} \alpha &= 2\cosh^{-1}\left(\frac{\omega}{\omega_c}\right) = 2\cosh^{-1}\left(\frac{f}{f_c}\right) \\ \beta &= 2\sin^{-1}\left(\frac{\omega}{\omega_c}\right) = 2\sin^{-1}\left(\frac{f}{f_c}\right) \end{aligned}}$$



variation of attenuation with frequency



variation of phase with frequency

e) Design equations of prototype LPF:

The design impedance (R_o) and cutoff frequency (f_c) can be given in terms of L and C as follows:

$$R_o = \sqrt{\frac{L}{C}} \quad ; \quad f_c = \frac{1}{\pi\sqrt{LC}}$$

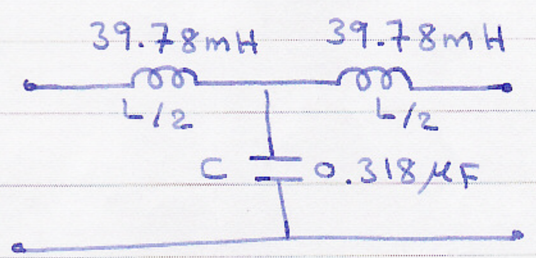
$$\therefore \quad \boxed{L = \frac{R_o}{\pi f_c} \quad ; \quad C = \frac{1}{\pi f_c R_o}}$$

Ex. Design a prototype LPF section if design impedance R_o is 500 Ω and cutoff frequency f_c is 2 KHz.

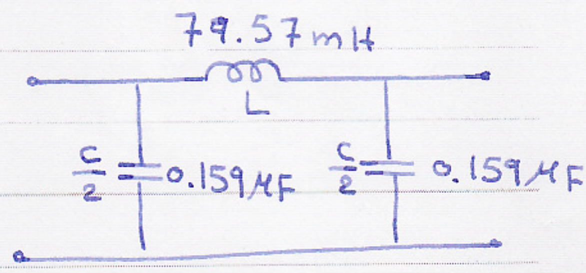
sol. Using LPF design equations:

$$L = \frac{R_o}{\pi f_c} = \frac{500}{\pi(2000)} = 79.57 \text{ mH}$$

$$C = \frac{1}{\pi f_c R_o} = \frac{1}{\pi(2000)(500)} = 0.318 \ \mu F$$



T-section LPF



π-section LPF

Ex. A constant-k, T-section LPF consists a series arm of 18mH inductor and a shunt arm of 0.1 μF capacitor. Calculate: a) cutoff frequency. b) design impedance ; c) c/cs impedance at 1 KHz and 8 KHz. Comment on the results.

sol. Total series arm inductance $= 2\left(\frac{L}{2}\right) = 2(18 \times 10^{-3})$
$= 36 \text{ mH}$

$$a) \ f_c = \frac{1}{\pi\sqrt{LC}} = \frac{1}{\pi\sqrt{(36 \times 10^{-3})(0.1 \times 10^{-6})}} = 5.3 \text{ KHz}$$

Ex. continued:

b) $R_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{36 * 10^{-3}}{0.1 * 10^{-6}}} = 600\ \Omega$

c) $Z_{oT} = R_o \sqrt{1-(\frac{f}{f_c})^2} = 600\sqrt{1-(\frac{f}{5300})^2}$

at $f = 1000$ Hz $\Rightarrow Z_{oT} = 589.24\ \Omega$

at $f = 8000$ Hz $\Rightarrow Z_{oT} = j677.21\ \Omega$

At $f = 1$ KHz (within pass band) the c/cs impedance is pure resistive ($Z_o = 589.24\ \Omega$).

At $f = 8$ KHz (within stopband) the c/cs impedance is pure reactive ($Z_o = j677.21\ \Omega$).

Ex. Design a prototype T-section LPF to be terminated with 600 Ω load, having cutoff frequency of 3 KHz.

Determine: a) Attenuation in neper and dB at 5KHz and 8KHz

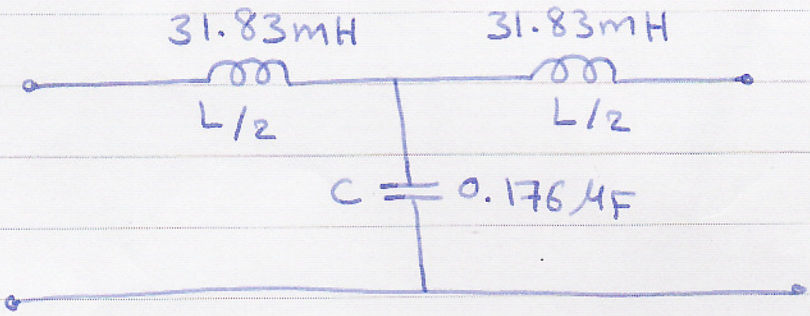
b) The c/cs impedance and phase constant at 2KHz

c) The frequency at which the filter attenuation is 17.372 dB

Sol. For the design of T-section LPF:

$$L = \frac{R_o}{\pi f_c} = \frac{600}{\pi(3000)} = 63.66\ mH$$

$$C = \frac{1}{\pi f_c R_o} = \frac{1}{\pi(3000)(600)} = 0.176\ \mu F$$



a) $\alpha = 2\cosh^{-1}\left(\frac{f}{f_c}\right) = 2\cosh^{-1}\left(\frac{f}{3000}\right)$

At $f = 5\,KHz \Rightarrow \alpha = 2.197$ neper

At $f = 8\,KHz \Rightarrow \alpha = 3.273$ neper

attenuation in neper = 0.115 * attenuation in dB

At $f = 5\,KHz \Rightarrow \alpha(dB) = \frac{2.197}{0.115} = 19.1\,dB$

$f = 8\,KHz \Rightarrow \alpha(dB) = \frac{3.273}{0.115} = 28.46\,dB$

b) $Z_{oT} = R_o\sqrt{1-\left(\frac{f}{f_c}\right)^2} = 600\sqrt{1-\left(\frac{2000}{3000}\right)^2} = 447.21\,\Omega$

$\beta = 2\sin^{-1}\left(\frac{f}{f_c}\right) = 2\sin^{-1}\left(\frac{2000}{3000}\right) = 83.62^\circ$

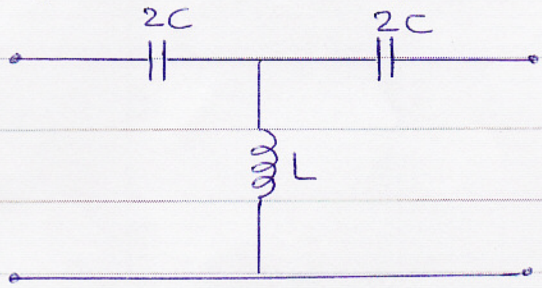
c) Attenuation in neper = 0.115 * attenuation in dB

= 0.115 * 17.372 = 2 neper

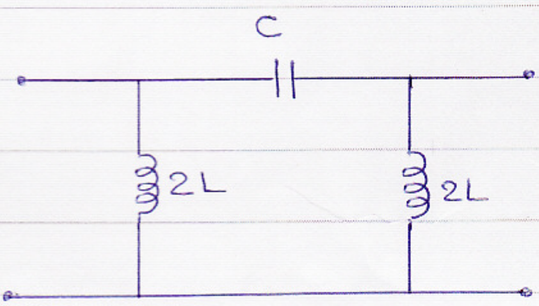
$\alpha = 2\cosh^{-1}\left(\frac{f}{f_c}\right) \Rightarrow 2 = 2\cosh^{-1}\left(\frac{f}{3000}\right) \Rightarrow f = 4.629\,KHz$

2. Highpass Filter:

A prototype HPF consists of capacitive series arm and inductive shunt arm, as shown.



T-section HPF



π-section HPF

a) Design impedance:

$R_o = \sqrt{Z_1 Z_2}$; $Z_1 = \frac{1}{j\omega C}$; $Z_2 = j\omega L$

$\therefore \boxed{R_o = \sqrt{\frac{L}{C}}}$

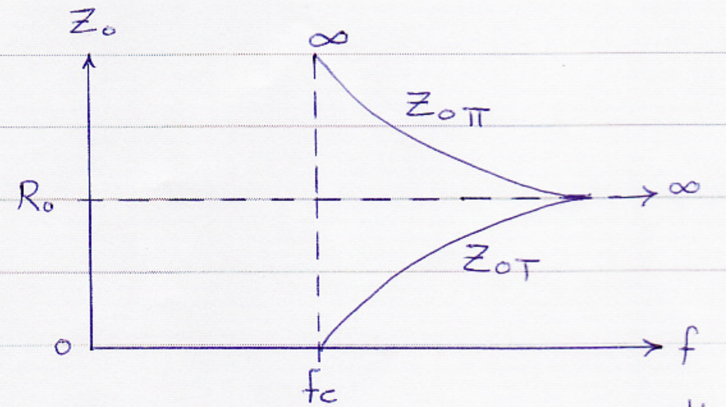
b) cutoff frequency:

The prototype HPF passes all frequencies above $\omega = \frac{1}{2\sqrt{LC}}$ while attenuates all frequencies below this value.

$$\therefore \quad \omega_c = \frac{1}{2\sqrt{LC}} \quad \text{or} \quad \boxed{f_c = \frac{1}{4\pi\sqrt{LC}}}$$

c) Variation of Z_{oT} and $Z_{o\pi}$ with frequency:

$$\boxed{Z_{oT} = R_o \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \quad ; \quad Z_{o\pi} = \frac{R_o}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}}$$

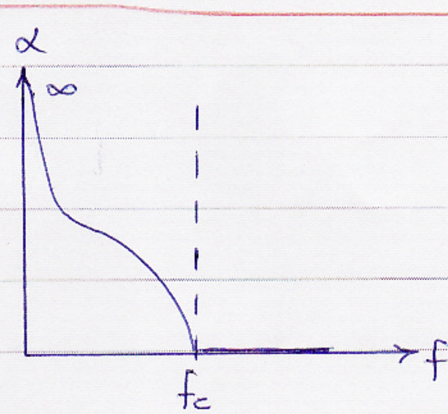


variation of c/cs impedance with frequency

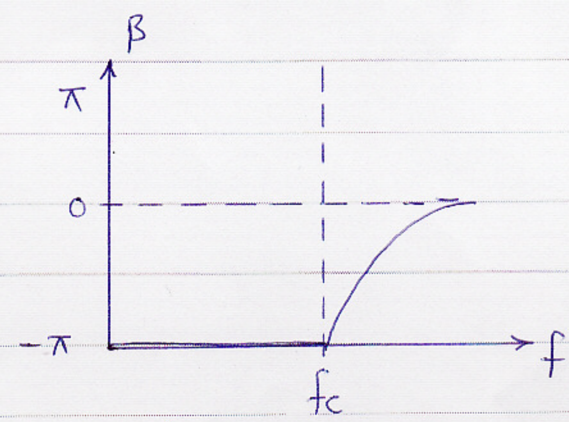
d) Variation of attenuation constant and phase shift with frequency:

$$\alpha = 2\cosh^{-1}\left(\frac{f_c}{f}\right) \quad \text{neper}$$

$$\beta = 2\sin^{-1}\left(\frac{f_c}{f}\right) \quad \text{degree}$$



variation of attenuation with frequency



variation of phase with frequency

e) Design equations of prototype HPF:

$$R_o = \sqrt{\frac{L}{C}} \quad ; \quad f_c = \frac{1}{4\pi\sqrt{LC}}$$

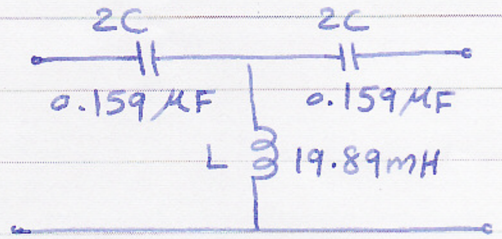
$$\therefore \boxed{L = \frac{R_o}{4\pi f_c} \quad ; \quad C = \frac{1}{4\pi f_c R_o}}$$

ex. Design a prototype HPF section if $R_o = 500\,\Omega$ and $f_c = 2\,KHz$.

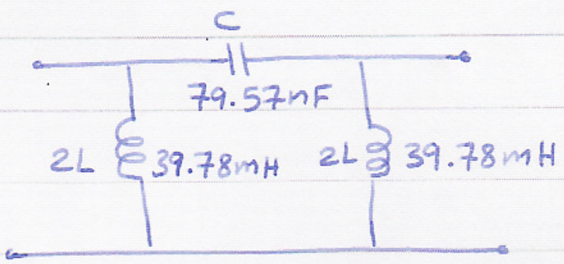
sol.

$$L = \frac{R_o}{4\pi f_c} = \frac{500}{4\pi(2000)} = 19.89\,mH$$

$$C = \frac{1}{4\pi f_c R_o} = \frac{1}{4\pi(2000)(500)} = 79.57\,nF$$



T-section HPF



π-section HPF

ex. Design a constant-k, π-section HPF to work into impedance of $500\,\Omega$ and $f_c = 1\,KHz$. For this filter calculate:

a) c/cs impedance at 2 KHz.

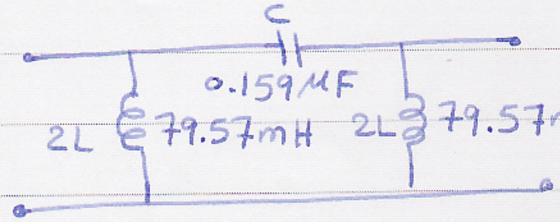
b) attenuation at 0.9 KHz

c) phase angle at 1.5 KHz

sol.

$$L = \frac{R_o}{4\pi f_c} = \frac{500}{4\pi(1000)} = 39.78\,mH$$

$$C = \frac{1}{4\pi f_c R_o} = \frac{1}{4\pi(1000)(500)} = 0.159\,\mu F$$



ex. continued:

a) $Z_{o\pi} = \dfrac{R_o}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \dfrac{500}{\sqrt{1-\left(\frac{1000}{2000}\right)^2}} = 577.35\ \Omega$

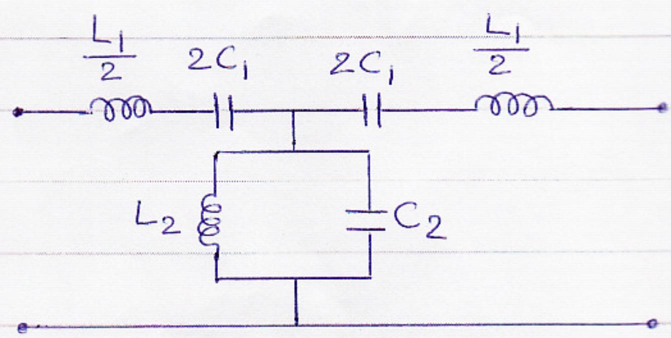
b) $\alpha = 2\cosh^{-1}(f_c/f) = 2\cosh^{-1}(1000/900) = 0.934$ neper

c) $\beta = 2\sin^{-1}(f_c/f) = 2\sin^{-1}(1000/1500) = -83.62^\circ$

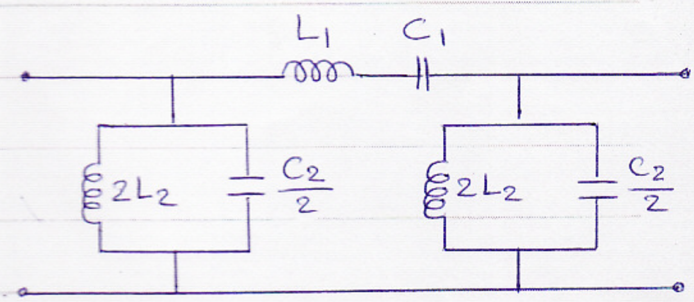
تضيف اشارة السالب لان قيم β محصورة بين 0 و $-\pi$

3. Band pass Filter:

A prototype BPF can be obtained either as a T-section or a π-section if series arm contains a series resonant circuit while the shunt arm contains a parallel resonant circuit, as shown in figure.



T– section BPF



π–section BPF

This filter is designed so that the resonant frequency of the series arm equals the resonant frequency of shunt arm.

i.e. for series arm: $\omega_o = \dfrac{1}{\sqrt{\frac{L_1}{2}\cdot 2C_1}} = \dfrac{1}{\sqrt{L_1C_1}} \Rightarrow \omega_o^2 L_1 C_1 = 1$

for shunt arm: $\omega_o = \dfrac{1}{\sqrt{L_2C_2}} \Rightarrow \omega_o^2 L_2 C_2 = 1$

$\therefore\ \omega_o^2 L_1 C_1 = \omega_o^2 L_2 C_2$

$$\boxed{L_1 C_1 = L_2 C_2}$$

a) Design impedance:

$R_o = \sqrt{Z_1 Z_2}$; Z_1 = total series arm impedance

Z_2 = total shunt arm impedance

$$Z_1 = X_{L_1} + X_{C_1} = j\omega L_1 + \frac{1}{j\omega C_1} = j\left(\frac{\omega^2 L_1 C_1 - 1}{\omega C_1}\right)$$

$$Z_2 = X_{L_2} // X_{C_2} = j\omega L_2 // \frac{1}{j\omega C_2} = \frac{-j\omega L_2}{\omega^2 L_2 C_2 - 1}$$

$\because L_1 C_1 = L_2 C_2$

$$\therefore \boxed{R_o = \sqrt{Z_1 Z_2} = \sqrt{\frac{L_2}{C_1}} = \sqrt{\frac{L_1}{C_2}}}$$

b) cutoff frequency:

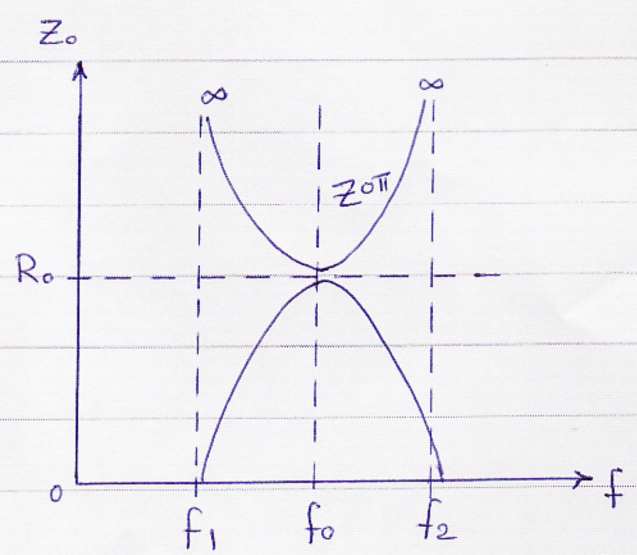
$$\boxed{f_o = \sqrt{f_1 f_2}}$$

where f_1 = lower cutoff frequency

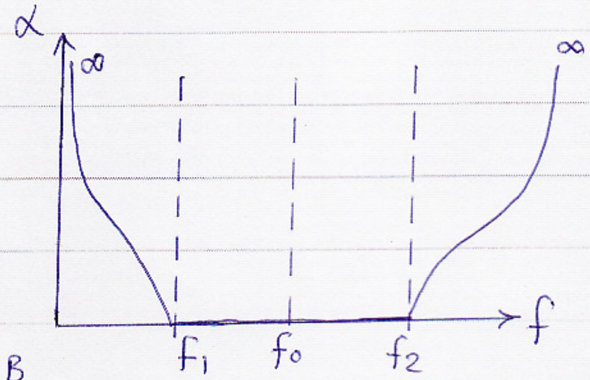
f_2 = upper cutoff frequency

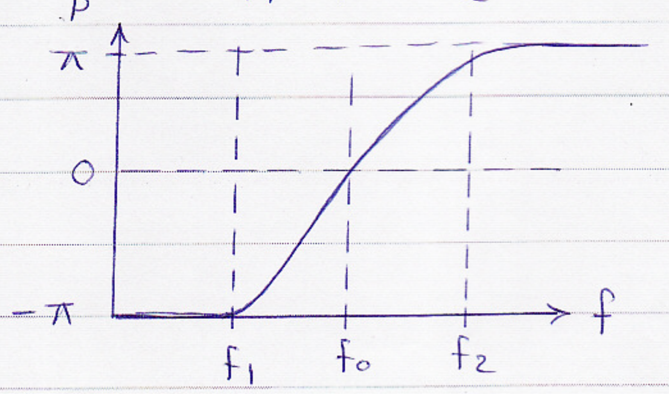
This equation indicates that the resonant frequency of the individual arms is the geometric mean of two cutoff frequencies.

c) Variation of Z_{oT} and $Z_{o\pi}$ with frequency:



d) Variation of attenuation constant and phase shift with frequency:





d) Design equations of prototype BPF:

$$R_o = \sqrt{\frac{L_2}{C_1}} = \sqrt{\frac{L_1}{C_2}}$$

$$f_o = \sqrt{f_1 f_2} = \frac{1}{2\pi\sqrt{L_1 C_1}} = \frac{1}{2\pi\sqrt{L_2 C_2}}$$

$$C_1 = \frac{f_2 - f_1}{4\pi R_o f_o^2} \quad ; \quad L_1 = \frac{R_o}{\pi(f_2 - f_1)}$$

$$C_2 = \frac{1}{\pi R_o (f_2 - f_1)} \quad ; \quad L_2 = \frac{R_o (f_2 - f_1)}{4\pi f_o^2}$$

ex. Design a prototype BPF if $R_o = 4k\Omega$ and passband is between 1.25 KHz and 2 KHz.

sol. $R_o = 4k\Omega$; $f_1 = 1.25$ KHz ; $f_2 = 2$ KHz

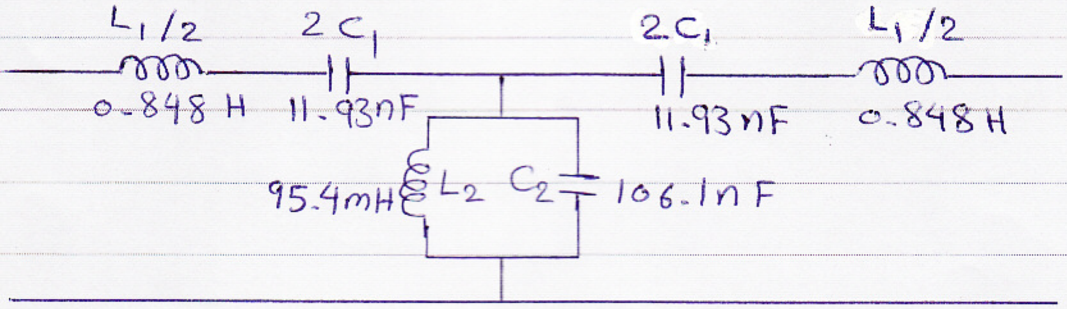
$$C_1 = \frac{f_2 - f_1}{4\pi R_o f_o^2} = \frac{2000 - 1250}{4\pi(4000)(1250 * 2000)} = 5.96\ nF$$

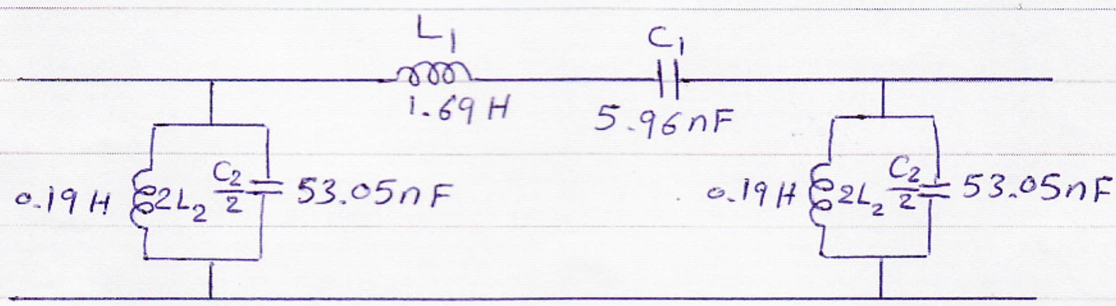
ex. continued:

$$L_1 = \frac{R_o}{\pi(f_2 - f_1)} = \frac{4000}{\pi(2000-1250)} = 1.69\ H$$

$$C_2 = \frac{1}{\pi R_o (f_2 - f_1)} = \frac{1}{\pi(4000)(2000-1250)} = 106.1\ nF$$

$$L_2 = \frac{R_o(f_2 - f_1)}{4\pi f_o^2} = \frac{4000(2000-1250)}{4\pi(1250 * 2000)} = 95.4\ mH$$





ex. The series arm of a T-section filter consists a total of 0.5 μF capacitance in series with 0.35 H inductance. If $R_o = 500\ \Omega$, determine type of filter and the elements in the shunt arm. Also find frequency of resonance (f_o) and passband frequencies.

Sol. As the series arm consists of a capacitor in series with an inductor, then it is a prototype BPF.

$$R_o^2 = \frac{L_2}{C_1} = \frac{L_1}{C_2}$$

لا نقسم C_1 على 2 لان المعطى هو

total series arm capacitance $= C_1$

$$\therefore L_2 = R_o^2 C_1 = (500)^2 (0.5 * 10^{-6}) = 0.125\ H$$

$$C_2 = \frac{L_1}{R_o^2} = \frac{0.35}{(500)^2} = 1.4\ \mu F$$

ex. continued:

$$f_o = \frac{1}{2\pi\sqrt{L_1C_1}} = \frac{1}{2\pi\sqrt{(0.35)(0.5*10^{-6})}} = 380.45\ Hz$$

To find passband frequencies (f_1 and f_2):

$$L_1 = \frac{R_o}{\pi(f_2-f_1)} \Rightarrow f_2 - f_1 = \frac{R_o}{\pi L_1} = \frac{500}{\pi(0.35)} = 454.72\ Hz$$

also: $f_o = \sqrt{f_1f_2} \Rightarrow f_o^2 = f_1f_2 = (380.45)^2 = 144.74*10^3$

$$\therefore f_1 = \frac{144.74*10^3}{f_2}$$

$$\therefore f_2 - \frac{144.74*10^3}{f_2} = 454.72\ Hz$$

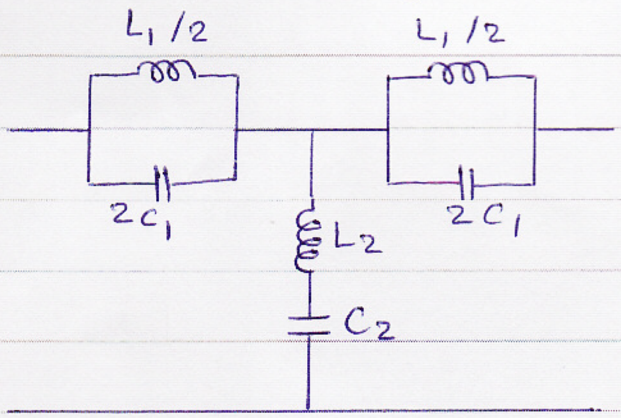
$$f_2^2 - 454.72 f_2 - 144.74*10^3 = 0$$

solve to get: $f_2 = 670.58\ Hz$

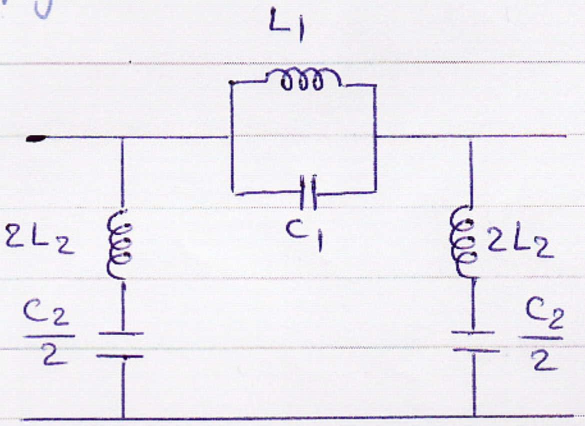
$\therefore f_1 = f_2 - 454.72 = 670.58 - 454.72 = 215.85\ Hz$

4. Band stop Filter:

A prototype BSF can be obtained if series arm contains parallel resonant circuit while shunt arm contains series resonant circuit, as shown in figure.



T- section BSF



Π - section BSF

The series resonant frequency of the shunt arm is selected same as the resonant frequency of the series arm.

i.e. For series arm: $\omega_o = \dfrac{1}{\sqrt{\frac{L_1}{2} \cdot 2C_1}} = \dfrac{1}{\sqrt{L_1 C_1}} \Rightarrow \omega_o^2 L_1 C_1 = 1$

For shunt arm: $\omega_o = \dfrac{1}{\sqrt{L_2 C_2}} \Rightarrow \omega_o^2 L_2 C_2 = 1$

$\therefore \ \omega_o^2 L_1 C_1 = \omega_o^2 L_2 C_2 \Rightarrow \boxed{L_1 C_1 = L_2 C_2}$

a) design impedance: (R_o)

$R_o = \sqrt{Z_1 Z_2}$

$Z_1 = X_{L_1} // X_{C_1} = j\omega L_1 // \dfrac{1}{j\omega C_1} = \dfrac{\omega L_1}{j(\omega^2 L_1 C_1 - 1)}$

$Z_2 = X_{L_2} + X_{C_2} = j\omega L_2 + \dfrac{1}{j\omega C_2} = j\left(\dfrac{\omega^2 L_2 C_2 - 1}{\omega C_2}\right)$

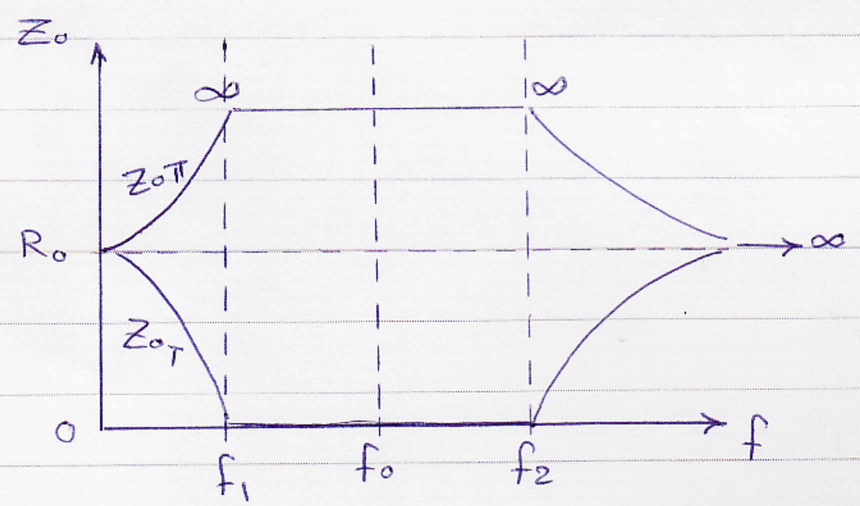
$\because \ L_1 C_1 = L_2 C_2$

$\therefore \ \boxed{R_o = \sqrt{\dfrac{L_2}{C_1}} = \sqrt{\dfrac{L_1}{C_2}}}$

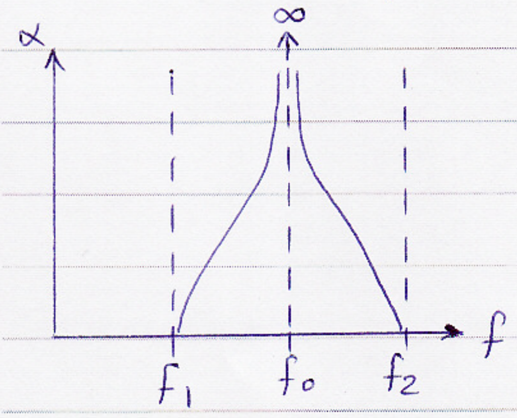
b) cut off frequency:

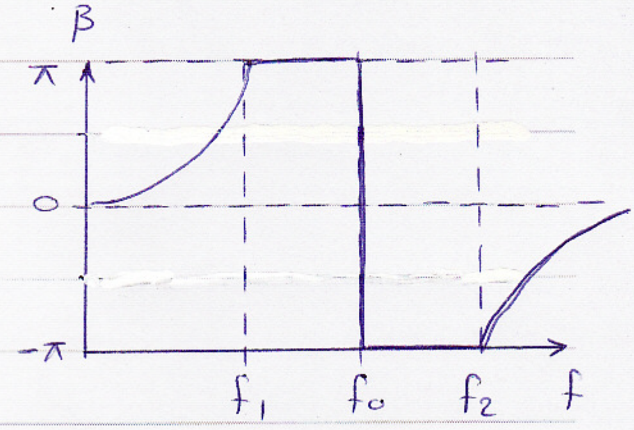
$\boxed{f_o = \sqrt{f_1 f_2}}$

c) variation of Z_{oT} and $Z_{o\pi}$ with frequency:



d) variation of attenuation constant and phase constant with frequency:





e) design equations of prototype BSF:

$$R_o = \sqrt{\frac{L_2}{C_1}} = \sqrt{\frac{L_1}{C_2}}$$

$$f_o = \sqrt{f_1 f_2} = \frac{1}{2\pi\sqrt{L_1 C_1}} = \frac{1}{2\pi\sqrt{L_2 C_2}}$$

$$\boxed{C_1 = \frac{1}{4\pi R_o (f_2 - f_1)} \quad ; \quad L_1 = \frac{R_o (f_2 - f_1)}{\pi f_o^2}}$$

$$\boxed{C_2 = \frac{f_2 - f_1}{\pi R_o f_o^2} \quad ; \quad L_2 = \frac{R_o}{4\pi (f_2 - f_1)}}$$

ex. Design a prototype band elimination filter sections if design impedance is 400 Ω and cutoff frequencies are 1250 Hz and 2000 Hz.

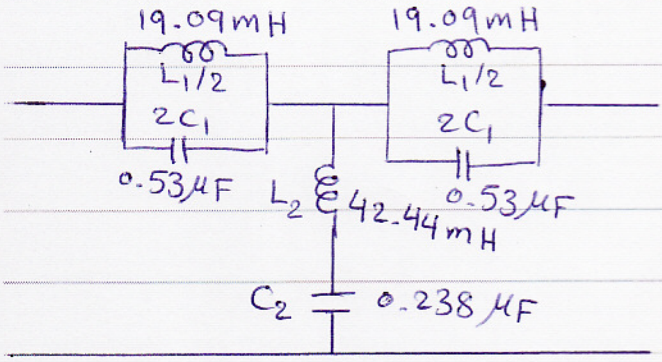
Sol.

$$L_1 = \frac{R_o (f_2 - f_1)}{\pi f_o^2} = \frac{400(2000-1250)}{\pi(1250 * 2000)} = 38.197 \text{ mH}$$

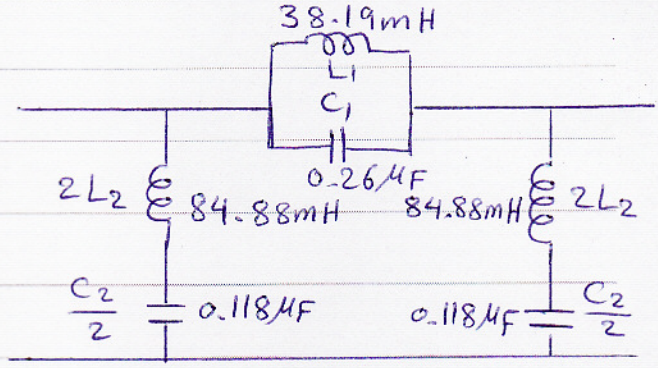
$$C_1 = \frac{1}{4\pi R_o (f_2 - f_1)} = \frac{1}{4\pi(400)(2000-1250)} = 0.265 \, \mu F$$

$$L_2 = \frac{R_o}{4\pi(f_2 - f_1)} = \frac{400}{4\pi(2000-1250)} = 42.44 \text{ mH}$$

$$C_2 = \frac{f_2 - f_1}{\pi R_o f_o^2} = \frac{2000-1250}{\pi(400)(1250 * 2000)} = 0.238 \, \mu F$$



T- section BSF



π - section BSF

Disadvantages of prototype Filter sections:

Constant - K filters has two main drawbacks:

1. In the stopband, the attenuation doesnot increase rapidly with frequency. Hence, frequencies near cutoff frequency are passed throughout the filter.

2. In the passband, the c/cs impedance (Z_o) doesnot remain constant but varies with frequency. Hence, filter cannot be terminated properly.

m-derived filter sections:

To overcome the disadvantages of a particular filter section, a new section derived is called m-derived section.

Derivation of m-derived sections:

a) T-section: in the m-derived T-section, the series arm is of same type as that in prototype section but its value is $mZ_1/2$, where m is a constant.

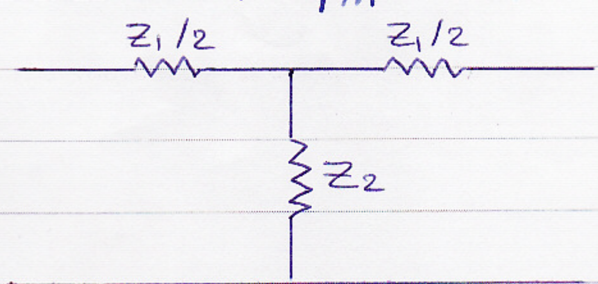
Z_2 is changed to Z_2' so that the value of Z_o for both the sections is same.

$$Z_{oT} = \sqrt{\frac{Z_1^2}{4} + Z_1 Z_2} \quad \text{for prototype section}$$

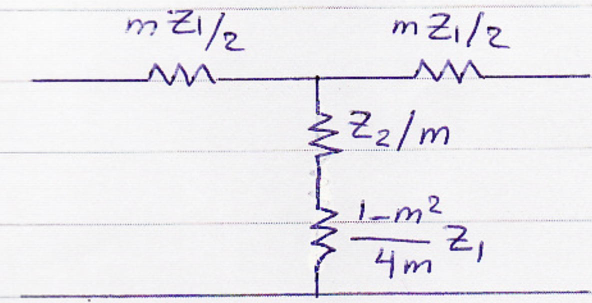
$$Z_{oT} = \sqrt{\frac{(mZ_1)^2}{4} + (mZ_1) Z_2'} \quad \text{for m-derived section}$$

$$\frac{Z_1^2}{4} + Z_1 Z_2 = \frac{(mZ_1)^2}{4} + (mZ_1) Z_2'$$

$$\therefore Z_2' = \left(\frac{1-m^2}{4m}\right) Z_1 + \frac{Z_2}{m}$$



prototype T-section



m-derived T-section

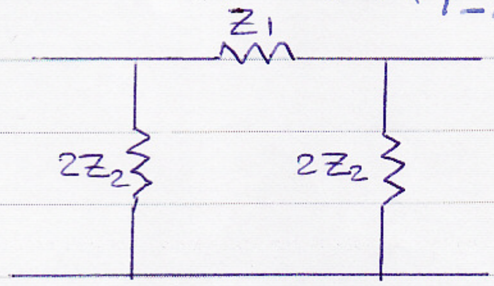
b) π-section: The shunt arm of a π-section is of same type but having a value $2Z_2/m$. Z_1 is changed to Z_1' so that Z_o for both sections is same.

$$Z_{o\pi} = \frac{Z_1 Z_2}{\sqrt{\frac{Z_1^2}{4} + Z_1 Z_2}} \quad \text{for prototype section}$$

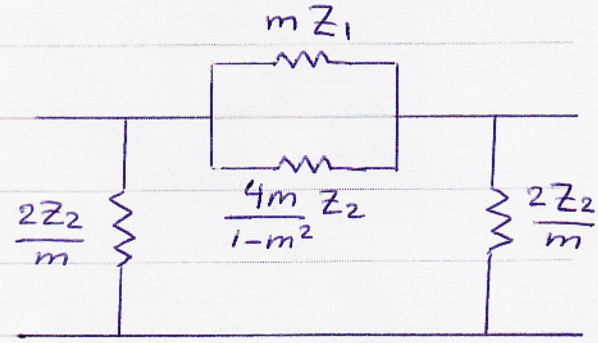
$$Z_{o\pi} = \frac{Z_1' \frac{Z_2}{m}}{\sqrt{\frac{Z_1'^2}{4} + Z_1'\left(\frac{Z_2}{m}\right)}}$$ for m-derived section

$$\frac{Z_1 Z_2}{\sqrt{\frac{Z_1^2}{4} + Z_1 Z_2}} = \frac{Z_1' \frac{Z_2}{m}}{\sqrt{\frac{Z_1'^2}{4} + Z_1'\left(\frac{Z_2}{m}\right)}}$$

$$\therefore Z_1' = \frac{(mZ_1)\left(\frac{4m}{1-m^2} Z_2\right)}{(mZ_1) + \left(\frac{4m}{1-m^2} Z_2\right)}$$



Prototype π-section

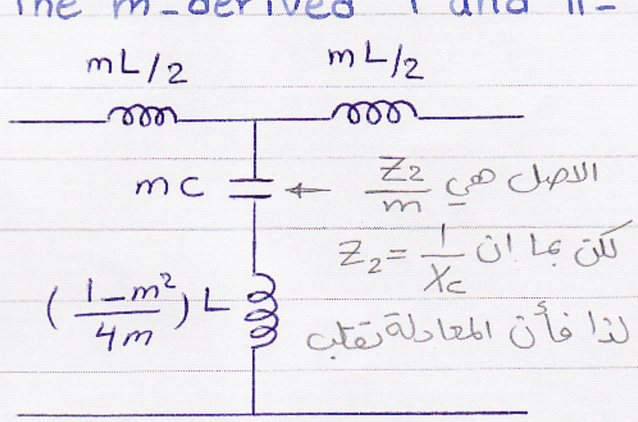


m-derived π-section

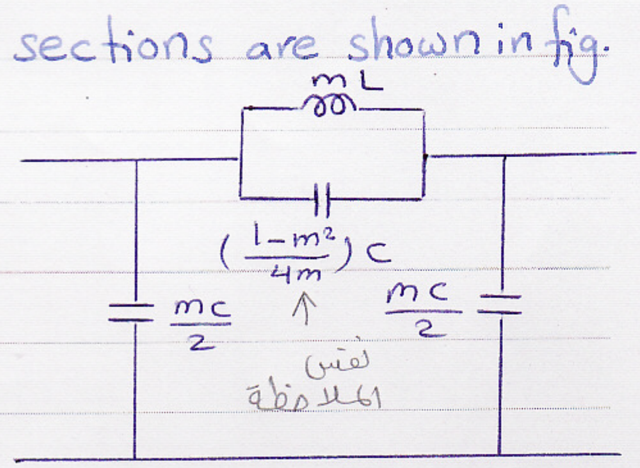
In m-derived filters, attenuation constant is improved in stopband by using series resonant circuit in the shunt arm of a T-section m-derived and parallel resonant circuit in the series arm of a π-section m-derived filter.
The frequency at which the attenuation increases to ∞ is called "frequency of infinite attenuation" (f_∞).
The value of m must be always selected as $0 < m < 1$

1. m-derived LPF:

The m-derived T and π- LPF sections are shown in fig.



T- section



π - section

The frequency of infinite attenuation is selected just above f_c

$$f_c = \frac{1}{\pi\sqrt{LC}}$$

$$f_\infty = \frac{1}{2\pi\sqrt{(mL)\left(\frac{1-m^2}{4m}\right)C}} = \frac{1}{\pi\sqrt{(1-m^2)LC}}$$

$$\boxed{\therefore\ f_\infty = \frac{f_c}{\sqrt{1-m^2}}}$$

كلما صغرت m كان الفلتر افضل

$$\sqrt{1-m^2} = \frac{f_c}{f_\infty} \Longrightarrow \boxed{m = \sqrt{1-\left(\frac{f_c}{f_\infty}\right)^2}}$$

ex. Design an m_derived T-type LPF to work into load of 500 Ω and cut off frequency at 4 KHz and infinite attenuation at 4.5 KHz.

sol. $R_o = 500\ \Omega$; $f_c = 4\ KHz$; $f_\infty = 4.5\ KHz$

from design equations of prototype LPF:

$$L = \frac{R_o}{\pi f_c} = \frac{500}{\pi(4000)} = 39.78\ mH$$

$$C = \frac{1}{\pi f_c R_o} = \frac{1}{\pi(4000)(500)} = 0.159\ \mu F$$

$$m = \sqrt{1-\left(\frac{f_c}{f_\infty}\right)^2} = \sqrt{1-\left(\frac{4000}{4500}\right)^2} = 0.458$$

∴ The actual values of components are:

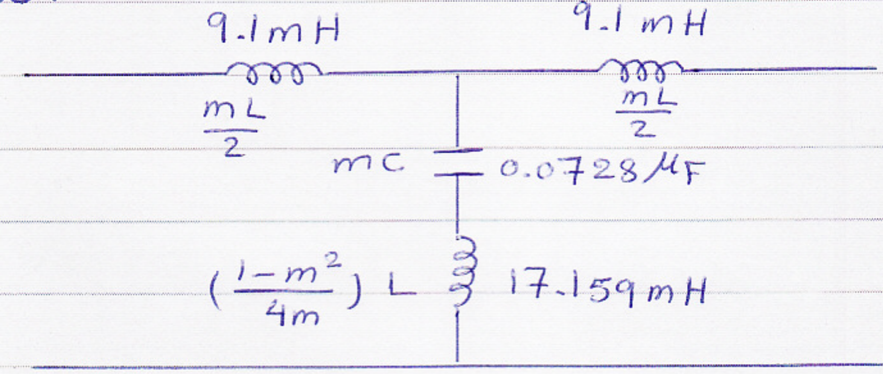
$$\frac{mL}{2} = \frac{1}{2}(0.458)(39.78 * 10^{-3}) = 9.1\ mH \quad \text{(series arm)}$$

$$mC = (0.458)(0.159 * 10^{-6}) = 0.0728\ \mu F \quad \text{(shunt arm)}$$

$$\left(\frac{1-m^2}{4m}\right)L = \left(\frac{1-(0.458)^2}{4(0.458)}\right)(39.78 * 10^{-3}) = 17.159\ mH$$

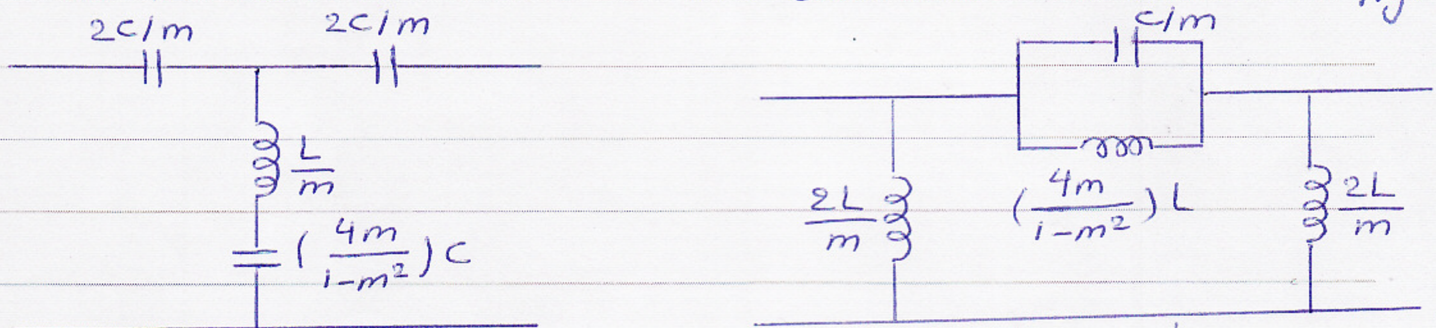
(shunt arm)

ex. continued:



2. m-derived HPF:

The m-derived T and π HPF sections are shown in fig.



T- section π- section

The frequency of infinite attenuation is selected just below f_c.

$$f_c = \frac{1}{4\pi\sqrt{LC}}$$

$$f_\infty = \frac{1}{2\pi\sqrt{\left(\frac{L}{m}\right)\left(\frac{4m}{1-m^2}\right)C}} = \frac{1}{2\pi\sqrt{\frac{4LC}{1-m^2}}} = \frac{\sqrt{1-m^2}}{4\pi\sqrt{LC}}$$

$$\boxed{\therefore\ f_\infty = f_c\sqrt{1-m^2}}$$

$$\frac{f_\infty}{f_c} = \sqrt{1-m^2} \quad \Longrightarrow \quad \boxed{m = \sqrt{1-\left(\frac{f_\infty}{f_c}\right)^2}}$$

ex. Design an m-derived, π-section HPF to work into load of 600Ω with cutoff frequency of $\frac{1000}{\pi}$ Hz and peak attenuation frequency of 300 Hz.

sol.
$$L = \frac{R_o}{4\pi f_c} = \frac{600}{4\pi(1000/\pi)} = 150\,mH$$

$$C = \frac{1}{4\pi f_c R_o} = \frac{1}{4\pi(1000/\pi)(600)} = 0.416\,\mu F$$

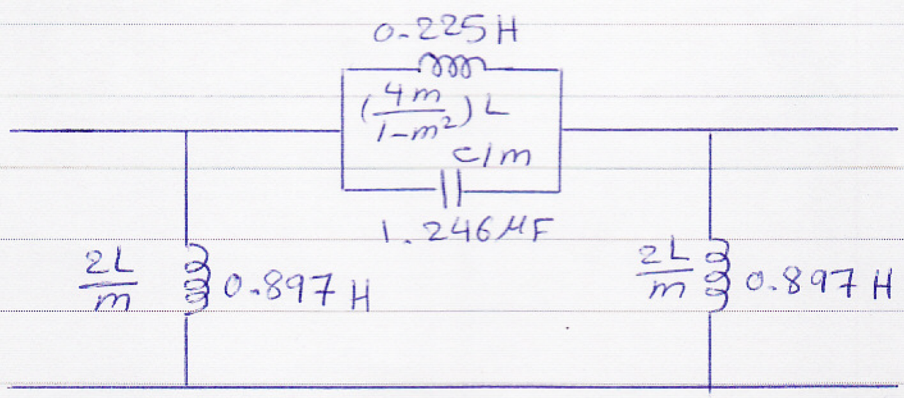
$$m = \sqrt{1-\left(\frac{f_\infty}{f_c}\right)^2} = \sqrt{1-\left(\frac{300}{1000/\pi}\right)^2} = 0.334$$

The actual values of elements are:

$2L/m = 2(150*10^{-3})/0.334 = 0.897$ H (shunt arm)

$\left(\frac{4m}{1-m^2}\right)L = \left(\frac{4(0.334)}{1-(0.334)^2}\right)(150*10^{-3}) = 0.225$ H (series arm)

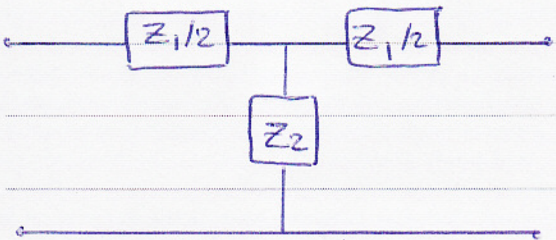
$C/m = 0.416*10^{-6}/0.334 = 1.246\,\mu F$ (series arm)



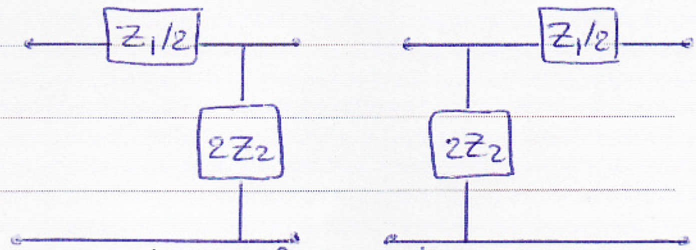
m-derived half sections:

A half section filter is needed for impedance matching at both ends of a filter network.

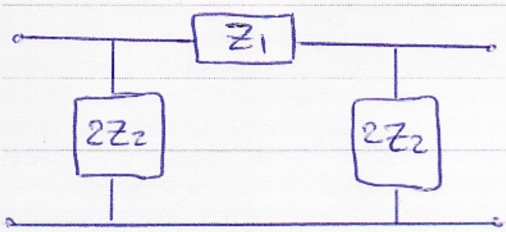
If a T or π-section is bisected longitudinally, we get two half sections as shown in fig.



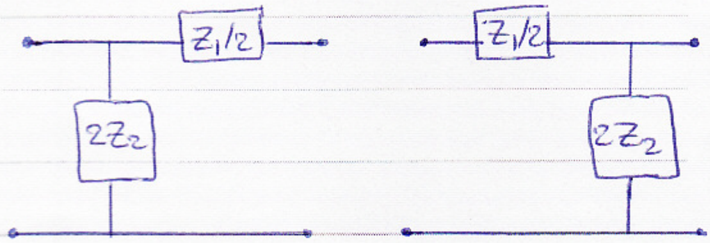
T- section



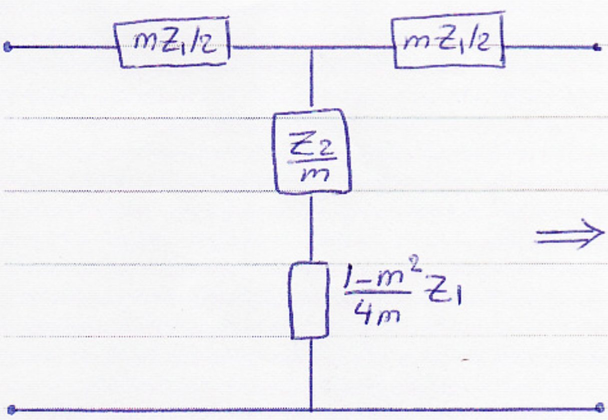
Half sections



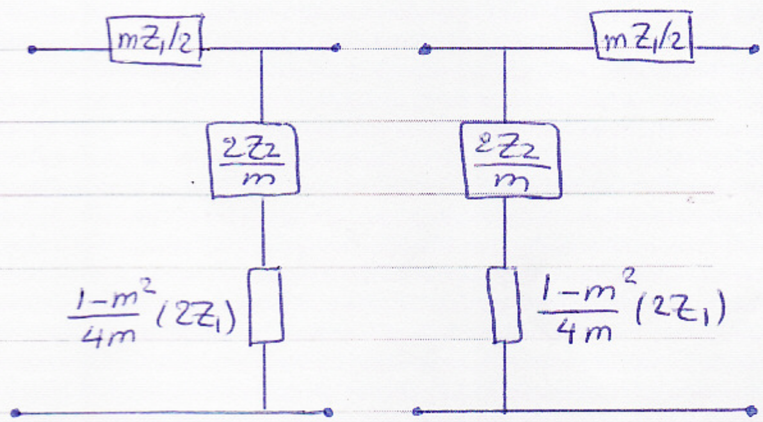
π- section



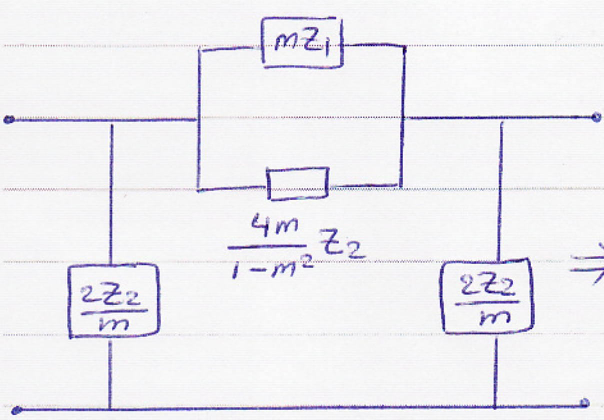
half sections



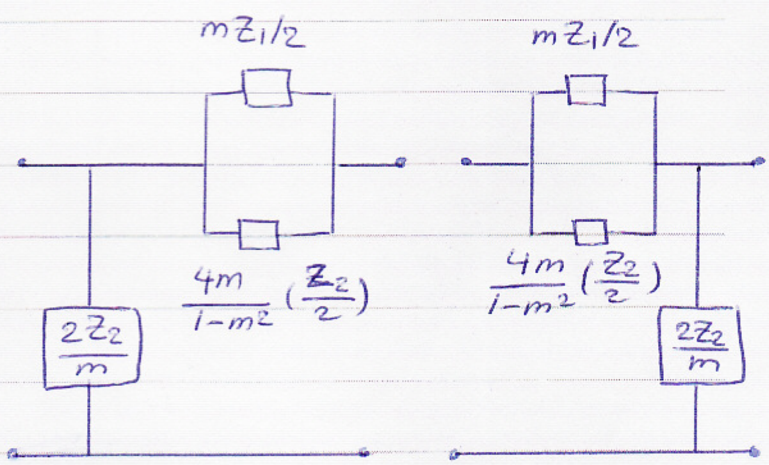
m-derived T-section



m-derived half sections

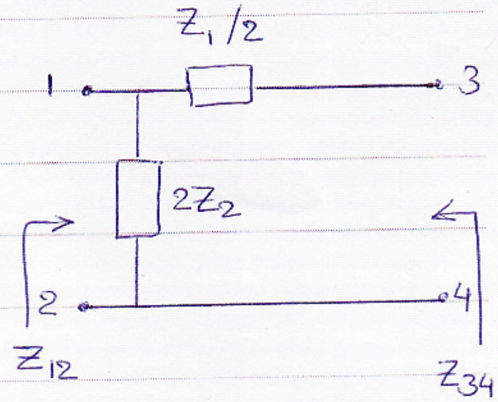


m-derived π-section



half sections

Consider the half section shown.



The image impedance can be found by:

$$Z_{12} = \sqrt{Z_{oc} Z_{sc}} = \sqrt{(2Z_2)\left(\frac{2Z_2 \frac{Z_1}{2}}{2Z_2 + \frac{Z_1}{2}}\right)}$$

$$= \sqrt{\frac{4Z_1 Z_2^2}{Z_1 + 4Z_2}} = Z_{o\pi K} \quad (1)$$

$$Z_{34} = \sqrt{Z_{oc} Z_{sc}} = \sqrt{\left(\frac{Z_1}{2}\right)\left(\frac{2Z_2 \frac{Z_1}{2}}{2Z_2 + \frac{Z_1}{2}}\right)} = \sqrt{Z_1 Z_2 + \frac{Z_1^2}{4}} = Z_{oTK} \quad (2)$$

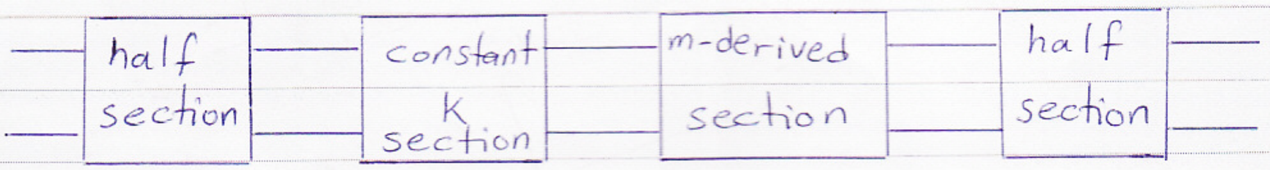
As seen from eqs (1) and (2), the half section has the impedance of π_section at terminals 1-2 and the impedance of T-section at terminals 3-4.

Thus, it can be used to match a π-section to a T-section or match a filter to a terminating impedance.

Composite Filter:

A composite Filter has four sections, as follows:

1. terminating half sections: needed for impedance matching.
2. constant_k section: gives more attenuation at frequencies beyond f_∞ (which not exists in m_derived).
3. m_derived section: gives a sharp cutoff (which is not given in constant_k).



composite filter

Design procedure of a composite filter:

1. determine the elements of the constant_k section (found from design equations).
2. determine the value of m for the full m_derived section, then the elements values can be found. (for a sharp cutoff c/cs, m is chosen between 0.2 ~ 0.3).
3. evaluate the m_derived terminating half-section elements (for proper impedance matching, choose m=0.6).

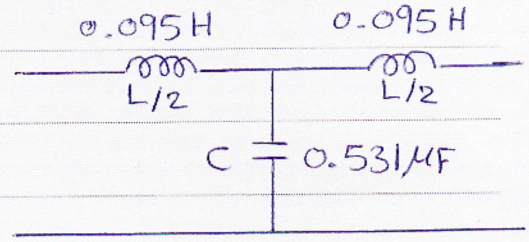
ex. Design a composite LPF to have a cutoff frequency of 1KHz and a characteristic impedance of 600Ω. Use one constant_k, T-section, one m-derived T-section and two terminating half sections with m=0.6. The frequency of infinite attenuation is 1050 Hz.

sol. For the constant_k section:

$$L = \frac{R_o}{\pi f_c} = \frac{600}{\pi(1000)} = 0.191 \text{ H}$$

ex. continued:

$$C = \frac{1}{\pi R_o f_c} = \frac{1}{\pi(600)(1000)} = 0.531\,\mu F$$



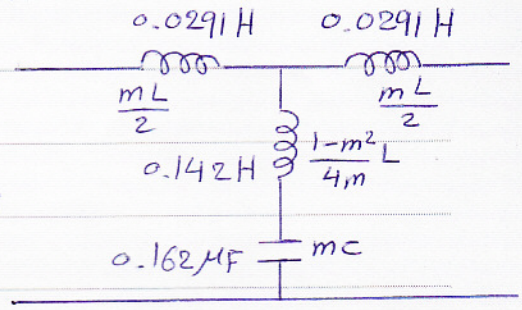
For the m-derived full section:

$$m = \sqrt{1 - \left(\frac{f_c}{f_\infty}\right)^2} = 0.305$$

then: $\frac{mL}{2} = \frac{1}{2}(0.305)(0.191) = 0.0291\ H$

$$\frac{1-m^2}{4m}L = \frac{1-(0.305)^2}{4(0.305)}(0.191) = 0.142\ H$$

$$mC = (0.305)(0.531 * 10^{-6}) = 0.162\,\mu F$$



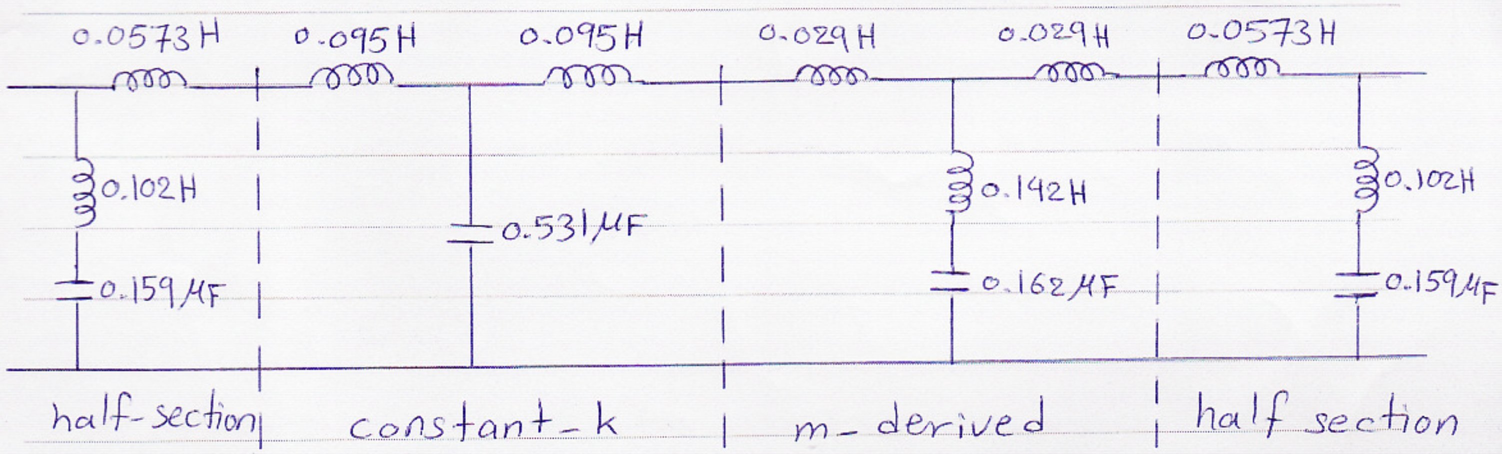
For each terminating half section:

$$\frac{mL}{2} = \frac{1}{2}(0.6)(0.191) = 0.0573\ H$$

$$\frac{1-m^2}{4m}L = \frac{1-(0.6)^2}{4(0.6)}(0.191)(2) = 0.102\ H$$

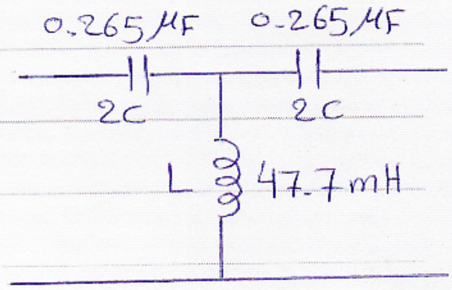
$$\frac{mC}{2} = \frac{1}{2}(0.6)(0.531 * 10^{-6}) = 0.159\,\mu F$$

So, the complete composite filter will be as shown:



ex. Design a composite T-section HPF to work into load of 600 Ω and cutoff frequency of 1 KHz. Terminate the filter properly.

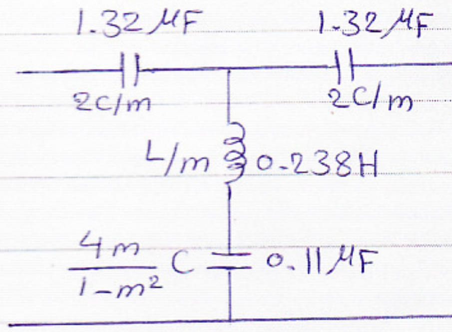
sol. For the constant-k section:



$$L = \frac{R_o}{4\pi f_c} = \frac{600}{4\pi(1000)} = 47.7\,mH$$

$$C = \frac{1}{4\pi R_o f_c} = \frac{1}{4\pi(600)(1000)} = 0.132\,\mu F$$

For the m-derived full section: choose m = 0.2



$$\frac{2C}{m} = 2\left(\frac{1}{0.2}\right)(0.132 * 10^{-6}) = 1.32\,\mu F$$

$$\frac{L}{m} = \frac{1}{(0.2)}(47.7 * 10^{-3}) = 0.238\,H$$

$$\frac{4m}{1-m^2}C = \frac{4(0.2)}{1-(0.2)^2}(0.132 * 10^{-6}) = 0.11\,\mu F$$

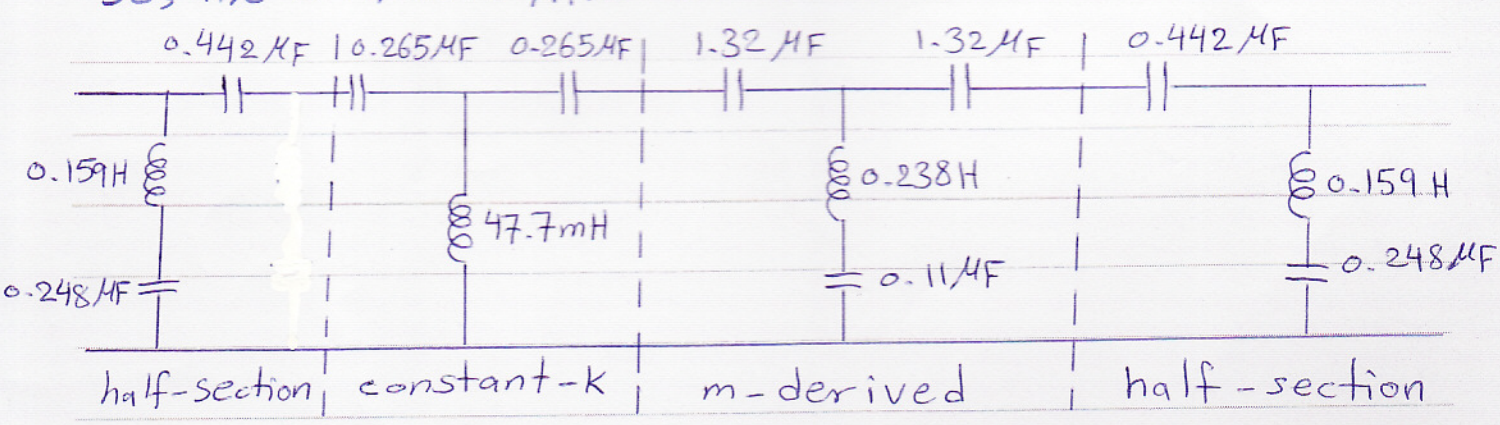
For the terminating half section: choose m = 0.6 for proper termination, then:

$$\frac{2C}{m} = 2\left(\frac{1}{0.6}\right)(0.132 * 10^{-6}) = 0.442\,\mu F$$

$$\frac{2L}{m} = 2\left(\frac{1}{0.6}\right)(47.7 * 10^{-3}) = 0.159\,H$$

$$\frac{4m}{1-m^2}\left(\frac{C}{2}\right) = \frac{4(0.6)}{1-(0.6)^2}\left(\frac{0.132 * 10^{-6}}{2}\right) = 0.248\,\mu F$$

So, the composite filter will be as shown:



Active Filters:

The passive filters have become almost obsolete due to many limitations, suchas:

1. A filter section consists of many components, so it becomes bulky.
2. Large size of inductors are required.
3. An external amplifier is required to provide suitable gain.

An active filter has an operational amplifier as the main element. The advantages of active filters are:

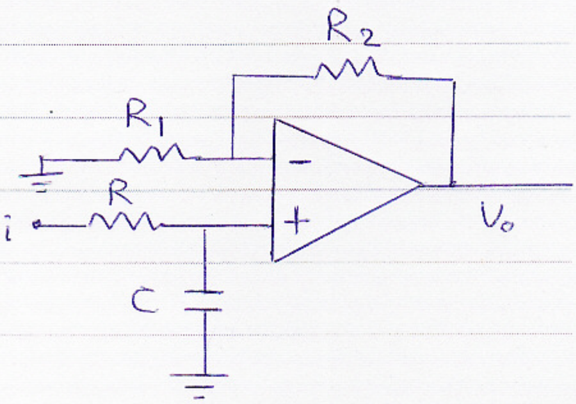
1. All elements are integrated, hence there is reduction in size and weight.
2. The cost is lower than passive components.
3. The inductors are absent in active filters.
4. Active filters can provide voltage gain.

Limitation of active filters:

1. Finite gain - bandwidth product place a limit on the highest frequency of operation (approximately up to 500 KHz, while passive filters can be used up to 500 MHz).
2. Sensitivity to temperature (thermal stability problems).
3. Requirement of dc power supply.

First-order active LPF:

The resistances R_1 and R_2 set the gain of the filter at the passband.



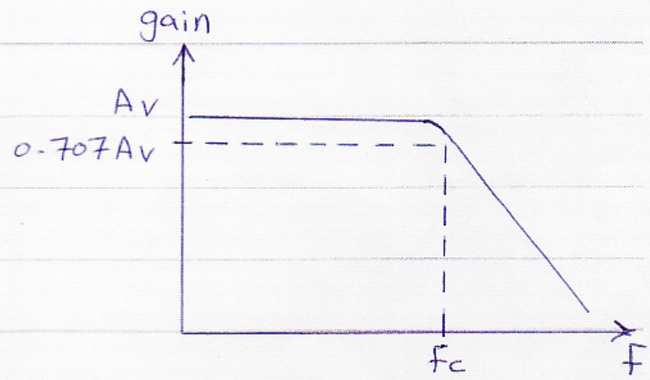
first-order active LPF

$$f_c = \frac{1}{2\pi RC}$$

$$A_v = \frac{V_o}{V_i} = 1 + \frac{R_2}{R_1} \quad \text{(passband gain)}$$

$$\left| \frac{V_o}{V_i} \right| = \frac{A_v}{\sqrt{1 + \left(\frac{f}{f_c}\right)^2}}$$

$$\phi = -\tan^{-1}\left(\frac{f}{f_c}\right) \quad \text{(degree)} \quad \text{(phase angle)}$$



frequency response of LPF

ex. Design an active LPF with $f_c = 15.9\ KHz$ and passband gain of 1.5

sol. Choose C usually between 1 nF and 1 μF, so let C = 1 nF

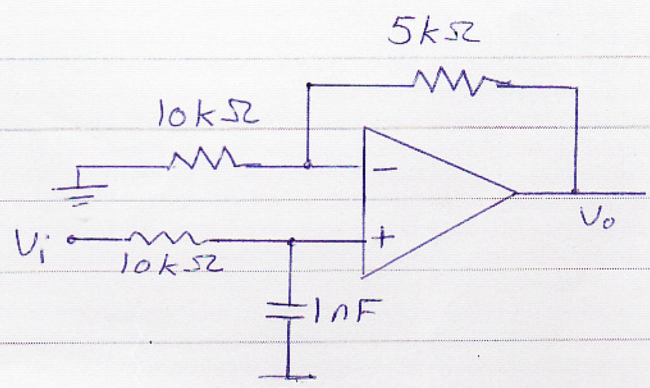
$$f_c = \frac{1}{2\pi RC} \Longrightarrow R = \frac{1}{2\pi(1*10^{-9})(15.9*10^{3})} = 10\,k\Omega$$

$$A_v = 1 + \frac{R_2}{R_1}$$

Let $R_1 = 10\,k\Omega$

$$\therefore 1.5 = 1 + \frac{R_2}{10*10^3}$$

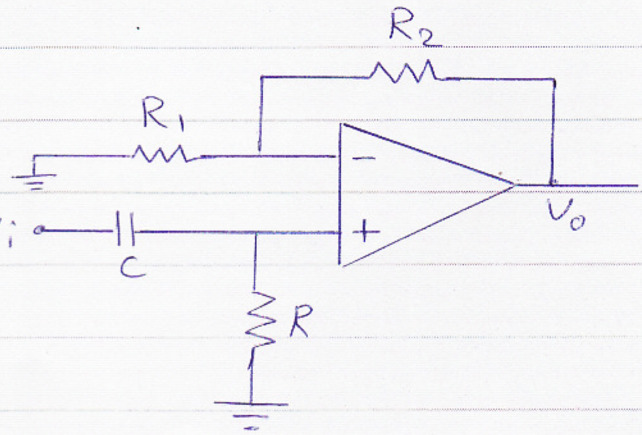
$$\therefore R_2 = 5\,k\Omega$$



First_order active HPF:

$$f_c = \frac{1}{2\pi RC}$$

$$A_v = 1 + \frac{R_2}{R_1}$$



First-order active HPF

ex. A first-order active HPF with $R = 1k\Omega$, $C = 0.02\,\mu F$ $R_1 = 10k\Omega$; $R_2 = 5k\Omega$. Find: a) midband gain b) gain at f_c ; c) cutoff frequency.

sol.

a) $A_v = 1 + \frac{R_2}{R_1} = 1 + \frac{5000}{10\,000} = 1.5$

b) A_v at $f_c = 0.707\,A_v = (0.707)(1.5) = 1.06$

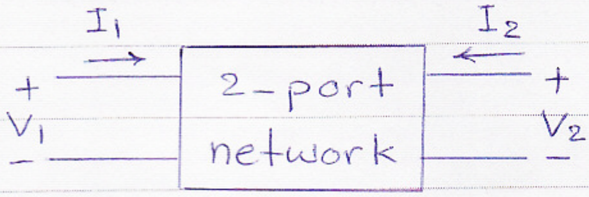
c) $f_c = \frac{1}{2\pi RC} = \frac{1}{2\pi(1000)(0.02 * 10^{-6})} = 7.95\ kHz$

Two - port Networks:

A two - port network is a network in which a signal is fed at one pair of terminals and output obtained from another pair of terminals, such as transmission lines, filters, transformers and amplifiers.

The figure below shows a two - port network with four variables V_1, V_2, I_1, and I_2. Two of these are independent variables while the remaining two are dependent.

There are six different ways (each way consists of two equations) of describing the behaviour of two - port networks.



The six sets of equations are known as: impedance and admittance parameters, hybrid and inverse hybrid parameters, transmission and inverse transmission parameters.

1. Open circuit impedance parameters: (Z - parameters)

In this set, I_1 and I_2 are independent variables while V_1 and V_2 are dependent variables.

i.e. $V_1 = f(I_1, I_2)$; $V_2 = f(I_1, I_2)$

The defining equations are:

$$V_1 = Z_{11} I_1 + Z_{12} I_2 \quad \text{———} (1)$$

$$V_2 = Z_{21} I_1 + Z_{22} I_2 \quad \text{———} (2)$$

Z - parameters are defined as follows:

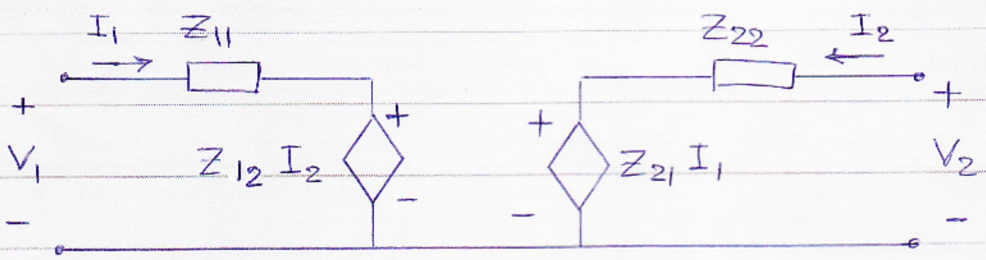
$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2 = 0}$$ open-circuit input impedance

$Z_{22} = \left.\dfrac{V_2}{I_2}\right|_{I_1=0}$ open-circuit output impedance

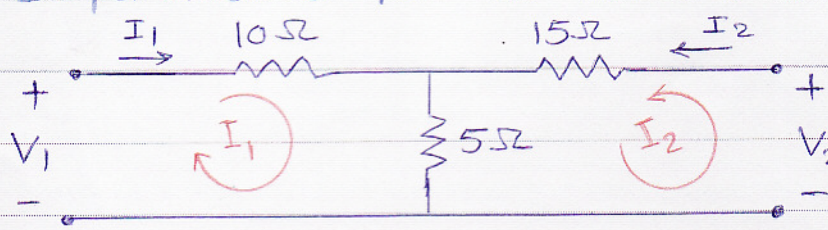
$Z_{12} = \left.\dfrac{V_1}{I_2}\right|_{I_1=0}$ open-circuit forward transfer impedance

$Z_{21} = \left.\dfrac{V_2}{I_1}\right|_{I_2=0}$ open-circuit reverse transfer impedance

The equivalent circuit of a two-port network in terms of open-circuit impedance parameters according to equations ① and ② is as shown in figure.



ex. Find the z-parameters for the network shown.



sol. Applying KVL on loops 1 and 2 yields:

$V_1 = 10 I_1 + 5(I_1 + I_2)$ ——①

$V_2 = 15 I_1 + 5(I_1 + I_2)$ ——②

from eq.①: let $I_1 = 0 \Rightarrow V_1 = 5 I_2 \Rightarrow Z_{12} = \dfrac{V_1}{I_2} = 5\ \Omega$

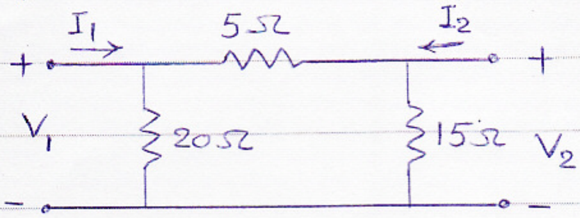
let $I_2 = 0 \Rightarrow V_1 = 15 I_1 \Rightarrow Z_{11} = \dfrac{V_1}{I_1} = 15\ \Omega$

from eq.②: let $I_1 = 0 \Rightarrow V_2 = 5 I_2 \Rightarrow Z_{22} = \dfrac{V_2}{I_2} = 5\ \Omega$

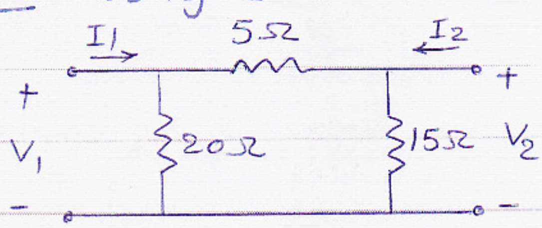
let $I_2 = 0 \Rightarrow V_2 = 20 I_1 \Rightarrow Z_{21} = \dfrac{V_2}{I_1} = 20\ \Omega$

$$\therefore [Z] = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ 20 & 5 \end{bmatrix}\ \Omega$$

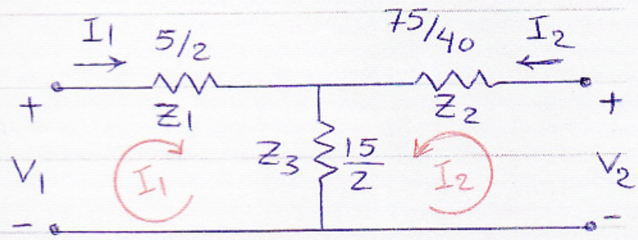
ex. Find the z-parameters for the circuit shown.



sol. Using Δ-Y conversion:



$\Rightarrow$



$$Z_1 = \frac{100}{40} = \frac{5}{2}\ \Omega \quad ; \quad Z_2 = \frac{75}{40}\ \Omega \quad ; \quad Z_3 = \frac{300}{40} = \frac{15}{2}\ \Omega$$

applying KVL:

$$V_1 = \frac{5}{2} I_1 + \frac{15}{2}(I_1 + I_2) \quad \text{——} ①$$

$$V_2 = \frac{75}{40} I_2 + \frac{15}{2}(I_1 + I_2) \quad \text{——} ②$$

from eq. ①: let $I_1 = 0 \Rightarrow V_1 = \frac{15}{2} I_2 \Rightarrow Z_{12} = \frac{V_1}{I_2} = \frac{15}{2}\ \Omega$

let $I_2 = 0 \Rightarrow V_1 = \frac{20}{2} I_1 \Rightarrow Z_{11} = \frac{V_1}{I_1} = 10\ \Omega$

from eq. ②: let $I_1 = 0 \Rightarrow V_2 = 9.375\ I_2 \Rightarrow Z_{22} = \frac{V_2}{I_2} = 9.375\ \Omega$

let $I_2 = 0 \Rightarrow V_2 = \frac{15}{2} I_1 \Rightarrow Z_{21} = \frac{V_2}{I_1} = \frac{15}{2}\ \Omega$

$$\therefore [Z] = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 10 & 7.5 \\ 7.5 & 9.375 \end{bmatrix} \Omega$$

2. Short circuit admittance parameters: (Y_ parameters)

In this set, V_1 and V_2 are independent variables while I_1 and I_2 are dependent variables.

i.e. $I_1 = f(V_1, V_2)$; $I_2 = f(V_1, V_2)$

The defining equations are:

$$I_1 = Y_{11} V_1 + Y_{12} V_2 \quad \text{———} (3)$$

$$I_2 = Y_{21} V_1 + Y_{22} V_2 \quad \text{———} (4)$$

Y_ parameters are defined as follows:

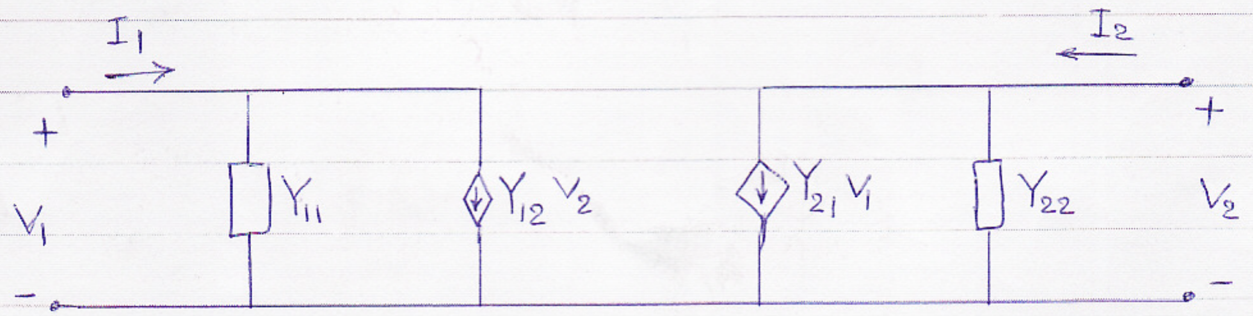
$Y_{11} = \frac{I_1}{V_1}\Big|_{V_2=0}$ short cct. i/p admittance

$Y_{12} = \frac{I_1}{V_2}\Big|_{V_1=0}$ short cct. reverse transfer admittance

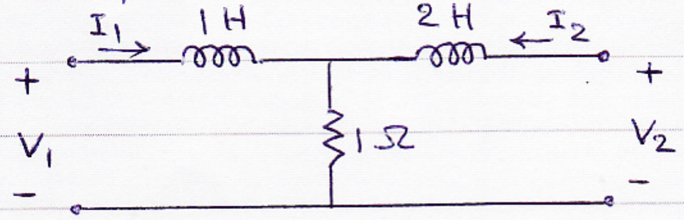
$Y_{21} = \frac{I_2}{V_1}\Big|_{V_2=0}$ short cct. forward transfer admittance

$Y_{22} = \frac{I_2}{V_2}\Big|_{V_1=0}$ short cct. o/p admittance

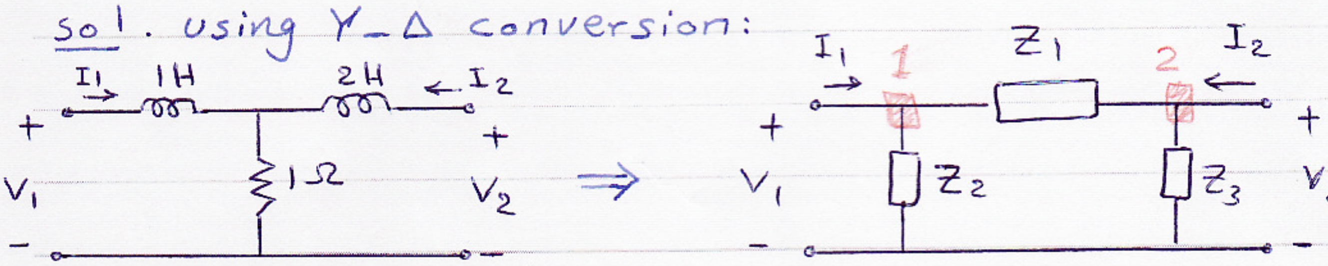
The equivalent circuit of a two port network in terms of Y_ parameters (according to eqs (3) and (4)) is as shown.



ex. Find the y-parameters for the circuit shown.



sol. using Y_Δ conversion:



$$Z_1 = j\omega + j2\omega + \frac{(j\omega)(j2\omega)}{1} = -2\omega^2 + j3\omega$$

$$Z_2 = j\omega + 1 + \frac{(j\omega)(1)}{j2\omega} = \frac{3}{2} + j\omega = \frac{3+j2\omega}{2}$$

$$Z_3 = j2\omega + 1 + \frac{(j2\omega)(1)}{j\omega} = 3 + j2\omega$$

Applying nodal analysis:

at node 1: $I_1 - \frac{V_1}{Z_2} - \frac{(V_1 - V_2)}{Z_1} = 0$

$$I_1 = \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right)V_1 - \frac{1}{Z_1}V_2 \quad \text{——} (1)$$

at node 2: $I_2 - \frac{V_2}{Z_3} - \frac{(V_2 - V_1)}{Z_1} = 0$

$$I_2 = -\frac{1}{Z_1}V_1 + \left(\frac{1}{Z_1} + \frac{1}{Z_3}\right)V_2 \quad \text{——} (2)$$

from eq. (1): let $V_1 = 0 \Rightarrow Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -\frac{1}{Z_1}$

let $V_2 = 0 \Rightarrow Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{1}{Z_1} + \frac{1}{Z_2}$

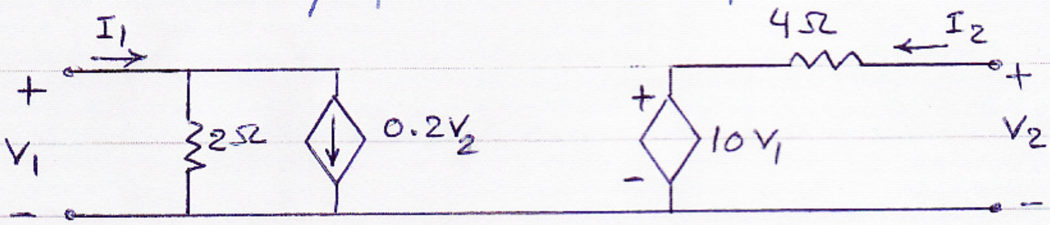
from eq. (2): let $V_1 = 0 \Rightarrow Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{1}{Z_1} + \frac{1}{Z_3}$

let $V_2 = 0 \Rightarrow Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = -\frac{1}{Z_1}$

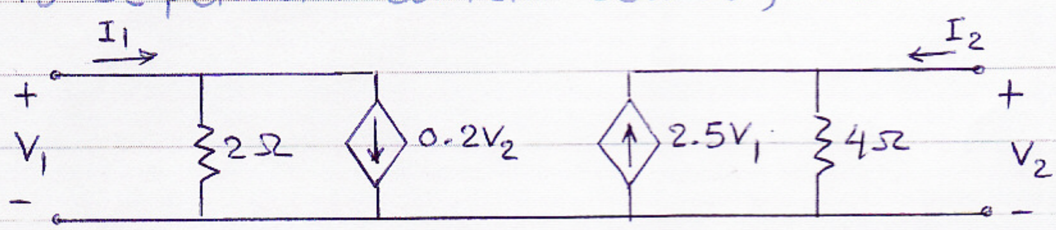
ex. continued:

$$\therefore [Y] = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{1+j2\omega}{-2\omega^2+j3\omega} & \dfrac{-1}{-2\omega^2+j3\omega} \\ \dfrac{-1}{-2\omega^2+j3\omega} & \dfrac{1+j\omega}{-2\omega^2+j3\omega} \end{bmatrix} \mho$$

ex. Find the y-parameters of the network shown.



sol. converting the dependent voltage source $(10V_1)$ into dependent current source; then:



Comparing the resulting network with the y-parameters equivalent circuit, then:

$Y_{11} = \frac{1}{2} = 0.5\ \mho$; $Y_{12} = 0.2\ \mho$

$Y_{21} = -2.5\ \mho$; $Y_{22} = \frac{1}{4} = 0.25\ \mho$

$$\therefore [Y] = \begin{bmatrix} 0.5 & 0.2 \\ -2.5 & 0.25 \end{bmatrix} \mho$$

م/ يوجد حل اخر بتحويل مصدر التيار الى مصدر فولتية ثم نأخذ KVL

3. Hybrid parameters:

In this set, I_1 and V_2 are independent variables while I_2 and V_1 are dependent variables.

i.e. $V_1 = f(I_1, V_2)$; $I_2 = f(I_1, V_2)$

The defining equations are:

$$V_1 = h_{11} I_1 + h_{12} V_2 \qquad \text{—— (5)}$$

$$I_2 = h_{21} I_1 + h_{22} V_2 \qquad \text{—— (6)}$$

h-parameters are defined as follows:

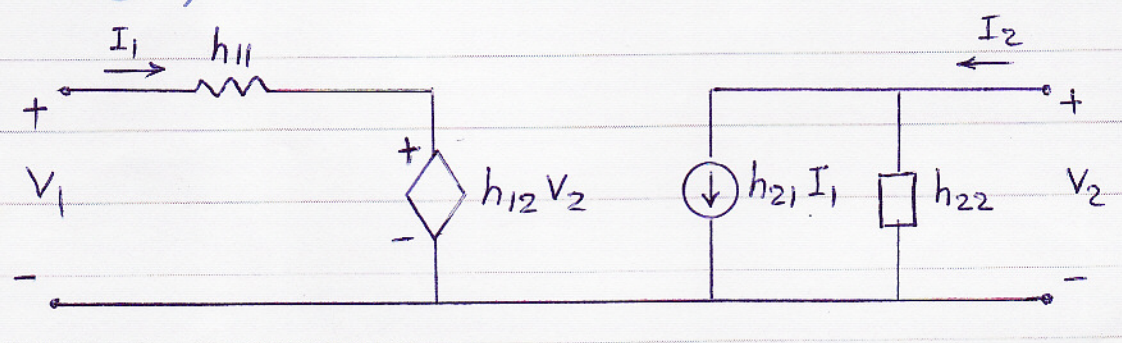
$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}$ short cct. i/p impedance

$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}$ open cct. reverse voltage gain

$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}$ short cct. forward current gain

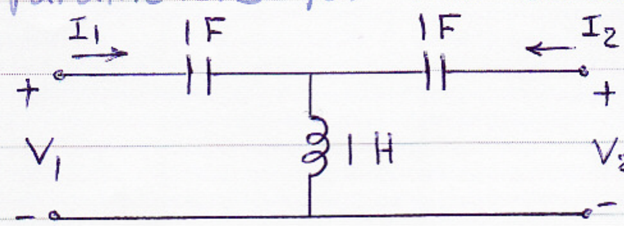
$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0}$ open cct. o/p admittance

The equivalent circuit of a two-port network in terms of hybrid parameters (according to eqs. (5) and (6)) is as shown.

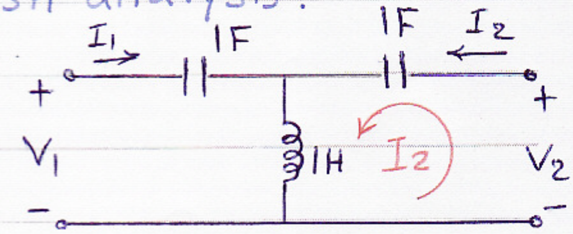


لإيجاد معاملات معادلة (5) نحوّل الدائرة الى π ثم نحل nodal

لإيجاد معاملات معادلة (6) نحوّل الدائرة الى T ثم نحل mesh ex. Find h-parameters for the circuit shown.



sol. since it is T-network, then we could find h_{21} and h_{22} directly by applying mesh analysis:



$$V_2 - \frac{I_2}{j\omega} - (I_1+I_2)\,j\omega = 0$$

$$V_2 - j\omega I_1 = \left(\frac{1}{j\omega} + j\omega\right) I_2$$

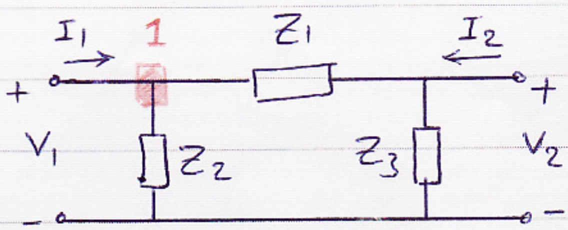
let $j\omega = s$, then: $V_2 - sI_1 = \left(\frac{1}{s} + s\right) I_2$

$$V_2 - sI_1 = \left(\frac{1+s^2}{s}\right) I_2 \Rightarrow I_2 = \left(\frac{s}{1+s^2}\right)V_2 - \left(\frac{s^2}{1+s^2}\right) I_1$$

$$\therefore\ h_{21} = -\frac{s^2}{1+s^2} = \frac{\omega^2}{1-\omega^2}$$

$$h_{22} = \frac{s}{1+s^2} = \frac{j\omega}{1-\omega^2}\ \mho$$

To find h_{11} and h_{12}, convert the network into π, and apply nodal analysis.



again let $j\omega = s$:

$$Z_1 = \frac{1}{s} + \frac{1}{s} + \frac{1/s^2}{s} = \frac{1+2s^2}{s^3}$$

$$Z_2 = \frac{1}{s} + s + \frac{1}{1/s} = \frac{1+2s^2}{s}$$

$$Z_3 = \frac{1}{s} + s + \frac{1}{1/s} = \frac{1+2s^2}{s}$$

$$I_1 - \frac{V_1}{Z_2} - \frac{(V_1 - V_2)}{Z_1} = 0$$

$$I_1 + \frac{V_2}{Z_1} = \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right) V_1$$

ex. continued: $I_1 + \frac{s^3}{1+2s^2} V_2 = \left(\frac{s^3}{1+2s^2} + \frac{s}{1+2s^2} \right) V_1$

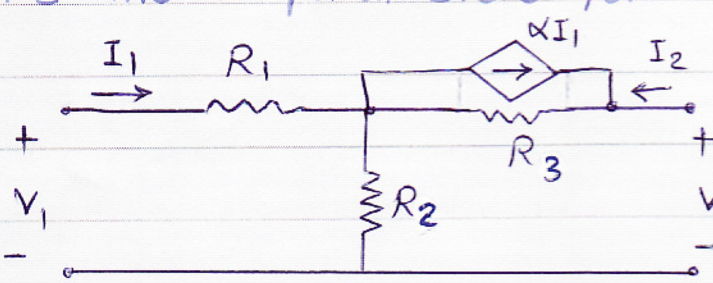
$$I_1 + \frac{s^3}{1+2s^2} V_2 = \left(\frac{s^3+s}{1+2s^2} \right) V_1$$

$$V_1 = \left(\frac{1+2s^2}{s^3+s} \right) I_1 + \frac{s^2}{s^2+1} V_2$$

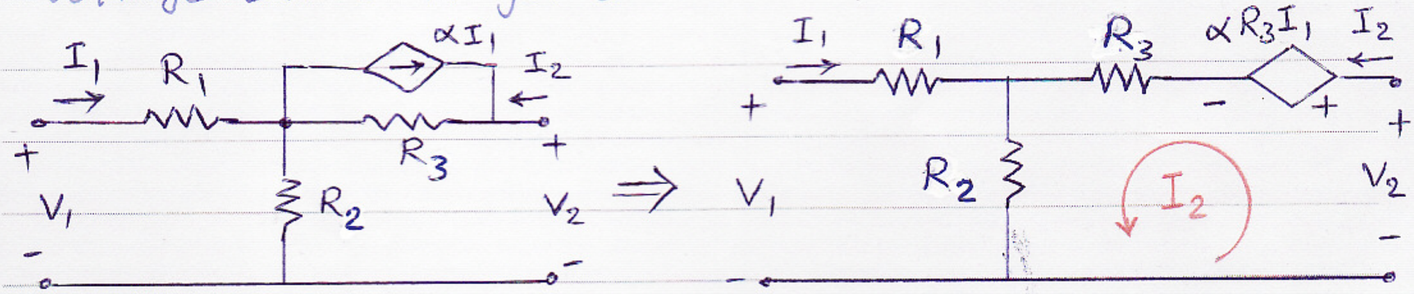
$$\therefore\ h_{11} = \frac{1+2s^2}{s^3+s} = \frac{1-2\omega^2}{-j\omega^3+j\omega}\ \Omega$$

$$h_{12} = \frac{s^2}{s^2+1} = \frac{-\omega^2}{-\omega^2+1}$$

ex. Find the h-parameters for the network shown.



sol. Convert the dependent current source into dependent voltage source to get a T-network.



Applying mesh analysis:

$$V_2 - \alpha R_3 I_1 - R_3 I_2 - R_2 (I_1 + I_2) = 0$$

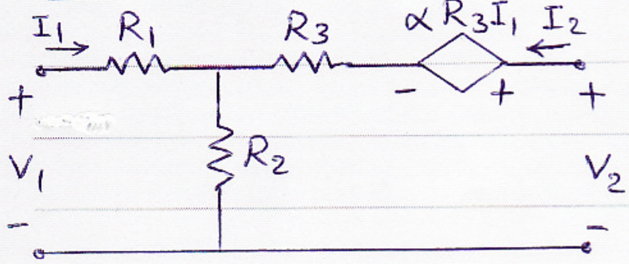
$$V_2 - (\alpha R_3 + R_2) I_1 = (R_2 + R_3) I_2$$

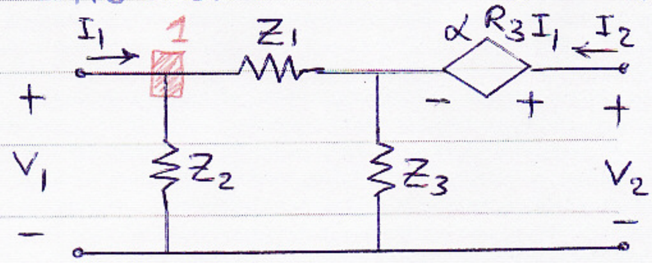
$$I_2 = \frac{1}{R_2+R_3} V_2 - \left(\frac{\alpha R_3 + R_2}{R_2 + R_3} \right) I_1$$

$$\therefore\ h_{21} = \frac{I_2}{I_1}\Big|_{V_2=0} = -\left(\frac{\alpha R_3 + R_2}{R_2+R_3} \right)$$

$$h_{22} = \frac{I_2}{V_2}\Big|_{I_1=0} = \frac{1}{R_2+R_3}\ \mho$$

ex. continued: convert the T-network into π-network





$$Z_1 = R_1 + R_3 + \frac{R_1 R_3}{R_2} = \frac{R_1R_2 + R_2R_3 + R_1R_3}{R_2}$$

$$Z_2 = R_1 + R_2 + \frac{R_1 R_2}{R_3} = \frac{R_1R_3 + R_2R_3 + R_1R_2}{R_3}$$

$$Z_3 = R_2 + R_3 + \frac{R_2 R_3}{R_1} = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_1}$$

Apply nodal analysis:

$$I_1 - \frac{V_1}{Z_2} - \frac{1}{Z_1}\left(V_1 - (V_2 - \alpha R_3 I_1)\right) = 0$$

$$I_1 - \frac{V_1}{Z_2} - \frac{V_1}{Z_1} + \frac{V_2}{Z_1} - \frac{\alpha R_3 I_1}{Z_1} = 0$$

$$\left(1 - \frac{\alpha R_3}{Z_1}\right) I_1 + \frac{V_2}{Z_1} = \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right) V_1$$

$$V_1 = \left(\frac{Z_1 Z_2}{Z_1 + Z_2}\right)\left(1 - \frac{\alpha R_3}{Z_1}\right) I_1 + \left(\frac{Z_2}{Z_1 + Z_2}\right) V_2$$

$$\therefore\ h_{11} = \frac{V_1}{I_1}\bigg|_{V_2=0} = \left(\frac{Z_1 Z_2}{Z_1 + Z_2}\right)\left(1 - \frac{\alpha R_3}{Z_1}\right)$$

$$= \frac{1}{R_3 + R_2}\left(R_1R_2 + R_1R_3 + R_2R_3(1-\alpha)\right)\ \Omega$$

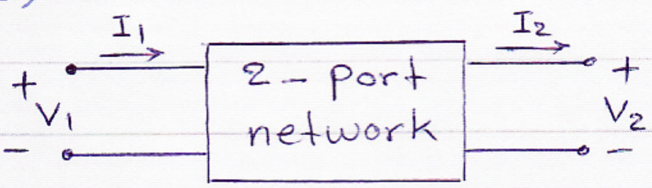
$$h_{12} = \frac{V_1}{V_2}\bigg|_{I_1=0} = \frac{Z_2}{Z_1 + Z_2} = \frac{R_2}{R_2 + R_3}$$

4. Transmission parameters: (ABCD parameters)

In this set, V_2 and I_2 are independent variables while V_1 and I_1 are dependent variables.

i.e. $V_1 = f(V_2, -I_2)$; $I_1 = f(V_2, -I_2)$

The negative sign of I_2 indicates that (only for ABCD parameters) the current I_2 is leaving port 2.



The defining equations are:

$$V_1 = A V_2 - B I_2 \qquad \text{—— (7)}$$

$$I_1 = C V_2 - D I_2 \qquad \text{—— (8)}$$

ABCD parameters are defined as follows:

$A = \left.\dfrac{V_1}{V_2}\right|_{I_2=0}$ open cct. reverse voltage gain

$B = \left.\dfrac{V_1}{-I_2}\right|_{V_2=0}$ short cct. transfer impedance

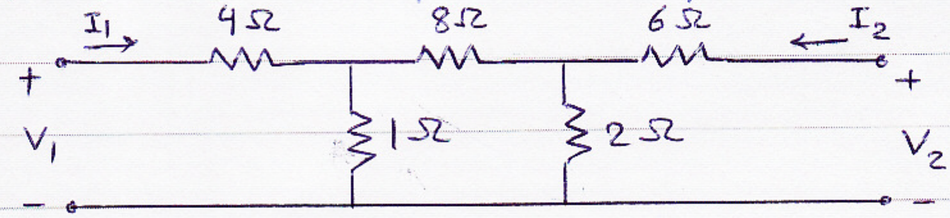
$C = \left.\dfrac{I_1}{V_2}\right|_{I_2=0}$ open cct. transfer admittance

$D = \left.\dfrac{I_1}{-I_2}\right|_{V_2=0}$ short cct. reverse current gain

م/ اذا كان تيار I_2 داخل للدائرة فأن المعادلات تكون:

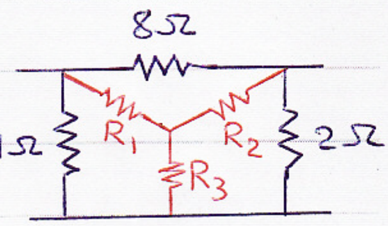
$V_1 = A V_2 + B I_2$; $I_1 = C V_2 + D I_2$

ex. Find the transmission parameters for the network shown.



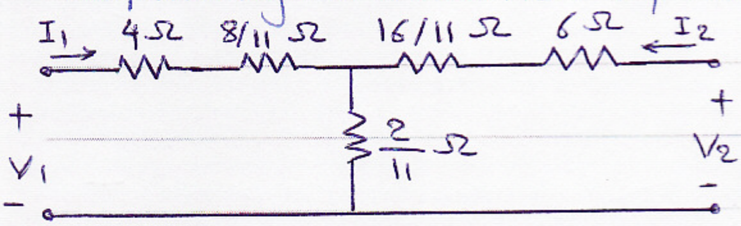
sol. Converting π- section resistances (1, 8 and 2Ω) into its equivalent T- section, then:

$$R_1 = \frac{1*8}{1+8+2} = \frac{8}{11}\ \Omega \ ; R_2 = \frac{8*2}{1+8+2} = \frac{16}{11}\ \Omega$$

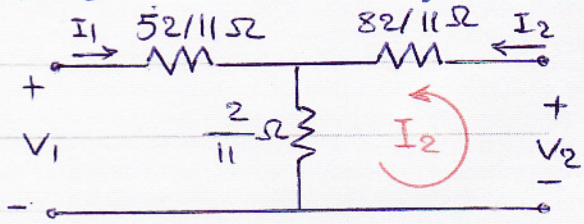


$$R_3 = \frac{1*2}{1+8+2} = \frac{2}{11}\ \Omega$$

Replacing π - section by its equivalent T- section, then:



⇒



To find C and D apply mesh analysis on loop 2 of the T- section:

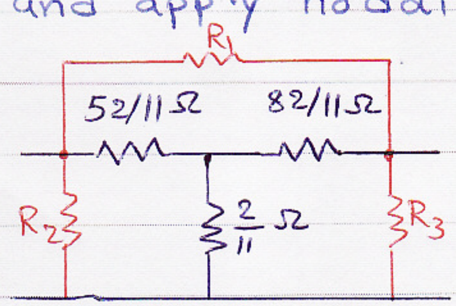
$$V_2 - \frac{82}{11} I_2 - \frac{2}{11}(I_1 + I_2) = 0 \Rightarrow \frac{2}{11} I_1 = V_2 - \frac{84}{11} I_2$$

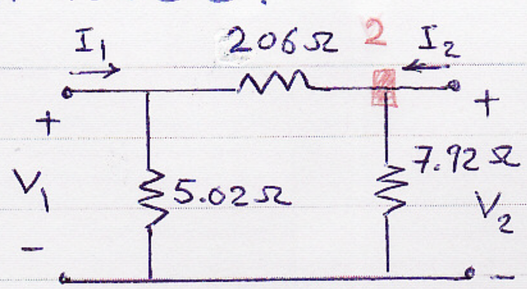
$$I_1 = \frac{11}{2} V_2 - 42 I_2$$

$$\therefore C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{11}{2}\ ℧$$

$$D = \left.\frac{I_1}{-I_2}\right|_{V_2=0} = 42$$

To find A and B, convert the T- section into π- section and apply nodal analysis on node 2:





ex. continued: $R_1 = \frac{52}{11} + \frac{82}{11} + \frac{(52/11)(82/11)}{(2/11)} = 206\ \Omega$

$$R_2 = \frac{52}{11} + \frac{2}{11} + \frac{(52/11)(2/11)}{(82/11)} = 5.024\ \Omega$$

$$R_3 = \frac{82}{11} + \frac{2}{11} + \frac{(82/11)(2/11)}{(52/11)} = 7.923\ \Omega$$

$$I_2 - \frac{V_2}{7.923} - \frac{(V_2 - V_1)}{206} = 0 \Rightarrow \frac{V_1}{206} = \frac{V_2}{7.923} + \frac{V_2}{206} - I_2$$

$$V_1 = 27 V_2 - 206 I_2$$

$$\therefore \quad A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = 27$$

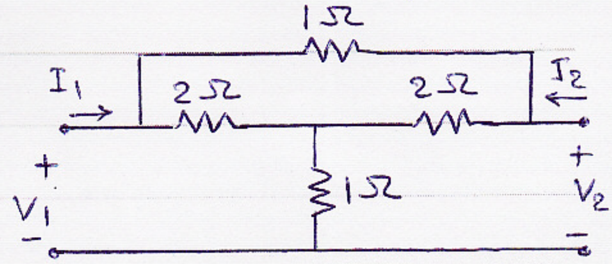
$$B = \left.\frac{V_1}{-I_2}\right|_{V_2=0} = 206\ \Omega$$

Interrelation between parameter sets:

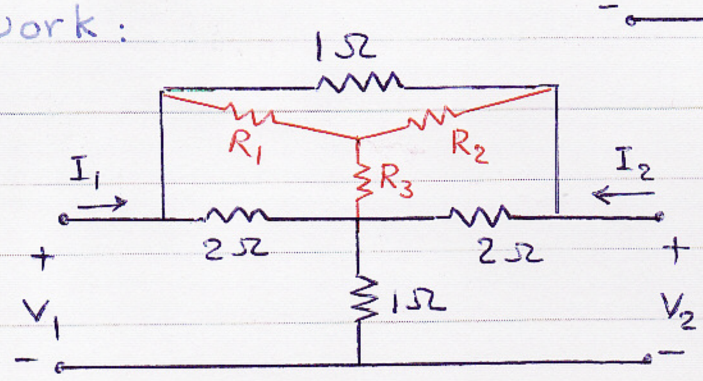
It is possible to express any parameter in terms of other parameters through a table of parameters conversion.

ملحق جدول التحويلات

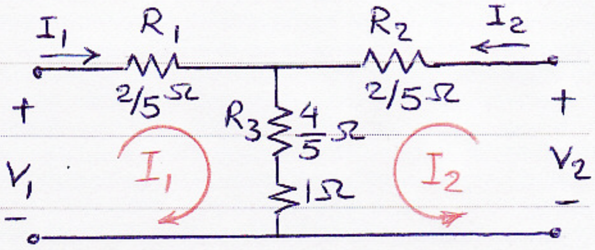
ex. For the bridged T-network shown: a) find Z-parameters b) y-parameters ; c) draw its π-network equivalent.



sol. a) Converting the bridged T-network into its equivalent T-network:



ex. continued:



$$R_1 = \frac{1*2}{1+2+2} = \frac{2}{5}\ \Omega$$

$$R_2 = \frac{1*2}{1+2+2} = \frac{2}{5}\ \Omega$$

$$R_3 = \frac{2*2}{1+2+2} = \frac{4}{5}\ \Omega$$

Apply mesh analysis, then:

$$V_1 - \frac{2}{5} I_1 - \frac{9}{5}(I_1 + I_2) = 0 \Rightarrow V_1 = \frac{11}{5} I_1 + \frac{9}{5} I_2 \quad \text{---} (*)$$

comparing eq. (*) with eq. (1):

$$Z_{11} = \frac{11}{5}\ \Omega \quad ; \quad Z_{12} = \frac{9}{5}\ \Omega$$

also: $V_2 - \frac{2}{5} I_2 - \frac{9}{5}(I_1 + I_2) = 0 \Rightarrow V_2 = \frac{9}{5} I_1 + \frac{11}{5} I_2$ --- (*)

comparing eq. (*) with eq. (2):

$$Z_{21} = \frac{9}{5}\ \Omega \quad ; \quad Z_{22} = \frac{11}{5}\ \Omega$$

$$\therefore\ [Z] = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 11/5 & 9/5 \\ 9/5 & 11/5 \end{bmatrix} \Omega$$

b) By using conversion table:

$$\Delta Z = Z_{11} Z_{22} - Z_{21} Z_{12} = \frac{11}{5} \cdot \frac{11}{5} - \frac{9}{5} \cdot \frac{9}{5} = \frac{40}{25} = \frac{8}{5}\ \Omega^2$$

$$Y_{11} = \frac{Z_{22}}{\Delta Z} = \frac{11/5}{8/5} = \frac{11}{8}\ \mho$$

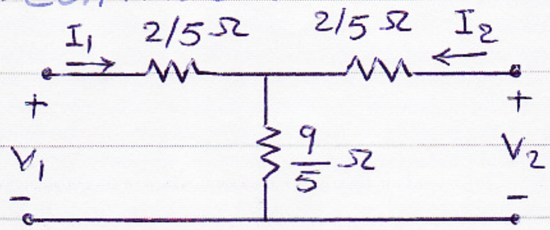
$$Y_{12} = -\frac{Z_{12}}{\Delta Z} = \frac{-9/5}{8/5} = -\frac{9}{8}\ \mho$$

$$Y_{21} = -\frac{Z_{21}}{\Delta Z} = \frac{-9/5}{8/5} = -\frac{9}{8}\ \mho$$

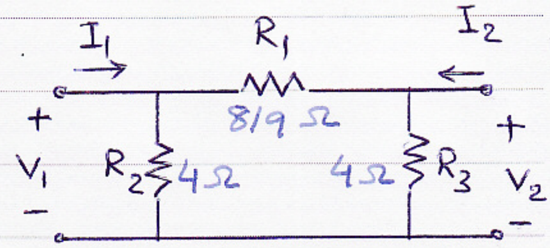
$$Y_{22} = \frac{Z_{11}}{\Delta Z} = \frac{11/5}{8/5} = \frac{11}{8}\ \mho$$

ex. continued:

c)



$\Rightarrow$



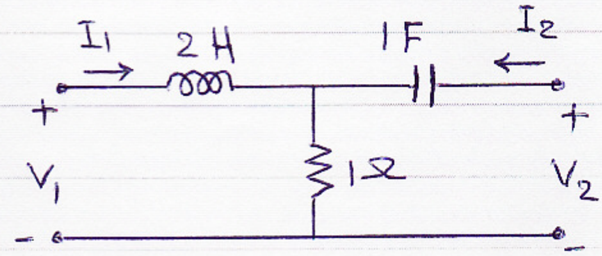
$$R_1 = \frac{2}{5} + \frac{2}{5} + \frac{4/25}{9/5} = \frac{8}{9}\ \Omega$$

equivalent π-network

$$R_2 = \frac{2}{5} + \frac{9}{5} + \frac{18/25}{2/5} = 4\ \Omega$$

$$R_3 = \frac{2}{5} + \frac{9}{5} + \frac{18/25}{2/5} = 4\ \Omega$$

ex. For the network shown, calculate: a) z-parameters b) transmission parameters.

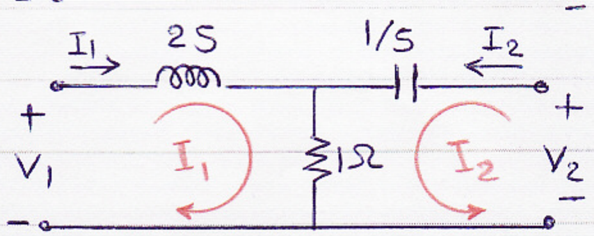


Sol. a)

let $j\omega = s$, then:

$$X_L = j\omega L = 2s\ \Omega$$

$$X_C = \frac{1}{j\omega C} = \frac{1}{s}\ \Omega$$



Applying mesh analysis:

$$V_1 - 2sI_1 - (I_1 + I_2) = 0 \Rightarrow V_1 = (2s+1)I_1 + I_2$$

$$\therefore\ z_{11} = 2s + 1 = 1 + j2\omega\ \Omega$$

$$z_{12} = 1\ \Omega$$

also: $$V_2 - \frac{I_2}{s} - (I_1 + I_2) = 0 \Rightarrow V_2 = I_1 + \left(1 + \frac{1}{s}\right) I_2$$

$$\therefore\ z_{21} = 1\ \Omega \quad ; \quad z_{22} = 1 + \frac{1}{s} = 1 + \frac{1}{j\omega}\ \Omega$$

b) using conversion table:

$$\Delta z = z_{11} z_{22} - z_{21} z_{12} = \frac{1 - 2\omega^2 + j2\omega}{j\omega}\ \Omega^2$$

$$A = \frac{z_{11}}{z_{21}} = \frac{1 + j2\omega}{1} = 1 + j2\omega$$

ex. continued:

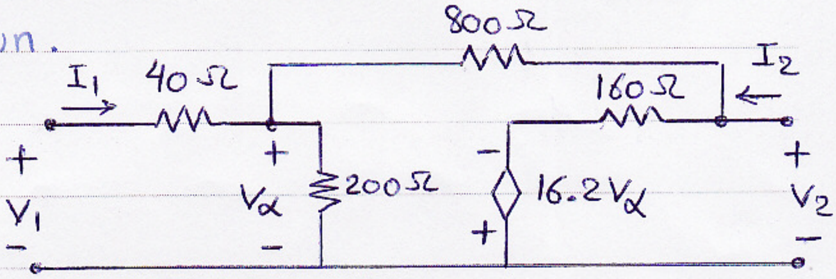
$$B = \frac{\Delta Z}{Z_{21}} = \frac{1 - 2\omega^2 + j2\omega}{j\omega} \ \Omega$$

$$C = \frac{1}{Z_{21}} = 1 \ \mho$$

$$D = \frac{Z_{22}}{Z_{21}} = 1 + j\omega$$

Further examples on two-port networks:

example One: Find the hybrid parameters for the Op-amp network shown.

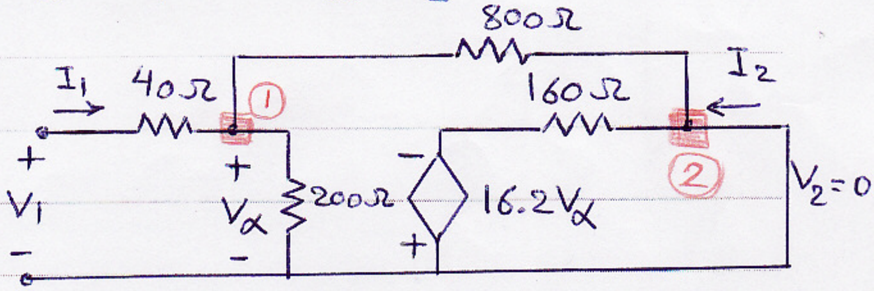


sol.

The defining equations for the h-parameters are:

$V_1 = h_{11} I_1 + h_{12} V_2$; $I_2 = h_{21} I_1 + h_{22} V_2$

– To find h_{11} and h_{21}, let $V_2 = 0$:



at node 1 :

V_2 اصلا

$$I_1 - \frac{V_\alpha}{200} - \left(\frac{V_\alpha - 0}{800}\right) = 0$$

$$I_1 - \frac{V_\alpha}{160} = 0 \implies V_\alpha = 160 I_1$$

$$\text{but } V_\alpha = V_1 * \frac{200 // 800}{(200 // 800) + 40} = 0.8 V_1$$

$$\therefore 0.8 V_1 = 160 I_1 \implies h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = 200 \ \Omega$$

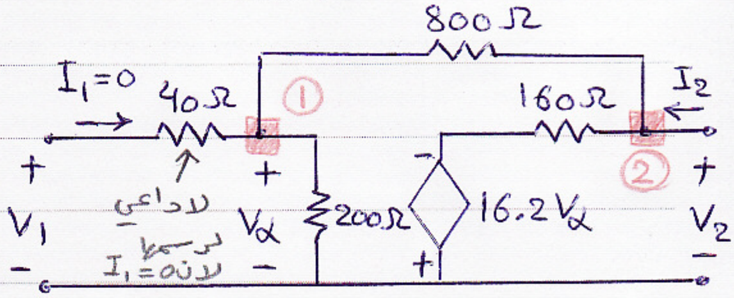
at node 2 :

$$I_2 - \left(\frac{0 - (-16.2 V_\alpha)}{160}\right) - \left(\frac{0 - V_\alpha}{800}\right) = 0$$

$$I_2 - \frac{V_\alpha}{10} = 0 \implies V_\alpha = 10 I_2$$

$$\therefore 160 I_1 = 10 I_2 \implies h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = 16$$

ex. continued:



To find h_{22} and h_{12}, let $I_1 = 0$:

at node 1:

$$-\frac{V_\alpha}{200} - \left(\frac{V_\alpha - V_2}{800}\right) = 0 \Rightarrow V_\alpha = \frac{V_2}{5}$$

هناكوجد $I_1 = 0$

but $V_\alpha = V_1 \Rightarrow V_1 = \frac{V_2}{5}$.

$$\therefore h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = 0.2$$

at node 2:

$$I_2 - \left(\frac{V_2-(-16.2V_\alpha)}{160}\right) - \left(\frac{V_2 - V_\alpha}{800}\right) = 0$$

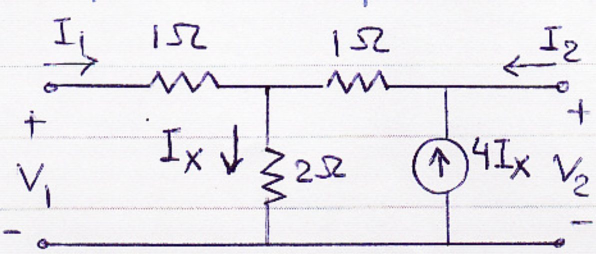
$$I_2 - \frac{V_2}{160} - \frac{16.2}{160}V_\alpha - \frac{V_2}{800} + \frac{V_\alpha}{800} = 0$$

$$I_2 - \frac{V_2}{160} - \frac{16.2}{160}\left(\frac{V_2}{5}\right) - \frac{V_2}{800} + \frac{1}{800}\left(\frac{V_2}{5}\right) = 0$$

$$\therefore I_2 = 0.0275\, V_2$$

$$\therefore h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = 0.0275\ \mho$$

Example Two: Find the ABCD parameters for the network shown:



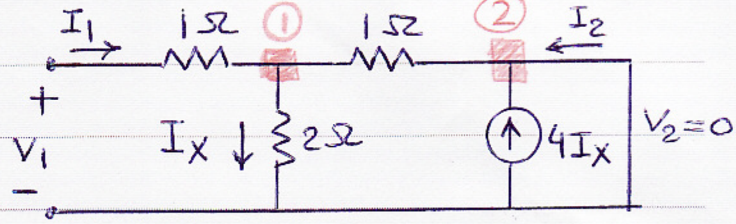
sol.

The defining equations for the ABCD parameters are:

$V_1 = A V_2 - B I_2$

$I_1 = C V_2 - D I_2$

ex. Continued:



- To find B and D, let $V_2=0$:

at node 1:

V2 اصلا

$$I_1 - I_x - \left(\frac{2I_x - 0}{1}\right) = 0$$

$$I_1 - 3I_x = 0 \Rightarrow I_x = \frac{1}{3} I_1$$

at node 2:

$$I_2 + 4I_x - \left(\frac{0 - 2I_x}{1}\right) = 0 \Rightarrow I_2 + 6I_x = 0$$

$$I_2 + 6\left(\frac{1}{3} I_1\right) = 0 \Rightarrow I_1 = -\frac{1}{2} I_2$$

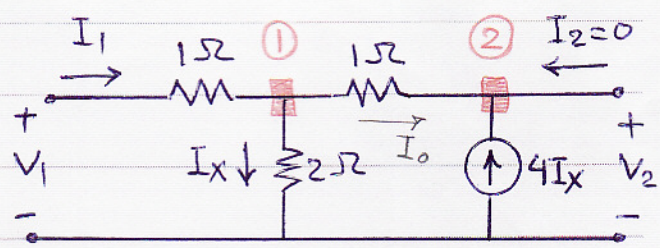
$$\therefore D = \left.\frac{I_1}{-I_2}\right|_{V_2=0} = \frac{1}{2}$$

Also: $I_x = \frac{1}{3} I_1 = \frac{1}{3}\left(\frac{-1}{2} I_2\right) = -\frac{1}{6} I_2$

$$V_1 - I_1 - 2I_x = 0 \qquad [\text{KVL at i/p loop}]$$

$$V_1 - \left(-\frac{1}{2} I_2\right) - 2\left(-\frac{1}{6} I_2\right) = 0 \Rightarrow V_1 = -0.833 I_2$$

$$\therefore B = \left.\frac{V_1}{-I_2}\right|_{V_2=0} = 0.833\ \Omega$$



- To find A and C, let $I_2=0$:

at node 2: $= I_o$

$$I_2 + 4I_x - \left(\frac{2I_x - V_2}{1}\right) = 0$$

0 = لأنو

$$2I_x + V_2 = 0 \Rightarrow I_x = -\frac{V_2}{2}$$

at node 1:

$$I_1 - I_x - \left(\frac{2I_x - V_2}{1}\right) = 0 \Rightarrow I_1 - 3I_x + V_2 = 0$$

$$I_1 - 3\left(-\frac{V_2}{2}\right) + V_2 = 0 \Rightarrow I_1 + \frac{5}{2} V_2 = 0$$

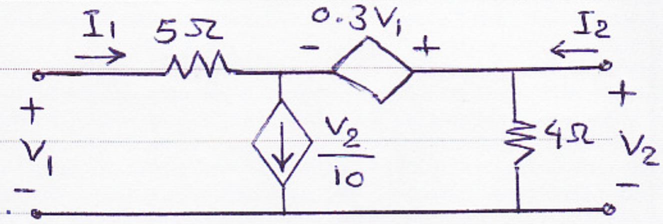
$$\therefore C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = -2.5\ \mho$$

also: $V_1 - I_1 - 2I_x = 0 \Rightarrow V_1 - (-2.5V_2) - 2\left(-\frac{V_2}{2}\right) = 0$

$$V_1 + 3.5V_2 = 0 \Rightarrow \therefore A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = -3.5$$

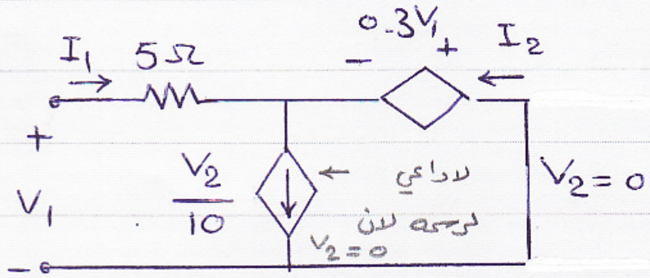
Example Three:

Find the transmission parameters for the network shown.



sol.

- To find B and D, let $V_2=0$:



$$I_2 + I_1 = \frac{V_2}{10}$$

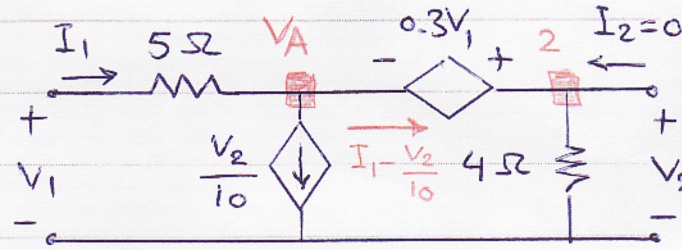
$$\therefore I_1 = -I_2$$

$$\therefore D = \frac{I_1}{-I_2}\Big|_{V_2=0} = 1$$

and $I_1 = \dfrac{V_1-(-0.3V_1)}{5} = \dfrac{1.3}{5}V_1 \Rightarrow -I_2 = \dfrac{1.3}{5}V_1$

$$\therefore B = \frac{V_1}{-I_2}\Big|_{V_2=0} = 3.8461\ \Omega$$

- To find A and C, let $I_2=0$:



at node 2:

$$\left(I_1 - \frac{V_2}{10}\right) + I_2 = \frac{V_2}{4}$$

$$I_1 = \frac{7}{20}V_2 \Rightarrow \therefore C = \frac{I_1}{V_2}\Big|_{I_2=0} = \frac{7}{20}\ \mho$$

Also:

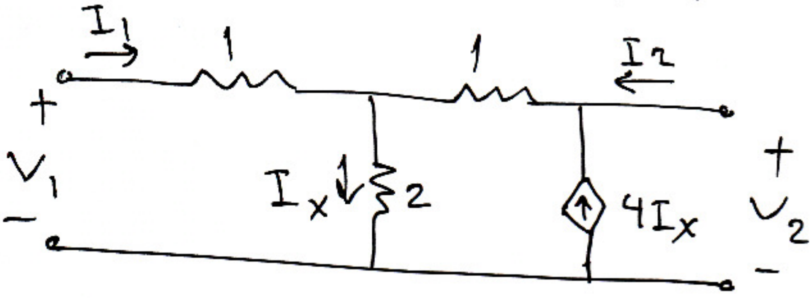
$$I_1 = \frac{V_1 - V_A}{5} = \frac{V_1 - (V_2 - 0.3V_1)}{5}$$

$$\frac{7}{20}V_2 = \frac{1.3V_1 - V_2}{5} \Rightarrow \frac{7}{4}V_2 = 1.3V_1 - V_2$$

$$\frac{11}{4}V_2 = 1.3V_1$$

$$\therefore A = \frac{V_1}{V_2}\Big|_{I_2=0} = 2.115$$

ex2. For the cct. shown, find y-parameters.



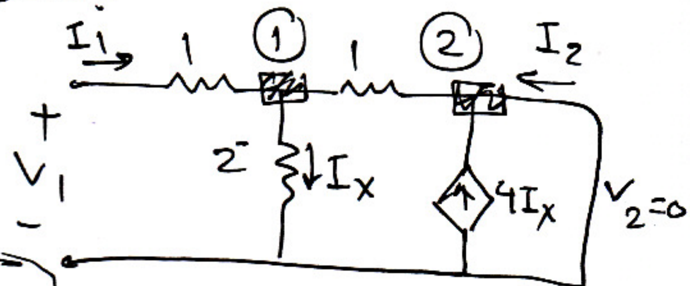
sol.

The defining eqs. are:

$$I_1 = y_{11} V_1 + y_{12} V_2$$

$$I_2 = y_{21} V_1 + y_{22} V_2$$

To find y_{11} and y_{21}, let $V_2 = 0$.



at node 1:

V_2 لانها

$$I_1 - I_x - \frac{(2I_x - 0)}{1} = 0$$

$$I_1 - 3I_x = 0 \Longrightarrow \boxed{I_x = \frac{I_1}{3}} \quad \text{—— (1)}$$

• at node 2: $I_2 + 4I_x - \frac{(0 - 2I_x)}{1} = 0$

$$I_2 + 6I_x = 0 \Longrightarrow \boxed{I_1 = -\frac{I_2}{2}} \quad \text{—— (2)}$$

Now, apply KVL on loop 1:

$$V_1 - (1) I_1 - 2I_x = 0$$

$$V_1 - I_1 - 2\left(\frac{I_1}{3}\right) = 0 \Longrightarrow V_1 = \frac{5}{3} I_1$$

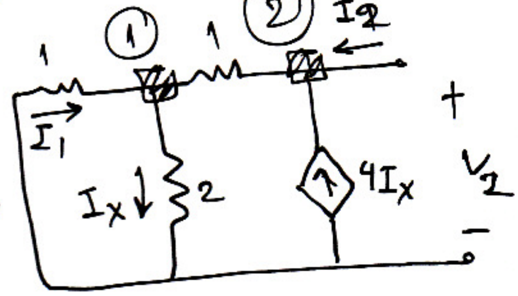
$$\therefore \; y_{11} = \frac{I_1}{V_1}\Big|_{V_2 = 0} = \boxed{0.6 \text{ S}}$$

ex. Continued:

also: $V_1 = \frac{5}{3}\left(-\frac{I_2}{2}\right)$

$\therefore Y_{21} = \frac{I_2}{V_1}\Big|_{V_2=0} = \boxed{-1.2\ S}$

To find Y_{12} and Y_{22}, let $V_1 = 0$:



at node 1:

$$I_1 - I_x - \left(\frac{2I_x - V_2}{1}\right) = 0$$

$$I_1 - 3I_x + V_2 = 0$$

but $-I_1 - 2I_x = 0 \Longrightarrow \boxed{I_x = -\frac{I_1}{2}}$ — 3

$\therefore I_1 - 3\left(-\frac{I_1}{2}\right) + V_2 = 0 \Longrightarrow \frac{5}{2} I_1 + V_2 = 0$

$\therefore Y_{12} = \frac{I_1}{V_2}\Big|_{V_1=0} = \boxed{-0.4\ S}$

at node 2:

$$I_2 + 4I_x - \frac{(V_2 - 2I_x)}{1} = 0$$

$$I_2 + 6I_x - V_2 = 0$$

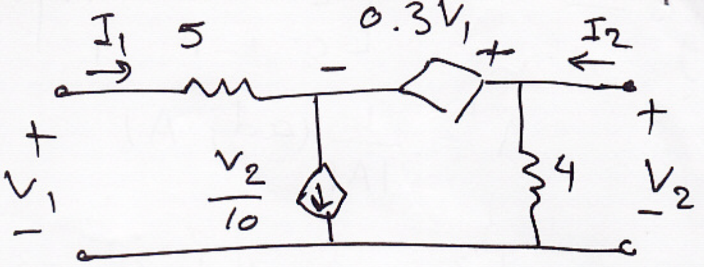
$I_2 + 6\left(-\frac{I_1}{2}\right) - V_2 = 0$ from 3

$I_2 - 3(-0.4V_2) - V_2 = 0$ from Y_{12}

$$I_2 + 0.2V_2 = 0$$

$\therefore Y_{22} = \frac{I_2}{V_2}\Big|_{V_1=0} = \boxed{-0.2\ S}$

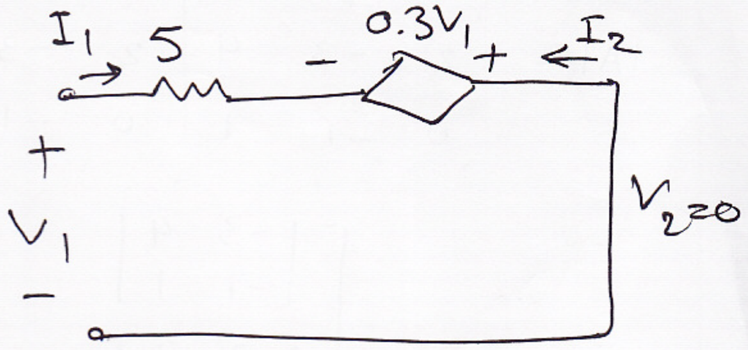
EX3. Find y-parameters



sol. To find y_{11} and y_{21},

let $V_2 = 0$:

apply KVL:



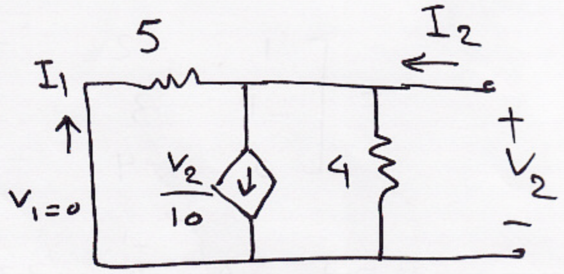
$$V_1 - 5I_1 + 0.3V_1 = 0$$

$$\therefore\ y_{11} = \frac{I_1}{V_1} = \boxed{0.26\ S}$$

and $I_1 = -I_2$

$$\therefore\ y_{21} = \frac{I_2}{V_1} = \boxed{-0.26\ S}$$

To find y_{12} and y_{22}, let $V_1 = 0$:



$$I_2 - \frac{V_2}{4} - \frac{V_2}{10} - \frac{(V_2 - 0)}{5} = 0$$

$$I_2 - 0.55V_2 = 0$$

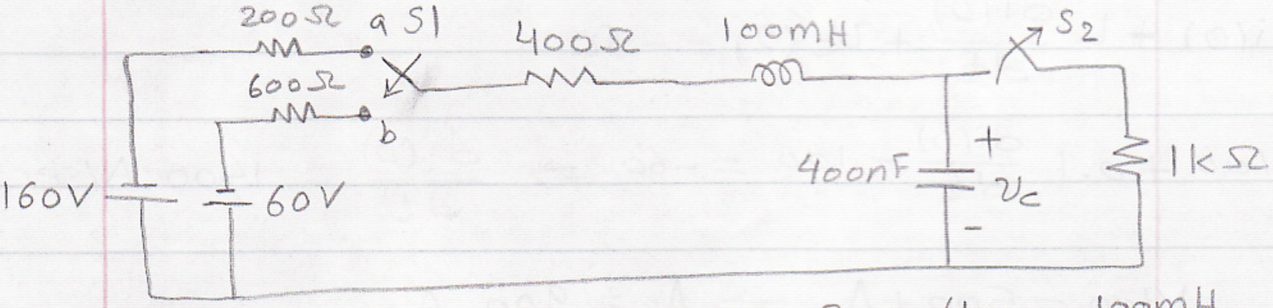
$$\therefore\ y_{22} = \frac{I_2}{V_2} = \boxed{0.55\ S}$$

and $V_2 = -5I_1$

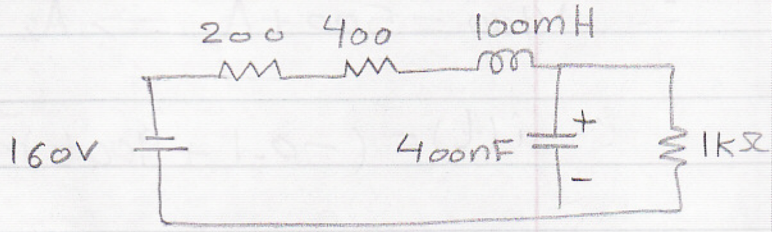
$$\therefore\ y_{12} = \boxed{-0.2\ S}$$

Further examples on transient response

Q1. The switches in the cct. shown operates synchronously. When switch 1 is in position (b), switch 2 is open. Switch 1 has been in position (a) for a long time. At $t=0$ it moves to position (b). Find $v_c(t)$ for $t>0$.



Sol. For $t<0$:

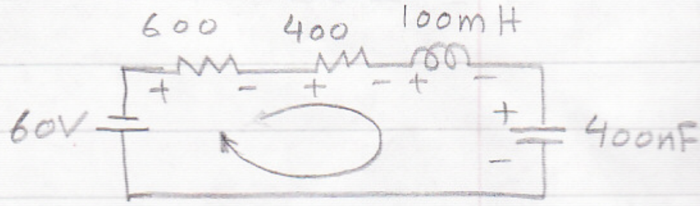


$$V_c(0^-) = -160\,\frac{1000}{1000+600}$$

$$= -100\text{ V}$$

$$i_L(0^-) = \frac{-160}{1000+600} = -0.1\text{ A}$$

For $t>0$:



$$\alpha = \frac{R}{2L} = \frac{1000}{0.2}$$

$$= 5000\text{ rad/sec}$$

$$\omega_n = \frac{1}{\sqrt{LC}} = 5000\text{ rad/sec}$$

$\because \alpha = \omega_n \Rightarrow \therefore$ The response is critically damped

$$i(t) = (A_1 + A_2 t)\,e^{-5000t}$$

$$i(0) = A_1 = -0.1$$

$$\therefore i(t) = (-0.1 + A_2 t)\,e^{-5000t}$$

also: $\dfrac{di(t)}{dt} = 500 e^{-5000t} - 5000 A_2 t e^{-5000t} + A_2 e^{-5000t}$

$\dfrac{di(0)}{dt} = 500 + A_2$ ——— (*)

$V_R + V_L + V_c = -60$

$R i(t) + L \dfrac{di(t)}{dt} + v_c(t) = -60$

$R i(0) + L \dfrac{di(0)}{dt} + v_c(0) = -60$

$-100 + 0.1 \dfrac{di(0)}{dt} - 100 = -60 \Rightarrow \dfrac{di(0)}{dt} = 1400$ A/sec

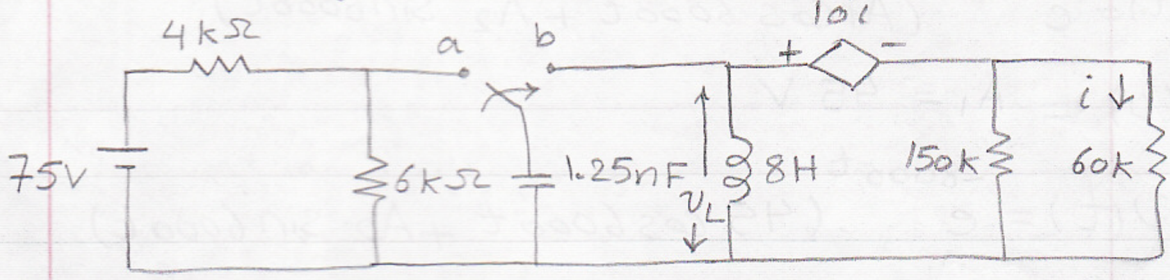
$\therefore 1400 = 500 + A_2 \Longrightarrow A_2 = 900$ A

$\therefore i(t) = (-0.1 + 900t) e^{-5000t}$

$v_c(t) = -60 - R i(t) - L \dfrac{di(t)}{dt}$

$= -60 - 40 e^{-5000t} - 450 \times 10^3 t e^{-5000t}$ volts

Q2. The switch in the cct. shown has been in position (a) for a long time. Find $v_L(t)$ for $t>0$.



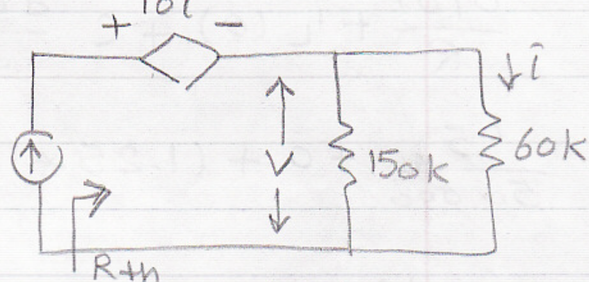
Sol. for $t<0$: $i_L(0^-)=0$

$$v_c(0^-) = 75 * \frac{6000}{4000+6000} = 45V$$

for $t>0$:
we should find Thévenin's equivalent resistance connected in parallel with the inductor and capacitor at $t>0$. we do this by connecting a 1A source as shown:

$$i = i_T \frac{150*10^3}{60*10^3+150*10^3}$$

$$= 1 * \frac{150}{60+150} = 0.714$$



∴ The voltage applied by the dependent source is

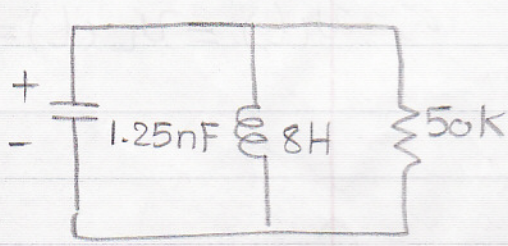
$10^4 i = 10^4(0.714) = 7142.85$ V

and its equivalent resistance is: $\frac{7142.85}{1} = 7142.85\Omega$

and: $R_{th} = 7142.85 + (150k // 60k) = 50k\Omega$

$$\alpha = \frac{1}{2RC} = 8000 \text{ rad/sec}$$

$$w_n = \frac{1}{\sqrt{LC}} = 10000 \text{ rad/sec}$$



∵ $\alpha < w_n$ ⇒ ∴ The response is underdamped

$\omega_d = \sqrt{\omega_n^2 - \alpha^2} = 6000$ rad/sec

$\therefore v(t) = e^{-8000t}(A_1 \cos 6000t + A_2 \sin 6000t)$

$v(0) = A_1 = 45$ V

$\therefore v(t) = e^{-8000t}(45 \cos 6000t + A_2 \sin 6000t)$

$$\frac{dv(t)}{dt} = 45(-8000e^{-8000t})\cos 6000t - 45e^{-8000t}(6000 \sin 6000t)$$
$$+ A_2(-8000e^{-8000t})\sin 6000t + A_2 e^{-8000t}(6000 \cos 6000t)$$

$$\frac{dv(0)}{dt} = -360\,000 + 6000 A_2$$

also: $I_R + I_L + I_C = 0 \Rightarrow \frac{v(t)}{R} + i_L(t) + C\frac{dv(t)}{dt} = 0$

$$\frac{v(0)}{R} + i_L(0) + C\frac{dv(0)}{dt} = 0$$

$$\frac{45}{50\,000} + 0 + (1.25 * 10^{-9})\frac{dv(0)}{dt} = 0$$

$$\therefore \frac{dv(0)}{dt} = -720\,000 \text{ rad/sec}$$

$\therefore -720\,000 = -360\,000 + 6000 A_2$

$A_2 = -60$ V

$\therefore v(t) = v_L(t) = e^{-8000t}(45 \cos 6000t - 60 \sin 6000t)$ V