Matrices:

Definition 1: A matrix is a rectangular array of numbers (or functions) enclosed in brackets. These numbers (or functions) are called entries (or elements) of the matrix.

For example:

① $$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1c} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2c} \\ \vdots & & & & \\ a_{r1} & a_{r2} & a_{r3} & \cdots & a_{rc} \end{bmatrix}$$

where M is a matrix of order $(r * c)$; r is number of rows; c is number of columns; a_{rc} is the entry of the matrix in the r'th row and c'th column.

② $$M = \begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}$$

where M is a square matrix (no. of rows = no. of columns) of order $(2*2)$ that has $a_{12} = 2x^2$.

③ $$M = \begin{bmatrix} 4 \\ \frac{1}{2} \\ -1 \end{bmatrix}$$

where M is a column matrix of order $(3*1)$ that has $a_{31} = -1$

④ $$M = \begin{bmatrix} 2 & 1 & -1 & 0 \end{bmatrix}$$

where M is a row matrix of order $(1*4)$.

Definition 2: A square matrix D is said to be a diagonal matrix if all its non-diagonal entries are zeros, that is:

$$D_{ij} = \begin{cases} 0 & i \neq j \\ \text{value} & i = j \end{cases}$$

For example:

① $$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

② A matrix $$L = \begin{bmatrix} 2 & 0 & 0 \\ 3 & -5 & 0 \\ -1 & 0 & 4 \end{bmatrix}$$ is called Lower Triangular matrix

③ A matrix $$U = \begin{bmatrix} 2 & 7 & 3 & 5 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ is called Upper Triangular matrix

Definition 3: A matrix M has a submatrix S which can be obtained from M by deleting certain rows and/or columns of M.

For example: if $$M = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

then the following is a submatrix of M:

$\text{Row}_2(M) = [5 \quad 6 \quad 7 \quad 8]$ the 2nd row of M

$\text{Col}_1(M) \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}$ the 1st column of M

$$\text{Min}_{22}(M) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 \\ 9 & 11 & 12 \end{bmatrix}$$ the minor of M of position (2,2)

The matrices can be obtained from the following:

1. System of linear algebraic equations.

ex. $2x + 6y - 3z = 2$

$4x + 3y - 9z = 11$

Associated with this system are the following matrices:

The coefficient matrix: $A = \begin{bmatrix} 2 & 6 & -3 \\ 4 & 3 & -9 \end{bmatrix}$

The input matrix: $k = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$

The output matrix: $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

The augmented matrix: $[A:k] = \begin{bmatrix} 2 & 6 & -3 & : 2 \\ 4 & 3 & -9 & : 11 \end{bmatrix}$

2. System of first order linear differential equations.

ex. Consider the system at which we use functions $y_1(t)$ and $y_2(t)$

$y_1'(t) - 3y_2'(t) = 4y_1(t) - 2y_2(t) + \cos(2t)$

$2y_1'(t) + 5y_2'(t) = 3y_1(t) + 7y_2(t) + 1$

$y_1(0) = 3 \;,\; y_2(0) = 0$

$A = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix}$; $B = \begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix}$ are coefficient matrices

$Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ is the output matrix

$F(t) = \begin{bmatrix} \cos 2t \\ 1 \end{bmatrix}$ is the input matrix

$Y(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ is the initial state matrix

$Y'(t) = \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix}$ is the derivative of output matrix

3. Analytic geometry and/or vector algebra.

Definition 4: Equality of matrices

Two matrices $A=[a_{jk}]$ and $B=[b_{jk}]$ are equal ($A=B$) if and only if they have the same size and the corresponding entries are equal, that is, $a_{11}=b_{11}$, $a_{12}=b_{12}$, and so on.

ex. $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B=\begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}$

then $A=B$ if and only if $a_{11}=4$, $a_{12}=0$, $a_{21}=3$, $a_{22}=-1$

Definition 5: Addition of matrices

If A and B are matrices of the same order $m*n$ then the sum $A \pm B$ is a matrix of order $m*n$ and has the entries $a_{jk} + b_{jk}$ obtained by adding the corresponding entries of A and B. (where $j=1,2,\dots m$ and $k=1,2,\dots n$).

ex. $A=\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 4 & 5 & 6 \end{bmatrix}$; $B=\begin{bmatrix} 2 & 0 & 1 \\ 9 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$$A+B=\begin{bmatrix} 1+2 & 2+0 & -1+1 \\ 0+9 & 3+3 & 1+2 \\ 4+0 & 5+0 & 6+1 \end{bmatrix}=\begin{bmatrix} 3 & 2 & 0 \\ 9 & 6 & 3 \\ 4 & 5 & 7 \end{bmatrix}$$

$$A-B=\begin{bmatrix} 1-2 & 2-0 & -1-1 \\ 0-9 & 3-3 & 1-2 \\ 4-0 & 5-0 & 6-1 \end{bmatrix}=\begin{bmatrix} -1 & 2 & -2 \\ -9 & 0 & -1 \\ 4 & 5 & 5 \end{bmatrix}$$

properties of addition:

1. $A+B = B+A$
2. $A + (B+C) = (A+B) + C$
3. If $A + B = A + C$ then $B = C$
4. $A + 0 = A$
5. $A + (-A) = 0$

Definition 6: Scalar Multiplication

If A is a matrix of order $m \times n$ then $cA = [c\, a_{jk}]$ where c is any scalar (number).

ex.

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ then $0A = \begin{bmatrix} 0\times1 & 0\times2 \\ 0\times3 & 0\times4 \\ 0\times5 & 0*6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$5A = \begin{bmatrix} 5\times1 & 5\times2 \\ 5\times3 & 5\times4 \\ 5\times5 & 5*6 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 15 & 20 \\ 25 & 30 \end{bmatrix}$$

properties of scalar multiplication:

1. $c(A+B) = cA + cB$
2. $(c+k)A = cA + kA$
3. $c(kA) = (ck)A$
4. $1 \cdot A = A$

Definition 7: Matrix Multiplication

Let A be an $(m \times n)$ and B be an $(n \times p)$ matrices, then the product $A \times B$ is an $(m \times p)$ matrix whose entries are defined by:

$$ent_{ij}(A \times B) = Row_i(A) \times Col_j(B)$$

$$= a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

ex. if $A=\begin{bmatrix} 2 & -1 & 0 \\ 4 & 3 & -2 \end{bmatrix}$, $B=\begin{bmatrix} -4 & 0 \\ 6 & -4 \\ 1 & 6 \end{bmatrix}$

$$AB=\begin{bmatrix} [2\ -1\ 0]\begin{bmatrix} -4 \\ 6 \\ 1 \end{bmatrix} & [2\ -1\ 0]\begin{bmatrix} 0 \\ -4 \\ 6 \end{bmatrix} \\ [4\ 3\ -2]\begin{bmatrix} -4 \\ 6 \\ 1 \end{bmatrix} & [4\ 3\ -2]\begin{bmatrix} 0 \\ -4 \\ 6 \end{bmatrix} \end{bmatrix}$$

$$=\begin{bmatrix} -8-6+0 & 0+4+0 \\ -16+18-2 & 0-12-12 \end{bmatrix}=\begin{bmatrix} -14 & 4 \\ 0 & -24 \end{bmatrix}$$

Similarly $BA=\begin{bmatrix} -8 & 4 & 0 \\ -4 & -18 & 8 \\ 26 & 17 & -12 \end{bmatrix}$

properties of matrix multiplication:

If A is of order $(m\times n)$ and B is of order $(s\times p)$, then:

1. $A\times B$ is defined if and only if $n=s$
2. $A\times B$ is a matrix of size $(m\times p)$
3. if $A\times B$ is defined then $B\times A$ is defined if and only if $p=m$ and of order $(s\times n)$.
4. $A\times B$ has the same size of $B\times A$ if and only if $m=n=s=p$.
5. $A\times B \neq B\times A$ in general

6. The product of non-zero matrices may give zero matrices.
7. $(A\times B)\times C = A\times(B\times C)$
8. $(A+B)\times C = A\times C + B\times C$

ex. Using definition 7, we can write the system of algebraic equation as follows:

$$3x_1 - x_2 + x_4 = 3$$
$$4x_1 + 3x_2 - 2x_3 + x_4 = 9$$
$$x_2 + x_3 - x_4 = 10$$

$$\begin{bmatrix} 3 & -1 & 0 & 1 \\ 4 & 3 & -2 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 10 \end{bmatrix}$$

Definition 8: Transposition of a matrix

if A is an $(m \times n)$ matrix, then the transpose of A (A^T) is an $(n \times m)$ matrix defined by:

$$\text{ent}ij(A^T) = \text{ent}ji(A)$$

ex① If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

② If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

properties of transposition:

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$ if $(A+B)$ is defined
3. $(aB)^T = aB^T$ where a is any scalar
4. $(A \times B)^T = B^T \times A^T$ if $(A \times B)$ is defined

Definition 9:

1. If A is an $(m \times n)$ complex matrix, then the conjugate of A (written $\bar{A}$) is $(m \times n)$ complex matrix defined by:

$$ent_{ij}(\bar{A}) = \overline{ent_{ij}(A)} = \overline{a_{ij}}$$

2. The tranjugate (Hermitian conjugate) of the matrix A (written A^*) is an $(n \times m)$ matrix defined by:

$$A^* = \bar{A}^T = \overline{A^T}$$

example.

$$A = \begin{bmatrix} 1 & 2+6j \\ 3 & 3j \\ 1-j & 2-j \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1 & 2-6j \\ 3 & -3j \\ 1+j & 2+j \end{bmatrix}$$ conjugate matrix

$$A^* = \bar{A}^T = \begin{bmatrix} 1 & 3 & 1+j \\ 2-6j & -3j & 2+j \end{bmatrix}$$ tranjugate matrix

The tranjugate matrix has the same properties as the transpose matrix.

Definition 10:

1. A is said to be symmetric matrix if $A^T = A$.
2. B is said to be skewsymmetric matrix if $B^T = -B$.
3. Complex matrix H is said to be Hermitian if $H^* = H$.
4. Complex matrix S is said to be skew-Hermitian if $S^* = -S$.

ex. ①
$$A = \begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix}$$ is symmetric

②
$$B = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$ is skew-symmetric

③
$$H = \begin{bmatrix} 4 & 1-3j \\ 1+3j & 7 \end{bmatrix}$$ is Hermitian

④
$$S = \begin{bmatrix} 3j & 2+j \\ -2+j & -j \end{bmatrix}$$ is skew-Hermitian.

Definition 11:

1. If a matrix consists of a single entry, then the determinant of the matrix is equal to the entry.

2. If A is a square matrix of order $n \geq 2$, then we define the determinant of A (written $|A|$) to be:

$$\det(A) = |A| = \sum_{k=1}^{n} a_k \, Cof_{1k}(A)$$

where, in general: $Cof_{ij} = (-1)^{i+j} \det(Minor_{ij}(A))$ is the cofactor of a_{ij}

ex. ① $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} \Rightarrow |A| = \begin{vmatrix}1 & 2\\ 3 & 4\end{vmatrix} = (1*4)-(2*3) = 4-6 = -2$

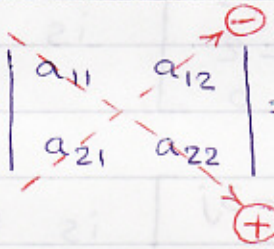
② $A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix} = 1*\begin{vmatrix}5 & 6\\ 8 & 9\end{vmatrix} - 2*\begin{vmatrix}4 & 6\\ 7 & 9\end{vmatrix} + 3*\begin{vmatrix}4 & 5\\ 7 & 8\end{vmatrix}$

$= -3 + 12 - 9 = 0$

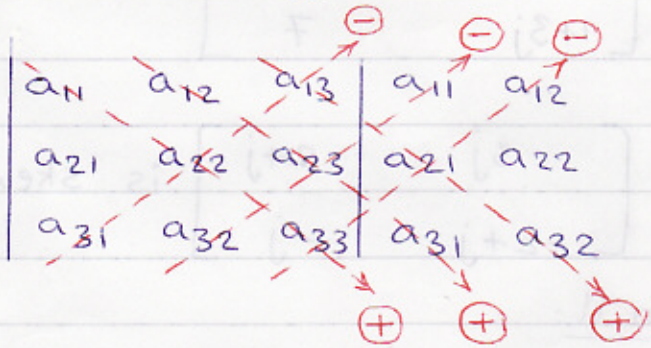
If $|A| = 0$ then A is said to be singular matrix.

Diagonal method of finding $|A|$:

① for $n=2 \Rightarrow |A| = \begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$



② for $n=3 \Rightarrow |A| =$



$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$

$- a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$

The diagonal method for finding $|A|$ is correct only for determinants of the second and third order, and will give incorrect results if applied on determinants of higher order.

Definition 12:

If A is a square matrix of order n, then the adjoint of A (written $Adj(A)$) is a square matrix defined by:

$$ent_{ij}(Adj(A)) = Cof_{ji}(A) = (-1)^{i+j} \det Min_{ji}(A)$$

ex.

$$B = \begin{bmatrix} 3 & 1 & -4 \\ 6 & 9 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$Adj(B) = \begin{bmatrix} \begin{vmatrix} 9 & -2 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 6 & -2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 6 & 9 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & -4 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & -4 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & -4 \\ 9 & -2 \end{vmatrix} & -\begin{vmatrix} 3 & -4 \\ 6 & -2 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 6 & 9 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 13 & -8 & 3 \\ -9 & 7 & -5 \\ 34 & -18 & 21 \end{bmatrix}^T = \begin{bmatrix} 13 & -9 & 34 \\ -8 & 7 & -18 \\ 3 & -5 & 21 \end{bmatrix}$$

Definition 13: Inverse of a matrix

1. If A is a square matrix and B satisfies $A \times B = B \times A = I$, where I is unit matrix, then B is a multiplicative inverse (or simply inverse of matrix) of A. (written A^{-1}).
2. If A has a multiplicative inverse then A is said to be nonsingular and when has not it is called singular matrix.

$$I = \begin{bmatrix} 1 \end{bmatrix} \quad ; \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad ; \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

properties of inverse of a matrix:

1. If A^{-1} exists then $(A^{-1})^{-1} = A$
2. If $b \neq 0$ and A^{-1} exists then $(bA)^{-1} = \frac{1}{b} A^{-1}$
3. If A^{-1} and B^{-1} both exists then $(A \times B)^{-1} = B^{-1} \times A^{-1}$

ex. Verify that $A^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 3 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ if $A = \begin{bmatrix} -3 & 5 & 6 \\ -1 & 2 & 2 \\ 1 & -1 & -1 \end{bmatrix}$

sol. If this is correct then $A^{-1} \times A = A \times A^{-1} = I$

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 3 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 5 & 6 \\ -1 & 2 & 2 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 & 2-2 \\ 3-3 & -5+6 & -6+6 \\ -3+2+1 & 5-4-1 & 6-4-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem:

The inverse of a nonsingular nxn matrix $A = [a_{jk}]$ is given by:

$$A^{-1} = \frac{1}{\det(A)} \operatorname{Adj}(A)$$

ex. Find B^{-1} if $B = \begin{bmatrix} 3 & 1 & -4 \\ 6 & 9 & -2 \\ 1 & 2 & 1 \end{bmatrix}$

sol.

From the example of definition 12:

$$\operatorname{Adj}(B) = \begin{bmatrix} 13 & -9 & 34 \\ -8 & 7 & -18 \\ 3 & -5 & 21 \end{bmatrix}$$

$$\det(B) = \begin{vmatrix} 3 & 1 & -4 \\ 6 & 9 & -2 \\ 1 & 2 & 1 \end{vmatrix} = 19 \quad \text{or} \quad B \times \operatorname{Adj}(B) = \det(B) I$$

$$\therefore B^{-1} = \frac{1}{19} \begin{bmatrix} 13 & -9 & 34 \\ -8 & 7 & -18 \\ 3 & -5 & 21 \end{bmatrix}$$

System of linear algebraic equations:

In the most general form, a system of linear algebraic equations is of the form:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

where n is the number of unknowns = number of equations

$b_1, b_2, b_3, \ldots, b_n$ are constants of the system.

If all constants are equal to zero, the system is said to be homogeneous. The system is nonhomogeneous if at least one of the constants $\neq 0$.

If we define the matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} ; X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} ; B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

the system can be written in the form $AX = B$

The solution is carried out most efficiently by operating on the rows of the augmented matrix $[AB]$ of the system rather than on the equations themselves. This type of operation is called elementary row operation that consists of the following:

1. An interchange of rows.
2. Multiplication of a row by a non-zero number.
3. Addition of a number times the elements of one row to the corresponding elements of another row.

Ex. Solve the system of linear equations:

$$3x_1 + 2x_2 = 4$$
$$2x_1 - x_2 = -9$$
$$x_1 + 3x_2 = 13$$

sol. The augmented matrix is : $\begin{bmatrix} 3 & 2 & 4 \\ 2 & -1 & -9 \\ 1 & 3 & 13 \end{bmatrix}$

① Interchange 1st and 3rd rows:

$$\begin{bmatrix} 1 & 3 & 13 \\ 2 & -1 & -9 \\ 3 & 2 & 4 \end{bmatrix}$$

② Add -2 times row 1 to row 2 then add -3 times row 1 to row 3 of the system :

$$\begin{bmatrix} 1 & 3 & 13 \\ 0 & -7 & -35 \\ 0 & -7 & -35 \end{bmatrix} \begin{matrix} \\ \text{row2} - 2\,\text{row1} \\ \text{row3} - 3\,\text{row1} \end{matrix} \Rightarrow \begin{matrix} x_1 + 3x_2 = 13 \\ -7x_2 = -35 \\ -7x_2 = -35 \end{matrix}$$

③ Add -1 times row 2 to row 3, and then multiply row 2 by $-1/7$ we get:

$$\begin{bmatrix} 1 & 3 & 13 \\ 0 & -7 & -35 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \text{row3} - \text{row2} \end{matrix} \sim \begin{bmatrix} 1 & 3 & 13 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} -\frac{1}{7}\,\text{row2}$$

$\therefore x_1 + 3x_2 = 13 \;;\; x_2 = 5 \Rightarrow x_1 = -2$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

H.W. Solve the following linear system of equations:

$2x_1 + 3x_2 - x_3 = 1$, $2x_1 - x_2 + x_3 = -3$

$3x_1 - 4x_2 + 3x_3 = -1$, $3x_1 + x_2 - 2x_3 = 4$

Inversematrix method for solving linear systems:

This method is ideally suited to the problem of solving linear system with nonsingular coefficient matrices.

$$AX = B$$
$$A^{-1}AX = A^{-1}B$$
$$IX = A^{-1}B$$
$$X = A^{-1}B$$

ex. Solve the following system:

$$2x_1 - x_2 + 3x_3 = 3$$
$$-x_1 + 2x_2 = 1$$
$$3x_1 - 5x_2 + 2x_3 = 1$$

Sol.

$$A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 0 \\ 3 & -5 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & 0 \\ 3 & -5 & 2 \end{vmatrix} = 2(4) + 1(-2) + 3(5-6) = 3$$

$$Adj(A) = \begin{bmatrix} 4 & -13 & -6 \\ 2 & -5 & -3 \\ -1 & 7 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} Adj(A) = \frac{1}{3}\begin{bmatrix} 4 & -13 & -6 \\ 2 & -5 & -3 \\ -1 & 7 & 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4 & -13 & -6 \\ 2 & -5 & -3 \\ -1 & 7 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -7 \\ -2 \\ 7 \end{bmatrix} = \begin{bmatrix} -7/3 \\ -2/3 \\ 7/3 \end{bmatrix}$$

Cramer's Rule for solving linear systems:

If $\det(A) \neq 0$, the system $AX = B$ of n linear equations has a unique solution whose j'th component given by:

$$X_j = \frac{|D_j|}{|A|} \qquad 1 \leq j \leq n$$

where the matrix D_j has B as its j'th column and the corresponding columns of A as its other columns.

ex. Use Cramer's rule to solve the system $AX = B$ where:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -2 & -1 & 2 \\ 3 & -1 & 2 & 2 \\ 1 & -1 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

sol.

$$|A| = 6$$

$$|D_1| = \begin{vmatrix} 3 & 1 & 1 & 1 \\ 0 & -2 & -1 & 2 \\ 2 & -1 & 2 & 2 \\ 0 & -1 & -2 & 1 \end{vmatrix} = 3 \quad \Rightarrow x_1 = \frac{3}{6}$$

$$|D_2| = \begin{vmatrix} 1 & 3 & 1 & 1 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 2 & 2 \\ 1 & 0 & -2 & 1 \end{vmatrix} = 9 \Rightarrow x_2 = \frac{9}{6}$$

$$|D_3| = 0 \Rightarrow x_3 = \frac{0}{6} \; ; \; |D_4| = 6 \Rightarrow x_4 = \frac{6}{6}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$$

Characteristic Value Problems:-

Many systems can be represented as a system of linear equations (mechanical vibration, alternating current and voltage and other oscillatory phenomena) of the type:

$$A\bar{X} = \lambda \bar{X} \quad \text{——①}$$

where A is $n \times n$ matrix, $\bar{X}$ is $n \times 1$ matrix, λ is a constant.

$$A\bar{X} = \lambda I \bar{X} \quad \text{——②}$$ where I is a unit matrix

$$(\lambda I - A)\bar{X} = 0 \quad \text{——③}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & -a_{13} \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & -a_{23} \cdots & -a_{2n} \\ \vdots & & & \\ -a_{n1} & -a_{n2} & -a_{n3} \cdots & \lambda - a_{nn} \end{vmatrix} \quad \text{——④}$$

$$P(\lambda) = \det(\lambda I - A) \quad \text{——⑤}$$ where $P(\lambda)$ is a polynomial of degree n

Equation (4) is called the c/cs equation of A.
Equation (5) is called the c/cs polynomial of A.
The n roots of $P(\lambda) = 0$ are called the c/cs values (eigenvalues).
The non-trivial vectors of 1, 2 and 3 are called c/cs vectors (eigenvectors) of A.

Example: Find the c/cs values and the corresponding c/cs vectors of the matrix:

$$A = \begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix}$$

Sol.

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-4 & 5 \\ -1 & \lambda+2 \end{vmatrix} = (\lambda-4)(\lambda+2) + 5 = \lambda^2 - 2\lambda - 3$$

$P(\lambda) = \det(\lambda I - A) = 0 \Rightarrow \lambda^2 - 2\lambda - 3 = 0$ c/cs equation

$(\lambda+1)(\lambda-3) = 0$

∴ The c/cs values of A are $\lambda = -1$ and $\lambda = 3$

If $\lambda = -1$, $(\lambda I - A)X = 0 \Rightarrow \begin{bmatrix} -5 & 5 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

which is equivalent to the set of linear equations:

$-5x_1 + 5x_2 = 0$ and $-x_1 + x_2 = 0$

The solution of this system is $x_2 = x_1$ with x_1 arbitrary.

So the c/cs vectors corresponding to $\lambda+1$ are:

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ where x_1 is arbitrary constant $\neq 0$

If $\lambda = 3$, $(\lambda I - A)X = 0 \Rightarrow \begin{bmatrix} -1 & 5 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$-x_1 + 5x_2 = 0$, $-x_1 + 5x_2 = 0$

$\therefore x_2 = \frac{1}{5}x_1$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \frac{1}{5}x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}$$

example: Find the c/cs values and the corresponding c/cs vectors of the matrix:

$$A=\begin{bmatrix} 3 & -2 & -5 \\ 4 & -1 & -5 \\ -2 & -1 & -3 \end{bmatrix}$$

Sol.

$$\det(\lambda I - A)=\begin{vmatrix} \lambda-3 & 2 & 5 \\ -4 & \lambda+1 & 5 \\ 2 & 1 & \lambda+3 \end{vmatrix} = \lambda^3+\lambda^2-16\lambda+20$$

$P(\lambda)=0 \Rightarrow \lambda^3+\lambda^2-16\lambda+20=0$

$(\lambda+5)(\lambda-2)^2=0 \Rightarrow \lambda=-5$ and $\lambda=2$ c/cs values

For $\lambda=-5$, $(\lambda I-A)X=0$

$$\begin{bmatrix} -8 & 2 & 5 \\ -4 & -4 & 5 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$-8x_1+2x_2+5x_3=0$

$-4x_1-4x_2+5x_3=0$

$2x_1+x_2-2x_3=0$

$$X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} x_1 \\ \frac{2}{3}x_1 \\ \frac{4}{3}x_1 \end{bmatrix}=x_1\begin{bmatrix} 1 \\ \frac{2}{3} \\ \frac{4}{3} \end{bmatrix}$$

For $\lambda = 2$, $(\lambda I - A)X = 0 \Rightarrow \begin{bmatrix} -1 & 2 & 5 \\ -4 & 3 & 5 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$-x_1 + 2x_2 + 5x_3 = 0$$
$$-4x_1 + 3x_2 + 5x_3 = 0$$
$$2x_1 + x_2 + 5x_3 = 0$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 3x_1 \\ -x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

Engineering Analysis

Differential Equations

A differential equation (D.E.) is an equation which involves the independent variable x, unknown function $y=f(x)$ and its derivates $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$

The order of a D.E. is the order of the highest derivative in the eq.

The degree of a D.E. is the degree (power) of the highest derivative in the eq.

i.e.

① $\frac{dy}{dx} = \frac{1+x^2}{1-y^2}$ 1st order

② $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 8y = 0$ 2nd order

③ $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$ 2nd order, 2nd degree

Second-Order Differential Equations:

The general form of a linear D.E. of the 2nd order is:

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

$P(x)$ and $Q(x)$ are coefficients of the equation (could be constants or variables).

$R(x)$ could be a constant of function of x.

if $R(x)=0$ then the equation is said to be homogeneous otherwise, it is nonhomogeneous.

Solution of Constant-Coefficient Homogeneous Linear D.E:

Consider the D.E.: $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$

where p and q are constants. The solution of this equation ($y=f(x)$) is given by the complementary function.

Theorem: If y_1 and y_2 are two independent solutions for a D.E., then $y = c_1 y_1 + c_2 y_2$ is also a solution.

Steps for solving homogeneous D.E.:

Given an equation: $y'' + py' + qy = 0$, the solution is carried out by these steps.

1- Find the characteristic equation: $\lambda^2 + p\lambda + q = 0$

2- Find the roots of the c/cs eq. (i.e. λ_1 and λ_2)

3. If λ_1 and λ_2 are:

(a) two distinct real roots ($p^2 - 4q > 0$), then:

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

(b) real double root ($p^2 - 4q = 0$), then:

$$y = c_1 e^{\lambda x} + c_2 x e^{\lambda x} = (c_1 + c_2 x) e^{\lambda x}$$

(c) Complex roots ($p^2 - 4q < 0$), then

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

where α = real part of λ ; β = imaginary part of λ

ex. Find a solution of the eq. $y'' - 4y' + 4y = 0$

sol. The c/cs eq. is $\lambda^2 - 4\lambda + 4 = 0 \Rightarrow (\lambda - 2)^2 = 0$

$\lambda_{1,2} = 2$ real double root

$\therefore y = (c_1 + c_2 x) e^{\lambda x} = (c_1 + c_2 x) e^{2x}$

where c_1 and c_2 are constants determined by initial condition.

ex. Find a solution for the eq. $y'' + 2y' + 5 = 0$

sol. the c/cs eq. is: $\lambda^2 + 2\lambda + 5 = 0$

$\lambda_1 = \frac{-2 + \sqrt{4 - 20}}{2} = -1 + j2$; $\lambda_2 = -1 - j2$

$\because \lambda_1$ and λ_2 are complex roots with $\alpha = -1$; $\beta = 2$

$\therefore y = e^{\alpha x} (A \cos \beta x + B \sin \beta x) = e^{-x} (A \cos 2x + B \sin 2x)$

where A and B are constants determined by initial condition.

Nonhomogeneous Linear Differential Equations:

Consider the D.E.: $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = r(x)$

where p and q are constants. $r(x)$ is a function of x

The solution of such eq. is of the form:

$$y = y_h + y_p$$

where y is the complete solution

y_h is the homogeneous solution

y_p is the particular solution

Solving nonhomogeneous equations by the method of variation of parameters: (Lagrange's Method)

Given the equation: $y'' + py' + qy = r(x)$, the solution $(y = y_h + y_p)$ is carried out by these steps:

* Finding the homogeneous solution:

1. Make the given D.E. homogeneous (i.e. $y'' + py' + qy = 0$)
2. Find the characteristic equation (i.e. $\lambda^2 + p\lambda + q = 0$)
3. Find the roots of the c/cs eq.
4. Depending on the type of roots, then y_h will be either:

a) $y_h = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

b) $y_h = (c_1 + c_2 x) e^{\lambda x}$

c) $y_h = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

* Finding the particular solution:

5. Replace the constants c_1 and c_2 from y_h by $u_1(x)$ and $u_2(x)$, then:

$$y = u_1 y_1 + u_2 y_2 \quad \text{---(1)} \quad \text{where } u_1, y_1, u_2, y_2 \text{ are all functions of } x$$

$$y' = u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2$$

Let $u_1' y_1 + u_2' y_2 = 0$ ---(2), then:

$$y' = u_1 y_1' + u_2 y_2' \quad \text{---(3)}$$

$$y'' = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2' \quad \text{---(4)}$$

6. substitute eqs. (3) and (4) into the given nonhomogeneous eq. $(y'' + py' + qy = r(x))$, then:

$$(u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2') + p(u_1 y_1' + u_2 y_2') + q(u_1 y_1 + u_2 y_2) = r(x)$$

rearranging:

$$u_1(y_1'' + py_1' + qy_1) + u_2(y_2'' + py_2' + qy_2) + u_1' y_1' + u_2' y_2' = r(x)$$

$\because y_1$ and y_2 are both solutions for the homogeneous eq., then:

$$y_1'' + py_1' + qy_1 = y_2'' + py_2' + qy_2 = 0$$

$$\therefore u_1' y_1' + u_2' y_2' = r(x) \quad \text{---(5)}$$

7. from equations ② and ⑤ Solve to get $u_1'(x)$ and $u_2'(x)$.

i.e. $u_1' y_1 + u_2' y_2 = 0$ —— ②

$u_1' y_1' + u_2' y_2' = r(x)$ —— ⑤

$$\therefore u_1'(x) = \frac{\begin{vmatrix} 0 & y_2 \\ r(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \; ; \; u_2'(x) = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

$$u_1(x) = \int u_1'(x)\,dx = \int \frac{-y_2 r(x)}{W}\,dx$$

$$u_2(x) = \int u_2'(x)\,dx = \int \frac{y_1 r(x)}{W}\,dx$$

where W is the Wronskian of y_1 and $y_2 = y_1 y_2' - y_1' y_2$

8. The particular solution will be:

$$y_p = u_1(x)\, y_1 + u_2(x)\, y_2$$

9. The complete solution will be:

$$y = y_h + y_p = c_1 y_1 + c_2 y_2 + u_1(x) y_1 + u_2(x) y_2$$

ex. Solve the nonhomogeneous D.E.: $y''+4y=\sin^2 2x$

sol.

* Homogeneous solution: $y''+4y=0$

1. Find the c/cs eq.: $\lambda^2+4=0 \Rightarrow \lambda_{1,2}=\pm j2$ complex roots

2. $y_h = e^{\alpha x}(A\cos\beta x + B\sin\beta x) = A\cos 2x + B\sin 2x$

where A and B are constants

* Particular solution:

3. Comparing $y_h = A\cos 2x + B\sin 2x$ with $y_p = u_1(x)y_1 + u_2(x)y_2$

then: $y_1 = \cos 2x$; $y_2 = \sin 2x$

4. $y_1' = -2\sin 2x$; $y_2' = 2\cos 2x$

5. $$u_1'(x) = \frac{\begin{vmatrix} 0 & y_2 \\ r(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} 0 & \sin 2x \\ \sin^2 2x & 2\cos 2x \end{vmatrix}}{\begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}} = \frac{-\sin^3 2x}{2\cos^2 2x + 2\sin^2 2x} = \frac{-\sin^3 2x}{2}$$

$$u_1(x) = \int u_1'(x)\,dx = \int \frac{-\sin^3 2x}{2}\,dx = -\frac{1}{2}\int \sin 2x \cdot \sin^2 2x\,dx$$

$$= -\frac{1}{2}\int \sin 2x\,(1-\cos^2 2x)\,dx = -\frac{1}{2}\left[\int \sin 2x\,dx - \int \cos^2 2x \sin 2x\,dx\right]$$

$$= -\frac{1}{2}\left[\frac{-\cos 2x}{2} + \frac{\cos^3 2x}{6}\right] = \frac{\cos 2x}{4} - \frac{\cos^3 2x}{12}$$

6. $$u_2'(x) = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \sin^2 2x \end{vmatrix}}{\begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}} = \frac{\sin^2 2x \cos 2x}{2\sin^2 2x + 2\cos^2 2x} = \frac{\sin^2 2x \cos 2x}{2}$$

$$u_2(x) = \int u_2'(x)\,dx = \int \frac{\sin^2 2x \cos 2x}{2}\,dx = \frac{1}{2}\,\frac{\sin^3 2x}{6} = \frac{\sin^3 2x}{12}$$

7. The complete solution is: $y = y_h + y_p = c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2$

$$= A\cos 2x + B\sin 2x + \left[\frac{\cos 2x}{4} - \frac{\cos^3 2x}{12}\right]\cos 2x + \frac{\sin^3 2x}{12}\sin 2x$$

ex. Solve the D.E.: $y'' - 4y' + 4y = e^{2x}\sec^2 x$

Sol.

* To find y_h: $\lambda^2 - 4\lambda + 4 = 0 \Rightarrow (\lambda-2)^2 = 0 \Rightarrow \lambda_{1,2} = 2$ double root

$\therefore y_h = (c_1 + c_2 x)e^{\lambda x} = (c_1 + c_2 x)e^{2x}$

* To find y_p:

$y_1 = e^{2x}$; $y_2 = xe^{2x}$; $y_1' = 2e^{2x}$; $y_2' = 2xe^{2x} + e^{2x}$

$$u_1(x) = \int \frac{-y_2\, r(x)}{W}dx = \int \frac{-xe^{2x}\cdot e^{2x}\sec^2 x}{e^{2x}(2xe^{2x}+e^{2x}) - 2e^{2x}\cdot xe^{2x}}dx = \int \frac{-x\sec^2 x}{2x+1-2x}dx$$

$$= -\int x\sec^2 x\,dx = -x\tan x + \log(\sec x)$$

$$u_2(x) = \int \frac{y_1\, r(x)}{W}dx = \int \frac{e^{4x}\sec^2 x}{(2x+1-2x)e^{4x}}dx = \int \sec^2 x\,dx = \tan x$$

$\therefore y_p = (-x\tan x + \log(\sec x))e^{2x} + \tan x \cdot xe^{2x}$

$\therefore y = y_h + y_p$

$= (c_1 + c_2 x)e^{2x} + (-x\tan x + \log(\sec x))e^{2x} + xe^{2x}\tan x$

$= (c_1 + c_2 x)e^{2x} + e^{2x}\log(\sec x)$

Higher order D.E-2

ex. solve the homogeneous D.E: $y''' - 5y'' - 22y' + 56y = 0$

sol. The clcs eq. is: $\lambda^3 - 5\lambda^2 - 22\lambda + 56 = 0$

solve to get $\lambda_1 = -4$; $\lambda_2 = 2$; $\lambda_3 = 7$

$\therefore y = c_1 e^{-4x} + c_2 e^{2x} + c_3 e^{7x}$

ex. solve: $2y'''' + 11y''' + 18y'' + 4y' - 8y = 0$

sol. The clcs eq. is: $2\lambda^4 + 11\lambda^3 + 18\lambda^2 + 4\lambda - 8 = 0$

$(2\lambda - 1)(\lambda + 2)^3 = 0$

solve to get $\lambda_1 = \frac{1}{2}$; $\lambda_{2,3,4} = -2$

$\therefore y = c_1 e^{\frac{1}{2}x} + c_2 e^{-2x} + c_3 x e^{-2x} + c_4 x^2 e^{-2x}$

ex. solve $(D^5 + 12D^4 + 104D^3 + 408D^2 + 1156D) = 0$

where $D = \frac{d}{dx}$

sol. The clcs eq. is: $\lambda^5 + 12\lambda^4 + 104\lambda^3 + 408\lambda^2 + 1156\lambda = 0$

$\lambda(\lambda^2 + 6\lambda + 34)^2 = 0$

$\lambda_1 = 0$; $\lambda_{2,3} = -3 + j5$; $\lambda_{4,5} = -3 - j5$

$\therefore y = c_1 + c_2 e^{-3x}\cos 5x + c_3 e^{-3x}\sin 5x + c_4 x e^{-3x}\cos 5x + c_5 x e^{-3x}\sin 5x$

Euler-Cauchy Equations:

Euler-Cauchy equations are ODE with variable coefficients (power of x) of the form:

$$x^2 y'' + a x y' + b y = 0$$

where a and b are constants. The solution is of the form $y = x^\lambda$. Its derivative is $y' = \lambda x^{\lambda-1}$ and $y'' = \lambda(\lambda-1) x^{\lambda-2}$, then substituting in the original equation:

$$x^2[\lambda(\lambda-1)x^{\lambda-2}] + ax[\lambda x^{\lambda-1}] + b x^\lambda = 0$$

$$\lambda(\lambda-1) x^\lambda + a\lambda x^\lambda + b x^\lambda = 0 \qquad \div x^\lambda$$

$$\therefore \lambda^2 + (a-1)\lambda + b = 0 \quad \text{auxiliary equation}$$

$y = x^\lambda$ is a solution if and only if λ is a root for the auxiliary equation.

Case I. Real distinct roots:

$$y = c_1 x^{\lambda_1} + c_2 x^{\lambda_2}$$

Case II. Real double roots:

$$y = (c_1 + c_2 \ln x) x^\lambda$$

Case III. Complex roots:

$$y = x^\alpha (A \cos(\beta \ln x) + B \sin(\beta \ln x))$$

where α and β are real and imaginary parts of λ respectively.

ex. Find a general solution for: $4x^2y'' + 4xy' - y = 0$

sol. divide the eq. by 4 to get standard form:

$$x^2y'' + xy' - \frac{1}{4}y = 0$$

$y = x^\lambda \Rightarrow y' = \lambda x^{\lambda-1}$; $y'' = \lambda(\lambda-1)x^{\lambda-2}$

$$\lambda(\lambda-1)x^2 x^{\lambda-2} + \lambda x x^{\lambda-1} - \frac{1}{4}x^\lambda = 0$$

$$\lambda^2 - \frac{1}{4} = 0 \Rightarrow \lambda = \pm\frac{1}{2}$$

$$\therefore y = c_1x^{\lambda_1} + c_2x^{\lambda_2} = c_1x^{\frac{1}{2}} + c_2x^{-\frac{1}{2}} = c_1\sqrt{x} + \frac{c_2}{\sqrt{x}}$$

ex. Solve the D.E.: $x^3y''' - 3x^2y'' + 7xy' - 8y = 0$

sol. $y = x^\lambda \Rightarrow y' = \lambda x^{\lambda-1}$; $y'' = \lambda(\lambda-1)x^{\lambda-2}$; $y''' = \lambda(\lambda-1)(\lambda-2)x^{\lambda-3}$

$$\lambda(\lambda-1)(\lambda-2)x^\lambda - 3\lambda(\lambda-1)x^\lambda + 7\lambda x^\lambda - 8x^\lambda = 0$$

$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$ auxiliary eq.

Solve to get: $\lambda = 2$ repeated triple roots

$$\therefore y = (c_1 + c_2\ln x + c_3(\ln x)^2)x^\lambda = (c_1 + c_2\ln x + c_3(\ln x)^2)x^2$$

ex- Solve the nonhomogeneous Euler-Cauchy equation:

$$x^2y'' + xy' - y = \frac{x^3}{1+x^2}$$

sol. * Homogeneous solution: $x^2y'' + xy' - y = 0$

Let $y = x^m$; $y' = mx^{m-1}$; $y'' = m(m-1)x^{m-2}$

$\therefore m(m-1)x^m + mx^m - x^m = 0 \Rightarrow m^2 - m + m - 1 = 0$

$\therefore m = \pm 1 \Rightarrow y_h = c_1x^{m_1} + c_2x^{m_2} = c_1x + c_2x^{-1} = c_1x + \frac{c_2}{x}$

* Particular solution:

$y_1 = x$; $y_1' = 1$

$y_2 = x^{-1}$; $y_2' = -x^{-2}$

To get the standard form of a nonhomogeneous D.E., divide the given eq. by x^2, then:

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = \frac{x}{1+x^2}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -x^{-1} - x^{-1} = -2x^{-1}$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ r(x) & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & x^{-1} \\ \frac{x}{1+x^2} & -x^{-1} \end{vmatrix}}{-2x^{-1}} = \frac{\frac{1}{1+x^2}}{\frac{2}{x}} = \frac{x}{2(1+x^2)}$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r(x) \end{vmatrix}}{W} = \frac{\begin{vmatrix} x & 0 \\ 1 & \frac{x}{1+x^2} \end{vmatrix}}{-2x^{-1}} = \frac{\frac{x^2}{1+x^2}}{\frac{-2}{x}} = \frac{x^3}{-2(1+x^2)}$$

$$u_1 = \int u_1'(x)\,dx = \int \frac{x}{2(1+x^2)}\,dx = \frac{1}{2}\left[\frac{1}{2}\ln(x^2+1)\right] = \frac{1}{4}\ln(x^2+1)$$

$$\text{Hint} : \int \frac{x\,dx}{x^2+a^2} = \frac{1}{2}\ln(x^2+a^2)$$

$$u_2 = \int u_2'(x)\,dx = \int \frac{x^3\,dx}{-2(1+x^2)} = -\frac{1}{2}\left[\frac{1}{2} + \frac{x^2}{2} - \frac{1}{2}\ln(x^2+1)\right]$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \frac{1}{4}\ln(x^2+1)\,(x) - \frac{1}{2}\left[\frac{1}{2} + \frac{x^2}{2} - \frac{1}{2}\ln(x^2+1)\right] x^{-1}$$

$$= \frac{x}{4}\ln(x^2+1) - \frac{1}{4x} - \frac{x}{4} + \frac{1}{4x}\ln(x^2+1)$$

Simultaneous differential equations:

A Simultaneous D.E. is an equation consists of one independent variable and two or more dependent variables.

i.e. $\frac{dx}{dt} + 4y = t$; $\frac{dy}{dt} - x = e^t$

where t is the independt variable and x, y are dependent variables.

The method of solving these equations is based on the process of elimination, similar to those of solving system of algebraic equations.

ex. Solve the system of equations: $\frac{dx}{dt} = -wy$; $\frac{dy}{dt} = wx$

sol.

x, y are dependent variables ; t is independent variable

w is a constant.

① Put $\frac{d}{dt} \equiv D$ in the equations.

$\therefore$ $Dx + wy = 0$ ——①

$Dy - wx = 0$ ——②

② multiply eq. ① by w and eq. ② by D.

$Dwx + w^2 y = 0$ ——③

$-Dwx + D^2 y = 0$ ——④

③ Add both eqs. ③ and ④ to get

$w^2 y + D^2 y = 0 \Rightarrow \frac{d^2y}{dt^2} + w^2 y = 0$

This is a 2nd order homogeneous ODE.

ex. continued:

The c/cs eq. is: $\lambda^2 + \omega^2 = 0 \Longrightarrow \lambda = \pm j\omega$

$\therefore \alpha = 0 \;;\; \beta = \omega$

$\therefore y = C_1 \cos\omega t + C_2 \sin\omega t$

$\frac{dy}{dt} = Dy = -C_1 \omega \sin\omega t + C_2 \omega \cos\omega t$ substitute in eq. ②

$\omega x = Dy = -C_1 \omega \sin\omega t + C_2 \omega \cos\omega t$

$\therefore x = C_2 \cos\omega t - C_1 \sin\omega t$

ex. Solve the system of equations:

$$\frac{dx}{dt}+4x+3y=t \;;\; \frac{dy}{dt}+2x+5y=e^{t}$$

sol.

$$Dx+4x+3y=t \Rightarrow (D+4)x+3y=t \quad *2$$

$$Dy+2x+5y=e^{t} \Rightarrow 2x+(D+5)y=e^{t} \quad *-(D+4)$$

$$2(D+4)x+6y=2t$$

$$-2(D+4)x-(D+5)(D+4)y=-(D+4)e^{t}$$

$$6y-(D+5)(D+4)y=2t-(D+4)e^{+t}$$

$$6y-D^2y-4Dy-5Dy-20y=2t-De^{t}-4e^{t}$$

$$-D^2y-9Dy-14y=2t-De^{t}-4e^{t}$$

$$D^2y+9Dy+14y=De^{t}+4e^{t}-2t$$

$$\frac{d^2y}{dt^2}+9\frac{dy}{dt}+14y=\frac{d}{dt}e^{t}+4e^{t}-2t=e^{t}+4e^{t}-2t=5e^{t}-2t$$

2nd order nonhomogeneous ODE:

* Homogeneous solution:

$$\lambda^2+9\lambda+14=0 \Rightarrow \lambda_{1,2}=-7,-2$$

$$\therefore y_h=c_1e^{-7t}+c_2e^{-2t}$$

* Particular solution:

$$y_1=e^{-7t} \Rightarrow y_1'=-7e^{-7t} \;;\; y_2=e^{-2t} \Rightarrow y_2'=-2e^{-2t}$$

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} e^{-7t} & e^{-2t} \\ -7e^{-7t} & -2e^{-2t} \end{vmatrix}=(-2e^{-9t})-(-7e^{-9t})=5e^{-9t}$$

$$u_1'=\frac{\begin{vmatrix} 0 & y_2 \\ r(x) & y_2' \end{vmatrix}}{W}=\frac{\begin{vmatrix} 0 & e^{-2t} \\ 5e^{t}-2t & -2e^{-2t} \end{vmatrix}}{5e^{-9t}}=\frac{-e^{-2t}(5e^{t}-2t)}{5e^{-9t}}=-e^{8t}+\frac{2}{5}te^{7t}$$

$$u_2'=\frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r(x) \end{vmatrix}}{W}=\frac{\begin{vmatrix} e^{-7t} & 0 \\ -7e^{-7t} & 5e^{t}-2t \end{vmatrix}}{5e^{-9t}}=\frac{(5e^{t}-2t)e^{-7t}}{5e^{-9t}}=e^{3t}-\frac{2}{5}te^{2t}$$

$u_1' = -e^{8t} + \frac{2}{5} t e^{7t}$; $u_2' = e^{3t} - \frac{2}{5} t e^{2t}$

$u_1(t) = \int u_1'(t)\,dt = \int (-e^{8t} + \frac{2}{5} t e^{7t})\,dt = -\frac{1}{8} e^{8t} + \frac{2}{5}\left[\frac{1}{7} t e^{7t} - \frac{1}{49} e^{7t}\right]$

$= \frac{-1}{8} e^{8t} + \frac{2}{35} t e^{7t} - \frac{2}{245} e^{7t} = -\frac{1}{8} e^{8t} - \frac{2}{245} e^{7t} + \frac{2}{35} t e^{7t}$

$u_2(t) = \int (e^{3t} - \frac{2}{5} t e^{2t})\,dt = \frac{1}{3} e^{3t} - \frac{2}{5}\left[\frac{1}{2} t e^{2t} - \frac{1}{4} e^{2t}\right]$

$= \frac{1}{3} e^{3t} - \frac{1}{5} t e^{2t} + \frac{1}{10} e^{2t} = \frac{1}{3} e^{3t} + \frac{1}{10} e^{2t} - \frac{1}{5} t e^{2t}$

$y_p = u_1 y_1 + u_2 y_2 = -\frac{1}{8} e^{t} - \frac{2}{245} + \frac{2}{35} t + \frac{1}{3} e^{t} + \frac{1}{10} - \frac{1}{5} t$

($y_1 \uparrow e^{-7t}$, $y_2 \uparrow e^{-2t}$)

$= -\frac{1}{7} t + \frac{9}{98} + \frac{5}{24} e^{t}$

∴ general solution for $y = y_h + y_p$ is:

$y = C_1 e^{-7t} + C_2 e^{-2t} - \frac{1}{7} t + \frac{5}{24} e^{t} + \frac{9}{98}$

Use this solution where coefficient of x is simple.

$2x + (D+5)y = e^{t}$

$Dy = \frac{dy}{dt} = -7C_1 e^{-7t} - 2C_2 e^{-2t} - \frac{1}{7} + \frac{5}{24} e^{t}$

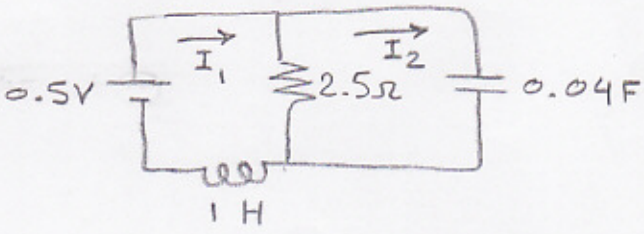
$\therefore 2x = -Dy - 5y + e^{t} = 7C_1 e^{-7t} + 2C_2 e^{-2t} + \frac{1}{7} - \frac{5}{24} e^{t} - 5C_1 e^{-7t} - 5C_2 e^{-2t}$

$+ \frac{5}{7} t - \frac{25}{24} e^{t} - \frac{45}{98} + e^{t}$

$2x = 2C_1 e^{-7t} - 3C_2 e^{-2t} - \frac{31}{98} - \frac{1}{4} e^{t} + \frac{5}{7} t$

$\therefore x = C_1 e^{-7t} - \frac{3}{2} C_2 e^{-2t} - \frac{31}{196} - \frac{1}{8} e^{t} + \frac{5}{14} t$

ex. Find the currents I_1 and I_2 For the cct. shown:



Sol. The equations for I_1 and I_2 are:

$$0.5 = 2.5(I_1 - I_2) + L\frac{dI_1}{dt}$$

$$0 = 2.5(I_2 - I_1) + \frac{1}{C}\int I_2\, dt$$

$$0.5 = 2.5(I_1 - I_2) + \frac{dI_1}{dt}$$

$$0 = 2.5\left(\frac{dI_2}{dt} - \frac{dI_1}{dt}\right) + \frac{1}{0.04} I_2$$

$$2.5 I_1 + DI_1 - 2.5 I_2 = 0.5 \Rightarrow (D+2.5) I_1 - 2.5 I_2 = 0.5$$

$$2.5\, DI_2 - 2.5\, DI_1 + 25 I_2 = 0 \quad \div 2.5$$

$$DI_2 - DI_1 + 10 I_2 = 0 \Rightarrow -DI_1 + (D+10) I_2 = 0$$

$$(D+2.5)(D+10) I_1 - 2.5(D+10) I_2 = (D+10)\, 0.5$$

$$-2.5 DI_1 + 2.5 (D+10) I_2 = 0$$

$$D^2 I_1 + 10 DI_1 + 2.5 DI_1 + 25 I_1 - 2.5 DI_1 = (D+10)\, 0.5$$

$$\therefore \frac{d^2 I_1}{dt^2} + 10\frac{dI_1}{dt} + 25 I_1 = 5$$

This is a 2nd order nonhomogeneous ODE.

$$\therefore I_1 = I_{1_h} + I_{1_p}$$

* Homogeneous solution: $\frac{d^2 I_1}{dt^2} + 10\frac{dI_1}{dt} + 25 I_1 = 0$

$$\lambda^2 + 10\lambda + 25 = 0 \Rightarrow \lambda_{1,2} = -5 \text{ repeated roots}$$

$$\therefore I_1 = (C_1 + C_2 t) e^{-5t}$$

* Particular solution:

$$y_1 = e^{-5t} \Rightarrow y_1' = -5 e^{-5t}$$

$$y_2 = t e^{-5t} \Rightarrow y_2' = -5t\, e^{-5t} + e^{-5t}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-5t} & te^{-5t} \\ -5e^{-5t} & -5te^{-5t} + e^{-5t} \end{vmatrix} = -5te^{-10t} + e^{-10t} + 5te^{-10t} = e^{-10t}$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ r(x) & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & te^{-5t} \\ 5 & -5te^{-5t} + e^{-5t} \end{vmatrix}}{e^{-10t}} = \frac{-5te^{-5t}}{e^{-10t}} = -5te^{5t}$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r(x) \end{vmatrix}}{W} = \frac{\begin{vmatrix} e^{-5t} & 0 \\ -5e^{-5t} & 5 \end{vmatrix}}{e^{-10t}} = \frac{5e^{-5t}}{e^{-10t}} = 5e^{5t}$$

$$\therefore u_1 = \int u_1'(t)\,dt = \int -5te^{5t}\,dt = -5\left[\frac{1}{5}te^{5t} - \frac{1}{25}e^{5t}\right] = \frac{e^{5t}}{5} - te^{5t}$$

$$u_2 = \int 5e^{5t}\,dt = e^{5t}$$

$$\therefore I_{1p} = u_1 y_1 + u_2 y_2 = \left(\frac{e^{5t}}{5} - te^{5t}\right)e^{-5t} + e^{5t}(te^{-5t})$$

$$= \frac{1}{5} - t + t = \frac{1}{5}$$

$$\therefore I_1 = (C_1 + C_2 t)e^{-5t} + \frac{1}{5}$$

To find I_2: $\frac{dI_1}{dt} = -5C_1e^{-5t} - 5C_2te^{-5t} + C_2e^{-5t}$

$$2.5I_2 = 2.5I_1 + \frac{dI_1}{dt} - 0.5$$

$$= 2.5C_1e^{-5t} + 2.5C_2te^{-5t} + 0.5 - 5C_1e^{-5t} - 5C_2te^{-5t} + C_2e^{-5t} - 0.5$$

$$= -2.5C_1e^{-5t} + C_2e^{-5t} - 2.5C_2te^{-5t}$$

$$\therefore I_2 = -C_1e^{-5t} + 0.4C_2e^{-5t} - C_2te^{-5t}$$

Series Solution of Differential Equations:

A series solution is a standard method for solving D.E.s with variable coefficients.

Review of power series:

A power series is an infinite series of the form:

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

where $a_0, a_1, a_2, \cdots$ are constants (coefficients), and x_0 is a fixed number (center of the series)

Four important power series are:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \quad ; \quad -1 < x < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!}$$

Ordinary point: Consider the equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = R(x)$$

where a_0, a_1 and a_2 are polynomials in x

If there is a point $x = x_0$ in which $P(x) = \frac{a_1(x)}{a_0(x)}$ and $Q(x) = \frac{a_2(x)}{a_0(x)}$ are analytic ($P(x_0) \neq \infty$ and $Q(x_0) \neq \infty$) then x_0 is said to be "Ordinary point".

ex.① $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0 \Rightarrow \frac{d^2y}{dx^2} + \frac{x}{1+x^2}\frac{dx}{dy} - \frac{1}{1+x^2}y = 0$

$P(x) = \frac{x}{1+x^2}$; $Q(x) = \frac{-1}{1+x^2}$

since $P(0) = 0$ and $Q(0) = -1$ then $x = 0$ is an ordinary point for this D.E.

ex② $x^2y'' + (2x^2-x)y' + y = 0$

$$P(x) = \frac{2x^2-x}{x^2} \quad ; Q(x) = \frac{1}{x^2}$$

since $P(0) = \infty$; $Q(x) = \infty$ then $x=0$ is not ordinary point

Singular Point:

If one (or both) of the coefficients $P(x)$ and $Q(x)$ are not analytic at $x = x_0$ ($P(x_0) = \infty$ and/or $Q(x_0) = \infty$) then x_0 is said to be "Singular Point"

ex. Analyze the following equations for ordinary and singular points

① $y'' + xy' + (x^2-4)y = 0$

$P(x) = x$; $Q(x) = x^2-4$

∴ $P(x)$ and $Q(x)$ are analytic everywhere
∴ all x are ordinary points

② $(x-1)y'' + xy' + \frac{1}{x}y = 0$

$$P(x) = \frac{x}{x-1} \quad ; Q(x) = \frac{1}{x(x-1)}$$

∴ $P(x)$ is not analytic at $x=1$, $Q(x)$ is not analytic at $x=0,1$

∴ $x=0,1$ are singular points

③ $x^2(x-2)^2y'' + 2(x-2)y' + (x+1)y = 0$

$$P(x) = \frac{2}{x^2(x-2)} \quad ; Q(x) = \frac{x+1}{x^2(x-2)^2}$$

∴ $P(x)$ and $Q(x)$ are not analytic at $x=0,2$

∴ $x=0,2$ are singular points

If x_0 is a singular point of the differential eq. and $(x-x_0)P(x)$ as well as $(x-x_0)^2 Q(x)$ are both analytic at x_0 then x_0 is said to be "regular singular point", otherwise it is "irregular singular point".

ex. Determine which of the singular points in the previous ex. are regular singular points.

① All points are ordinary points

② $x_0=0 \Rightarrow (x-x_0)P(x)=\frac{x^2}{x-1}$ analytic at $x_0=0$

$(x-x_0)^2 Q(x)=\frac{x}{x-1}$ analytic at $x_0=0$

$\therefore x_0=0$ is regular singular point

$x_0=1 \Rightarrow (x-x_0)P(x)=x$ analytic at $x_0=1$

$(x-1)^2 Q(x)=\frac{x-1}{x}$ analytic at $x_0=1$

$x_0=1$ is regular singular point

③ $x_0=0 \Rightarrow (x-x_0)P(x)=\frac{2}{x(x-2)}$ not analytic at $x_0=0$

$(x-x_0)^2 Q(x)=\frac{x+1}{(x-2)^2}$ analytic at $x_0=0$

$\therefore x_0=0$ is irregular singular point

$x_0=2 \Rightarrow (x-x_0)P(x)=\frac{2}{x^2}$ analytic at $x_0=2$

$(x-x_0)^2 Q(x)=\frac{x+1}{x^2}$ analytic at $x_0=2$

$\therefore x_0=2$ is regular singular point

H-W- Find the singular points of each of the equations, and determine whether they are regular or irregular.

① $e^x y'' + 2y' - xy = 0$

② $x^2 y'' + y' + y = 0$

③ $(1-x^2)y'' + y' + y = 0$

④ $x^2(1-x)y'' + (1-x)y' + y = 0$

Steps for solving a differential equation if $x_0=0$ is an ordinary point:

1. Assume the solution $y=\sum a_n x^n = a_0+a_1x+a_2x^2+\cdots$
2. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$
3. Substitute the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given D.E.
4. Equate to zero the coefficients of x^k to find $a_2, a_3, a_4, \cdots$ in terms of a_0 and a_1
5. Substitute $n=0,1,2,3,\cdots$ to get the values of $a_2, a_3, \cdots a_n$
6. Substitute the values of $a_2, a_3, \cdots a_n$ in the eq. of step(1).

ex. Find the power series solution of the D.E.: $y''+x^2y=0$

sol. $x_0=0$ is an ordinary point

Let $y=\sum a_n x^n$

$$\frac{dy}{dx}=\sum n a_n x^{n-1}$$

$$\frac{d^2y}{dx^2}=\sum n(n-1)a_n x^{n-2}$$

$$\sum n(n-1)a_n x^{n-2}+x^2\sum a_n x^n=0 \Rightarrow \sum n(n-1)\overset{a_n}{\uparrow} x^{n-2}+\sum a_n x^{n+2}=0$$

For the 1st power series: $\sum n(n-1)a_n x^{n-2}$

Let $n-2=k \Rightarrow n-1=k+1$, $n=k+2$

$\therefore \sum n(n-1)a_n x^{n-2} = \sum (k+2)(k+1)a_{k+2}x^k$

For the second power series: $\sum a_n x^{n+2}$

Let $n+2=k \Rightarrow n=k-2$

$\therefore \sum a_n x^{n+2} = \sum a_{k-2}x^k$

ex. continued: $\sum (k+2)(k+1)a_{k+2}x^k + \sum a_{k-2}x^k = 0$

$(k+2)(k+1)a_{k+2} + a_{k-2} = 0 \Rightarrow (k+2)(k+1)a_{k+2} = -a_{k-2}$

since a_{k+2} is given in terms of a_{k-2}, thus a_0 determines a_4 which in turn determines a_8 and so on. Also a_1 determines a_5 which in turn determines a_9 and so on.

$a_0 \longrightarrow a_4 \longrightarrow a_8 \longrightarrow a_{12}$ ...

$a_1 \longrightarrow a_5 \longrightarrow a_9 \longrightarrow a_{13}$ ...

$\therefore a_2 = 0$

$a_3 = 0$

$a_4 = \dfrac{-a_0}{4*3}$

$a_5 = \dfrac{-a_1}{5*4}$

$a_6 = 0$

$a_7 = 0$

$a_8 = \dfrac{-a_4}{8*7} = \dfrac{1}{8*7}\left(\dfrac{a_0}{4*3}\right)$

$a_9 = \dfrac{-a_5}{9*8} = \dfrac{1}{9*8}\left(\dfrac{a_1}{5*4}\right)$

$a_{10} = 0$

$a_{11} = 0$

$\vdots$

$\therefore y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$

$= a_0 + a_1 x + \left(\dfrac{-a_0}{12}\right)x^4 + \left(\dfrac{-a_1}{20}\right)x^5 + \dfrac{a_0}{672}x^8 + \dfrac{a_1}{1440}x^9 + \cdots$

$y = a_0\left(1 - \dfrac{x^4}{12} + \dfrac{x^8}{672} \cdots\right) + a_1\left(x - \dfrac{x^5}{20} + \dfrac{x^9}{1440} \cdots\right)$

ex. Find a series solution for the D.E.:

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$$

sol. $P(x) = \frac{x}{1-x^2}$ analytic at $x=0$

$Q(x) = \frac{4}{1-x^2}$ analytic at $x=0$

$\therefore x=0$ is ordinary point

Let $y = \sum a_n x^n$; $y' = \sum n a_n x^{n-1}$; $y'' = \sum n(n-1) a_n x^{n-2}$

$$(1-x^2)\sum n(n-1)a_n x^{n-2} - x\sum n a_n x^{n-1} + 4\sum a_n x^n = 0$$

$$\sum n(n-1)a_n x^{n-2} - \sum n(n-1)a_n x^n - \sum n a_n x^n + 4\sum a_n x^n = 0$$

$$\sum n(n-1)a_n x^{n-2} + \sum (4 - n - n^2 + n) a_n x^n = 0$$

$$\sum n(n-1)a_n x^{n-2} - \sum (n^2-4) a_n x^n = 0$$

For the 1st power series: $\sum n(n-1) a_n x^{n-2}$

Let $n-2 = k \Rightarrow n-1 = k+1$; $n = k+2$

$$\therefore \sum n(n-1) a_n x^{n-2} = \sum (k+2)(k+1)\, a_{k+2} x^k$$

For the second power series: $\sum (n^2-4) a_n x^n = \sum (k^2-4) a_k x^k$

$$\therefore \sum (k+2)(k+1) a_{k+2} x^k - \sum (k^2-4) a_k x^k = 0 \quad \div x^k$$

$$(k+2)(k+1)a_{k+2} - (k^2-4)a_k = 0$$

$$a_{k+2} = \frac{k^2-4}{(k+2)(k+1)} a_k$$

Since a_{k+2} is given in terms of a_k, thus:

$a_0 \longrightarrow a_2 \longrightarrow a_4 \longrightarrow a_6 \cdots$

$a_1 \longrightarrow a_3 \longrightarrow a_5 \longrightarrow a_7 \cdots$

$$\therefore a_2 = \frac{-4a_0}{2} = -2a_0$$

ex. continued: $a_3 = \frac{-3}{6} a_1 = -\frac{a_1}{2}$

$a_4 = 0$

$a_5 = \frac{5}{20} a_3 = \frac{a_3}{4} = \frac{1}{4}\left(-\frac{a_1}{2}\right) = -\frac{a_1}{8}$

$a_6 = \frac{12}{30} a_4 = 0$

$\therefore y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$

$= a_0 + a_1 x - 2a_0 x^2 - \frac{a_1}{2} x^3 + 0 - \frac{a_1}{8} x^5 + 0 + \cdots$

$= a_0\left(1 - 2x^2 + \cdots\right) + a_1\left(x - \frac{x^3}{2} - \frac{x^5}{8} + \cdots\right)$

ex. Find the first 8 terms in the power series solution of the D.E.: $y'' + 2xy' + (1+x^2)y = 0$

Sol. $P(x) = 2x$ analytic at $x=0$

$Q(x) = 1+x^2$ analytic at $x=0$

$\therefore x=0$ is ordinary point

Let $y = \sum a_n x^n$; $y' = \sum n a_n x^{n-1}$; $y'' = \sum n(n-1) a_n x^{n-2}$

$\sum n(n-1) a_n x^{n-2} + 2x \sum n a_n x^{n-1} + (1+x^2) \sum a_n x^n = 0$

$\sum n(n-1) a_n x^{n-2} + 2\sum n a_n x^n + \sum a_n x^n + \sum a_n x^{n+2} = 0$

$\sum n(n-1) a_n x^{n-2} + \sum (2n+1) a_n x^n + \sum a_n x^{n+2} = 0$

For the 1st power series: $\sum n(n-1) a_n x^{n-2}$

Let $n-2 = k \Rightarrow n-1 = k+1$; $n = k+2$

$\therefore \sum n(n-1) a_n x^{n-2} = \sum (k+2)(k+1) a_{k+2} x^k$

ex. Continued:

For the 2nd power series: $\sum (2n+1) a_n x^n = \sum (2k+1) a_k x^k$

For the third power series: $\sum a_n x^{n+2}$

Let $n+2=k \Rightarrow n=k-2$

$\therefore \sum a_n x^{n+2} = \sum a_{k-2} x^k$

$$\sum (k+2)(k+1) a_{k+2} x^k + \sum (2k+1) a_k x^k + \sum a_{k-2} x^k = 0$$

$$(k+2)(k+1) a_{k+2} + (2k+1) a_k + a_{k-2} = 0$$

$$a_{k+2} = \frac{-(2k+1)}{(k+2)(k+1)} a_k - \frac{1}{(k+2)(k+1)} a_{k-2}$$

$$a_2 = \frac{-1}{2} a_0$$

$$a_3 = \frac{-3}{6} a_1 = \frac{-1}{2} a_1$$

$$a_4 = \frac{-5}{12} a_2 - \frac{1}{12} a_0 = \frac{-5}{12}\left(-\frac{1}{2} a_0\right) - \frac{1}{12} a_0 = \frac{1}{8} a_0$$

$$a_5 = \frac{-7}{20} a_3 - \frac{1}{20} a_1 = \frac{-7}{20}\left(\frac{-1}{2} a_1\right) - \frac{1}{20} a_1 = \frac{1}{8} a_1$$

$$a_6 = \frac{-9}{30} a_4 - \frac{1}{30} a_2 = \frac{-9}{30}\left(\frac{1}{8} a_0\right) - \frac{1}{30}\left(-\frac{1}{2} a_0\right) = \frac{-1}{48} a_0$$

$$a_7 = \frac{-11}{42} a_5 - \frac{1}{42} a_3 = \frac{-11}{42}\left(\frac{1}{8} a_1\right) - \frac{1}{42}\left(\frac{-1}{2} a_1\right) = \frac{-1}{48} a_1$$

$$\therefore y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7$$

$$= a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{1}{2} a_1 x^3 + \frac{1}{8} a_0 x^4 + \frac{1}{8} a_1 x^5 - \frac{1}{48} a_0 x^6 - \frac{1}{48} a_0 x^7$$

$$= a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48}\right) + a_1\left(x - \frac{x^3}{2} + \frac{x^5}{8} - \frac{x^7}{48}\right)$$

steps for solving a differential equation if $x_0 = 0$ is a regular singular point (Frobenius Method):

If $x_0 = 0$ is a regular singular point of the equation:

$$y'' + P(x)\,y' + Q(x)\,y = 0$$

Then the series solution is $y = \sum a_n (x - x_0)^{n+r} = \sum a_n x^{n+r}$

1. Check whether $x_0 = 0$ is a regular singular point. If it is so then let $y = \sum a_n x^{n+r}$
2. Substitute the values of y, y' and y'' in the given eq.
3. Equate to zero the coefficients of lowest power of x. This gives a quadratic equation in r which is known as the indicial equation.
4. The indicial eq. will give two values of r. The series solution will depend on the nature of the roots of the indicial eq.

(A) When the roots are distinct and not differing by an integer (i.e. $r_1 - r_2 \neq 0$ or a positive integer):

$$y_1 = (x - x_0)^{r_1} \sum a_n (x - x_0)^n$$

$$y_2 = (x - x_0)^{r_2} \sum b_n (x - x_0)^n$$

(B) When the roots are equal (i.e. $r_1 = r_2$):

$$y_1 = (x - x_0)^{r} \sum a_n (x - x_0)^n$$

$$y_2 = (x - x_0)^{r+1} \sum b_n (x - x_0)^n + y_1 \ln (x - x_0)$$

(c) When the roots are distinct and differ by a positive integer (i.e. $r_1 - r_2 =$ integer , $r_1 > r_2$):

$$y_1 = (x-x_0)^{r_1} \sum a_n (x-x_0)^n$$

$$y_2 = (x-x_0)^{r_2} \sum b_n (x-x_0)^n + A\, y_1 \ln(x-x_0)$$

where A is a constant may or may not be zero.

ex. Find a general solution about $x=0$ for the D.E.:

$$3x\,y'' + 2y' + y = 0$$

sol. $y'' + \frac{2}{3x} y' + \frac{1}{3x} y = 0$

$P(x) = \frac{2}{3x}$ not analytic at $x=0$; $(x-x_0)P(x) = \frac{2}{3}$ analytic at $x=0$

$Q(x) = \frac{1}{3x}$ not analytic at $x=0$; $(x-x_0)^2 Q(x) = \frac{x}{3}$ analytic at $x=0$

$\therefore$ $x=0$ is regular singular point.

$$\therefore y = \sum a_n (x-x_0)^{n+r} = \sum a_n x^{n+r}$$

$$y' = \sum (n+r) a_n x^{n+r-1} \quad ; \quad y'' = \sum (n+r)(n+r-1) a_n x^{n+r-2}$$

$$\therefore 3x \sum (n+r)(n+r-1) a_n x^{n+r-2} + 2 \sum (n+r) a_n x^{n+r-1} + \sum a_n x^{n+r} = 0$$

$$\sum 3(n+r)(n+r-1) a_n x^{n+r-1} + \sum 2(n+r) a_n x^{n+r-1} + \sum a_n x^{n+r} = 0$$

$$\sum [3(n+r)(n+r-1) + 2(n+r)] a_n x^{n+r-1} + \sum a_n x^{n+r} = 0$$

The indicial eq. is obtained by putting $n=0$ and equating the coefficient of the lowest power of x to zero.

$$\sum [3r(r-1) + 2r] a_0 x^{r-1} + \sum a_0 x^r = 0$$

$\therefore 3r(r-1)+2r=0$ indicial eq.

$3r^2-3r+2r=0 \Rightarrow 3r^2-r=0 \Rightarrow r(3r-1)=0$

$r_1=0$, $r_2=\frac{1}{3}$; $r_2-r_1=\frac{1}{3}-0=\frac{1}{3}$

r_1 and r_2 are distinct roots not differing by an integer

$\therefore y_1=(x-x_0)^{r_1}\sum a_n(x-x_0)^n = x^0\sum a_n x^n = \sum a_n x^n$

$y_2=(x-x_0)^{r_2}\sum b_n(x-x_0)^n = x^{\frac{1}{3}}\sum b_n x^n$

$y=y_1+y_2=\sum a_n x^n + x^{\frac{1}{3}}\sum b_n x^n$

Ex. Solve $xy''+y'+x^2y=0$ around the origin

sol. $y''+\frac{1}{x}y'+xy=0$

$P(x)=\frac{1}{x}$ is not analytic at $x=0$; $xP(x)=1$ analytic at $x=0$

$Q(x)=x$ analytic at $x=0$; $x^2Q(x)=x^3$ analytic at $x=0$

$\therefore x=0$ is regular singular point

$\therefore y=\sum a_n(x-x_0)^{n+r}=\sum a_n x^{n+r}$

$y'=\sum(n+r)a_n x^{n+r-1}$; $y''=\sum(n+r)(n+r-1)a_n x^{n+r-2}$

$x\sum(n+r)(n+r-1)a_n x^{n+r-2}+\sum(n+r)a_n x^{n+r-1}+x^2\sum a_n x^{n+r}=0$

$\sum(n+r)(n+r-1)a_n x^{n+r-1}+\sum(n+r)a_n x^{n+r-1}+\sum a_n x^{n+r+2}=0$

$\sum[(n+r)(n+r-1)+(n+r)]a_n x^{n+r-1}+\sum a_n x^{n+r+2}=0$

Let $n=0$, the coefficient of the lowest power of x is:

$r(r-1)+r=0 \Rightarrow r^2=0 \Rightarrow r_1=r_2=0$

r_1 and r_2 are equal

ex. continued:

$$\therefore y_1 = (x-x_0)^r \sum a_n (x-x_0)^n = x^0 \sum a_n x^n = \sum a_n x^n$$

$$y_2 = (x-x_0)^{r+1} \sum b_n (x-x_0)^n + y_1 \ln(x-x_0)$$

$$= x^1 \sum b_n x^n + \ln(x) \sum a_n x^n$$

$$\therefore y = y_1 + y_2 = \sum a_n x^n + \ln x \sum a_n x^n + x \sum b_n x^n$$

$$= (1+\ln x) \sum a_n x^n + x \sum b_n x^n$$

ex. Find the power series solution of the D.E.:

$$x^2 y'' + 4xy' + (x^2+2)y = 0$$

sol. $P(x) = \frac{4}{x}$ not analytic at $x=0$; $xP(x) = 4$ analytic at $x=0$

$Q(x) = \frac{1}{x^2(x^2+2)}$ not analytic at $x=0$; $x^2Q(x) = \frac{1}{x^2+2}$ analytic at $x=0$

$\therefore x=0$ is a regular singular point

$$y = \sum a_n x^{n+r} \quad ; \quad y' = \sum (n+r) a_n x^{n+r-1} \quad ; \quad y'' = \sum (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x^2 \sum (n+r)(n+r-1) a_n x^{n+r-2} + 4x \sum (n+r) a_n x^{n+r-1} + (x^2+2) \sum a_n x^{n+r} = 0$$

$$\sum (n+r)(n+r-1) a_n x^{n+r} + \sum 4(n+r) a_n x^{n+r} + \sum a_n x^{n+r+2} + \sum 2a_n x^{n+r} = 0$$

$$\sum [(n+r)(n+r-1) + 4(n+r) + 2] a_n x^{n+r} + \sum a_n x^{n+r+2} = 0$$

Let $n=0$, the coefficient of the smallest power of x is:

$$r(r-1) + 4r + 2 = 0 \Rightarrow r^2 - r + 4r + 2 = 0 \Rightarrow r^2 + 3r + 2 = 0$$

$$r_1 = -1 \quad ; r_2 = -2 \quad ; \quad r_1 - r_2 = -1 + 2 = 1$$

r_1 and r_2 are distinct and differ by a positive integer

ex. continued:

$$\therefore y_1 = (x-x_0)^{r_1} \sum a_n (x-x_0)^n = x^{-1} \sum a_n x^n$$

$$y_2 = (x-x_0)^{r_2} \sum b_n (x-x_0)^n + A\, y_1 \ln(x-x_0)$$

$$= x^{-2} \sum b_n x^n + A\, x^{-1} \sum a_n x^n \ln x$$

$$\therefore y = y_1 + y_2 = x^{-1} \sum a_n x^n + x^{-2} \sum b_n x^n + A\, x^{-1} \ln x \sum a_n x^n$$

$$= (1 + A \ln x) \cdot x^{-1} \sum a_n x^n + x^{-2} \sum b_n x^n$$

Partial Differential Equations:

A PDE is an equation that contains partial differential coefficients, independent variables and dependent variables. A PDE involves at least two independent variables and one dependent variable.

i.e. $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$ 1st order PDE

$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ 2nd order PDE

The independent variables will be denoted by x and y and the dependent variable by z. The partial differential coefficients are denoted as follows:

$\frac{\partial z}{\partial x} = p$; $\frac{\partial z}{\partial y} = q$; $\frac{\partial^2 z}{\partial x^2} = r$; $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = s$

$\frac{\partial^2 z}{\partial y^2} = t$

Famous Partial Differential Equations:

PDE generally occurs in the problem of physics and engineering Some important eqs. are:

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ Laplace's equation

$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$ Wave equation

$\frac{\partial u}{\partial t} = c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$ Heat conduction equation

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Formation of Partial Differential Equations:

APDE is formed by two methods:

1. By elimination of arbitrary constants:

ex. Form a PDE from the equations by eliminating arbitrary constants.

i) $z = ax + by + ab$ ii) $x^2 + y^2 + (z-c)^2 = a^2$

iii) $z = ax e^y + \frac{1}{2} a^2 e^{2y} + b$ iv) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

sol. i) differentiating z partially w.r.t. x and y, then:

$$\frac{\partial z}{\partial x} = a \quad \text{or} \quad p = \frac{\partial z}{\partial x} = a$$

$$\frac{\partial z}{\partial y} = b \quad \text{or} \quad q = \frac{\partial z}{\partial y} = b$$

$$\therefore z = \frac{\partial z}{\partial x} x + \frac{\partial z}{\partial y} y + \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}$$

or $z = px + qy + pq$

ii) $x^2 + y^2 + (z-c)^2 = a^2$

differentiating (z) partially w.r.t. x:

$$2x + 2(z-c)\frac{\partial z}{\partial x} = 0 \Rightarrow 2x + 2(z-c)p = 0$$

$$x + (z-c)p = 0 \Rightarrow (z-c) = -\frac{x}{p}$$

differentiating z partially w.r.t. y:

$$2y + 2(z-c)\frac{\partial z}{\partial y} = 0 \Rightarrow 2y + 2(z-c)q = 0$$

$$y + (z-c)q = 0 \Rightarrow y - \frac{x}{p} q = 0 \Rightarrow py - qx = 0$$

ex. continued:

iii) $z = ax e^{y} + \frac{1}{2} a^2 e^{2y} + b$

$\frac{\partial z}{\partial x} = a e^{y} \Rightarrow p = a e^{y}$ and $p^2 = a^2 e^{2y}$

$\frac{\partial z}{\partial y} = ax e^{y} + a^2 e^{2y} \Rightarrow q = ax e^{y} + a^2 e^{2y}$

$\therefore q = (a e^{y}) x + a^2 e^{2y} = px + p^2 \Rightarrow q = px + p^2$

note that $p^2 = \left(\frac{\partial z}{\partial x}\right)^2 \neq \frac{\partial^2 z}{\partial x^2}$

iv) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

differentiating z partially w.r.t. x:

$\frac{2x}{a^2} + \frac{2z}{c^2} \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{2x}{a^2} + \frac{2z}{c^2} p = 0$

$\frac{x}{a^2} + \frac{z}{c^2} p = 0$ ——①

differentiating z partially w.r.t. y:

$\frac{2y}{b^2} + \frac{2z}{c^2} \frac{\partial z}{\partial y} = 0 \Rightarrow \frac{2y}{b^2} + \frac{2z}{c^2} q = 0 \Rightarrow \frac{y}{b^2} + \frac{z}{c^2} q = 0$ ——②

It is not possible to eliminate constants a, b and c from eqs. ① and ②, then differentiate eq-① partially w.r.t. x:

$\frac{1}{a^2} + \frac{z}{c^2} \frac{\partial p}{\partial x} + \frac{p}{c^2} \frac{\partial z}{\partial x} = 0$

where $\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = r$

$\therefore \frac{1}{a^2} + \frac{z}{c^2} r + \frac{p^2}{c^2} = 0$ ——③

Multiply eq-③ by $-x$ and add to eq-①:

$-\frac{x}{a^2} - \frac{zx}{c^2} r - \frac{p^2}{c^2} x = 0$ ——④

$\frac{x}{a^2} + \frac{z}{c^2} p = 0$ ——①

ex. continued:

$$\frac{z}{c^2}p - \frac{zx}{c^2}r - \frac{p^2}{c^2}x = 0 \Rightarrow zp - zxr - xp^2 = 0$$

* ممكن نعمل elimination عن طريق تطبيق نفس الخطوات على معادلة (2).

Note: From the above examples, we notice that if the number of constants to be eliminated is equal to the number of independent variables then the derived PDE is of the first order. But if the number of constants to be eliminated is greater than the number of independent variables (i.e. ex. iv), then the derived PDE is of second or higher order.

ex. Form a PDE by eliminating h and k from: $(x-h)^2+(y-k)^2+z^2=a^2$

sol. $2(x-h) + 2z\frac{\partial z}{\partial x} = 0 \Rightarrow (x-h) + zp = 0 \Rightarrow (x-h) = -zp$

$2(y-k) + 2z\frac{\partial z}{\partial y} = 0 \Rightarrow (y-k) + zq = 0 \Rightarrow (y-k) = -zq$

$\therefore z^2p^2 + z^2q^2 + z^2 = a^2$; $p^2 = (\frac{\partial z}{\partial x})^2 \neq \frac{\partial^2 z}{\partial x^2}$

$q^2 = (\frac{\partial z}{\partial y})^2 \neq \frac{\partial^2 z}{\partial y^2}$

Although the number of constants in the above ex. is 3 (h, k and a) but the constants to be eliminated are only two (h and k) then the order of the formed PDE is first order.

2- By elimination of arbitrary functions:

ex. Form a PDE from the equations by eliminating arbitrary functions.

i) $z = f(x^2 - y^2)$; where f is arbitrary function of x and y

ii) $z = x + y + f(xy)$

iii) $z = f_1(x+jy) + f_2(x-jy)$; where f_1 and f_2 are arbitrary funcs.

sol. i) differentiate z partially w.r.t. x :

$$\frac{\partial z}{\partial x} = f'(x^2-y^2)(2x) \Rightarrow p = f'(x^2-y^2)(2x)$$

$$\therefore f'(x^2-y^2) = \frac{p}{2x} \quad \text{---------} (1)$$

differentiate z partially w.r.t. y :

$$\frac{\partial z}{\partial y} = f'(x^2-y^2)(-2y) \Rightarrow q = f'(x^2-y^2)(-2y) \quad \text{---------} (2)$$

substitute (1) into (2) :

$$q = \frac{p}{2x}(-2y) \Rightarrow 2xq + 2py = 0 \Rightarrow qx + py = 0$$

ii) $$\frac{\partial z}{\partial x} = 1 + f'(xy)\,y \Rightarrow p = 1 + f'(xy)\,y \Rightarrow \frac{p-1}{y} = f'(xy) \quad \text{---------} (1)$$

$$\frac{\partial z}{\partial y} = 1 + f'(xy)\,x \Rightarrow q = 1 + f'(xy)\,x \quad \text{---------} (2)$$

substitute (1) into (2) yields:

$$q = 1 + x\left(\frac{p-1}{y}\right) \Rightarrow qy - y - px + x = 0 \Rightarrow px - qy = x - y$$

iii) $$\frac{\partial z}{\partial x} = f_1'(x+jy) + f_2'(x-jy)$$

$$\frac{\partial^2 z}{\partial x^2} = f_1''(x+jy) + f_2''(x-jy)$$ ← بما ان لدينا دالتين f_1 و f_2 لذا نكامل نشتق مرتين

$$\frac{\partial z}{\partial y} = f_1'(x+jy)(j) + f_2'(x-jy)(-j)$$

$$\frac{\partial^2 z}{\partial y^2} = f_1''(x+jy)(j^2) + f_2''(x-jy)(j^2) \Rightarrow -\frac{\partial^2 z}{\partial y^2} = f_1''(x+jy) + f_2''(x-jy)$$

ex. continued:

$$\frac{\partial^2 z}{\partial x^2} = -\frac{\partial^2 z}{\partial y^2} \Rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \Rightarrow r + t = 0$$

ex. Eliminate f_1 and f_2 to form a PDE from: $z = x f_1(x+y) + f_2(x+y)$

sol. $\frac{\partial z}{\partial x} = x f_1'(x+y) + f_1(x+y) + f_2'(x+y)$

$$\frac{\partial^2 z}{\partial x^2} = x f_1''(x+y) + f_1'(x+y) + f_1'(x+y) + f_2''(x+y)$$

$$r = x f_1''(x+y) + 2 f_1'(x+y) + f_2''(x+y) \quad \text{——①}$$

$$\frac{\partial z}{\partial y} = x f_1'(x+y) + f_2'(x+y)$$

$$\frac{\partial^2 z}{\partial y^2} = x f_1''(x+y) + f_2''(x+y)$$

$$t = x f_1''(x+y) + f_2''(x+y) \quad \text{——②}$$

substitute ② into ① to get:

$$r = t + 2 f_1'(x+y)$$

To eliminate $f_1'(x+y)$, then find $\frac{\partial^2 z}{\partial x \partial y}$:

$$\frac{\partial^2 z}{\partial x \partial y} = x f_1''(x+y) + f_1'(x+y) + f_2''(x+y)$$

$$s = t + f_1'(x+y) \Rightarrow f_1'(x+y) = s - t$$

$$\therefore r = t + 2(s-t) \Rightarrow r - 2s + t = 0$$

or $$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

Linear Homogeneous PDEs:

A PDE is said to be linear if it is of the first degree in the dependent variable z and its partial derivatives, otherwise it is nonlinear.

A linear PDE is homogeneous if each of its terms contains either z or one of its partial derivatives.

i.e.

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$ linear homogeneous PDE

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = f(x,y)$$ linear nonhomogeneous PDE

↖ because this term does not contain z or one of its partial derivatives.

$$\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial z}{\partial y} = 0$$ nonlinear PDE

Solution of PDE by direct integration method:

The equations which contain only one partial derivative can be solved by direct integration.

ex. Solve $t = \sin(xy)$

sol. $t = \frac{\partial^2 z}{\partial y^2} = \sin(xy)$

integrating w.r.t. y: $\frac{\partial z}{\partial y} = -\cos(xy)\cdot\frac{1}{x} + f(x)$

where $f(x)$ is arbitrary function (the constant is being taken as a function of x).

again integrating w.r.t. y:

$$z = -\frac{1}{x}\left[\sin(xy)\frac{1}{x}\right] + y f(x) + g(x)$$

$$\therefore z = -\frac{1}{x^2}\sin(xy) + y f(x) + g(x)$$ where $f(x)$ and $g(x)$ are arbitrary functions

ex. Solve $xy\,s = 1$

sol. $s = \dfrac{\partial^2 z}{\partial x \partial y} \Rightarrow \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{1}{xy}$

integrating w.r.t. y: $\dfrac{\partial z}{\partial x} = \dfrac{1}{x}\ln(y) + f(x)$

integrating w.r.t. x: $z = \ln(x)\ln(y) + F(x) + g(y)$

where $F(x) = \int f(x)\,dx$

ex. Solve $\dfrac{\partial^2 z}{\partial x \partial y} = x^2 y$

sol. integrating w.r.t. y: $\dfrac{\partial z}{\partial x} = x^2 \dfrac{y^2}{2} + f(x)$

integrating w.r.t. x: $z = \dfrac{y^2}{2}\cdot\dfrac{x^3}{3} + F(x) + g(y)$

$\therefore z = \dfrac{x^3 y^2}{6} + F(x) + g(y)$

where $F(x) = \int f(x)\,dx$

H-W. Solve:

① $\dfrac{\partial^2 z}{\partial x \partial y} = xy^2$ ans: $z = \dfrac{x^2 y^3}{6} + F(x) + g(y)$

② $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{y}{x} + 2$ ans: $z = \dfrac{y^2}{2}\ln(x) + 2xy + F(x) + g(y)$

③ $s = 4x\sin(3xy)$ ans: $z = -\dfrac{4}{9y}\sin(3xy) + f(x) + g(y)$

Laplace Transform:

The Laplace transform is a powerful method for solving differential equations and initial value problems. If $f(t)$ is a function defined for all $t \geq 0$, then its Laplace transform is given by:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt \qquad \text{where } s = \alpha + jw$$

and $$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{j2\pi} \int_{\alpha - j\infty}^{\alpha + j\infty} F(s)\, e^{st}\, ds$$

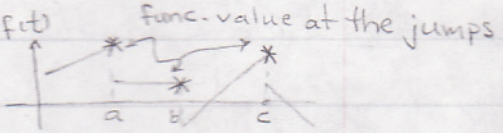
ex. Find L.T. of $f(t) = 1 \; ; t \geq 0$

Sol. $$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty 1 \cdot e^{-st}\, dt = -\frac{1}{s}\left(e^{-st}\right)\Big|_0^\infty = \frac{-1}{s}\left(e^{-\infty} - e^{0}\right) = \frac{1}{s}$$

ex. Find L.T. of $f(t) = e^{at}\, u(t)$

Sol. $$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{at} e^{-st}\, dt = \int_0^\infty e^{(a-s)t}\, dt = \frac{1}{a-s}\left(e^{-\infty} - e^{0}\right)$$

$$= \frac{1}{s-a}$$



Existence of Laplace Transform:

If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geq 0$ and $|f(t)| \leq Me^{kt}$ (where M and k are constants) for all $t \geq 0$, then the Laplace transform of $f(t)$ exists.

this means that $f(t)$ does not grow too fast

ex. Find L.T. of $f(t) = \cos \omega t \; ; t \geq 0$

sol: $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty \cos\omega t \, e^{-st} dt = \int_0^\infty \left[\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right] e^{-st} dt$

$= \frac{1}{2}\int_0^\infty e^{j\omega t} \cdot e^{-st} dt + \frac{1}{2}\int_0^\infty e^{-j\omega t} \cdot e^{-st} dt = \frac{1}{2}\int_0^\infty e^{-(s-j\omega)t} dt + \frac{1}{2}\int_0^\infty e^{-(s+j\omega)t} dt$

$= \frac{1}{2} \cdot \frac{-1}{s-j\omega}(e^{-\infty} - e^{0}) + \frac{1}{2} \cdot \frac{-1}{s+j\omega}(e^{-\infty} - e^{0}) = \frac{1}{2}\left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right)$

$= \frac{1}{2}\left(\frac{s+j\omega+s-j\omega}{s^2+j\omega s - j\omega s - j^2\omega^2}\right) = \frac{s}{s^2+\omega^2} \quad ; s > 0$

ex. Find L.T. of $f(t) = \sinh at \; ; t \geq 0$

sol: $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty \sinh at \, e^{-st} dt = \int_0^\infty \left[\frac{e^{at} - e^{-at}}{2}\right] e^{-st} dt$

$= \frac{1}{2}\int_0^\infty e^{at} \cdot e^{-st} dt - \frac{1}{2}\int_0^\infty e^{-at} \cdot e^{-st} dt = \frac{1}{2}\int_0^\infty e^{-(s-a)t} dt - \frac{1}{2}\int_0^\infty e^{-(s+a)t} dt$

$= \frac{1}{2} \cdot \frac{-1}{s-a}(e^{-\infty} - e^{0}) - \frac{1}{2} \cdot \frac{-1}{s+a}(e^{-\infty} - e^{0}) = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2-a^2} \quad ; s > 0$

H.W. Find L.T. of: ① $f(t) = \sin\omega t \, u(t)$; ② $f(t) = \cosh at \, u(t)$

ans: ① $F(s) = \frac{\omega}{s^2+\omega^2}$; ② $F(s) = \frac{s}{s^2-a^2}$

The Gamma Function: (Generalized factorial function)

$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$

To determine the relation of $\Gamma(x)$ to the factorial function $(x!)$:

Let $u = e^{-t}$; $dv = t^{x-1} dt \Rightarrow du = -e^{-t} dt$; $v = \frac{t^x}{x}$

$\therefore \Gamma(x) = e^{-t}\frac{t^x}{x}\Big|_0^\infty - \int_0^\infty \frac{t^x}{x}(-e^{-t})dt = +\frac{1}{x}\int_0^\infty t^x e^{-t} dt$

($\frac{t^x}{x} \to 0$ for $x > 0$)

but we have: $\Gamma(x+1) = \int_0^\infty e^{-t} t^x dt$

$\therefore \Gamma(x) = \frac{1}{x}\Gamma(x+1) \Rightarrow \Gamma(x+1) = x\,\Gamma(x) \; ; x > 0$

also:

$\Gamma(1) = \int_0^\infty e^{-t} t^{1-1} dt = \int_0^\infty e^{-t} dt = -e^{-\infty} + e^{0} = 1$

$\Gamma(1+1) = \Gamma(x+1) = x\Gamma(x)$

$\Gamma(2) = 1 \cdot \Gamma(1) = 1!$

$\Gamma(3) = 2 \cdot \Gamma(2) = 2 * 1! = 2!$

$\Gamma(4) = 3 \cdot \Gamma(3) = 3 * 2! = 3!$

$\Gamma(5) = 4 \cdot \Gamma(4) = 4 * 3! = 4!$

In general: $\Gamma(n+1) = n!$; n = +ve integer

Notes:

① $\Gamma(0) = \frac{1}{0}\Gamma(1) = \begin{cases} -\infty & \text{for } 0^- \\ +\infty & \text{for } 0^+ \end{cases}$

② $\Gamma(-1) = \frac{1}{-1}\Gamma(0) = \pm\infty$

$\Gamma(-2) = \pm\infty$

$\therefore \Gamma(x) = \pm\infty \quad x$ = -ve integer

ex. Find $\Gamma(4.7)$

Sol: $\Gamma(4.7) = 3.7\,\Gamma(3.7) = 3.7 * 2.7\,\Gamma(2.7) = 3.7 * 2.7 * 1.7\,\Gamma(1.7)$

where $\Gamma(1.7)$ can be found from certain mathematical table.

ex. Find $\Gamma(-2.7)$

Sol: $\Gamma(-2.7) = \frac{1}{-2.7}\Gamma(-1.7) = \frac{1}{-2.7} * \frac{1}{-1.7}\Gamma(-0.7) = \frac{1}{-2.7} * \frac{1}{-1.7} * \frac{1}{-0.7}\Gamma(0.3)$

$= \frac{1}{-2.7} * \frac{1}{-1.7} * \frac{1}{-0.7} * \frac{1}{0.3}\Gamma(1.3)$

where $\Gamma(1.3)$ can be found from certain mathematical table.

To find Laplace transform of t^n:

$\mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st}\,dt$

let $st = z \Rightarrow t = \frac{z}{s}$; $dt = \frac{1}{s}dz$

$\mathcal{L}\{t^n\} = \int_0^\infty (\frac{z}{s})^n e^{-z} \cdot \frac{1}{s}dz = \frac{1}{s^n} \cdot \frac{1}{s}\int_0^\infty z^n e^{-z}dz$

but we have $\int_0^\infty z^n e^{-z}dz = \Gamma(n+1)$

$$\therefore \mathcal{L}\{t^n\} = \begin{cases} \dfrac{\Gamma(n+1)}{s^{n+1}} & ; n > -1 \\ \dfrac{n!}{s^{n+1}} & ; n \text{ +ve integer} \end{cases}$$

if $\mathcal{L}\{y(t)\} = \frac{1}{s+3}$ find $y(t)$: الحل

Example. if $\mathcal{L}\{y(t)\} = \dfrac{s+1}{s^2+s-6}$, find $y(t)$. **Sol.** $y(t) = e^{-3t}$

Sol. Using partial fractions method:

$$\frac{s+1}{s^2+s-6} = \frac{s+1}{(s+3)(s-2)} = \frac{A}{s+3} + \frac{B}{s-2} = \frac{A(s-2)+B(s+3)}{(s+3)(s-2)}$$

$$= \frac{As-2A+Bs+3B}{(s+3)(s-2)} = \frac{(A+B)s+(3B-2A)}{(s+3)(s-2)}$$

$\therefore A+B=1$ and $3B-2A=1$ solving these equations yields: $A = \frac{2}{5}$; $B = \frac{3}{5}$

$$\therefore \mathcal{L}\{y(t)\} = \frac{2/5}{s+3} + \frac{3/5}{s-2}$$

$$\therefore y(t) = \frac{2}{5}e^{-3t} + \frac{3}{5}e^{+2t}$$

ex. if $\mathcal{L}\{y(t)\} = \frac{1}{s^4}$, find $y(t)$.

sol.

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \Rightarrow \frac{1}{n!}\mathcal{L}\{t^n\} = \frac{1}{s^{n+1}}$$

if $n=3 \Rightarrow \frac{1}{s^{3+1}} = \frac{1}{s^4} = \frac{1}{3!}\mathcal{L}\{t^3\}$

$$\therefore y(t) = \frac{1}{6}t^3$$

H.W. Find I.L.T. of: $\frac{1}{s^2+9}$; $\frac{2s+3}{s^2+9}$; $\frac{s+3}{(s+1)(s-3)}$

$\frac{j}{6}(e^{-j3t} - e^{j3t})$ ↙ $0.5e^{-t} + 1.5e^{3t}$ ↙

Linearity of the Laplace Transform:

if $\mathcal{L}\{f_1(t)\}$ and $\mathcal{L}\{f_2(t)\}$ both exist, then:

$$\mathcal{L}\{af_1(t)+bf_2(t)\} = a\mathcal{L}\{f_1(t)\} + b\mathcal{L}\{f_2(t)\}$$

where a and b are constants

Further General Theorems:

Theorem 1: If both $f(t)$ and $f'(t)$ are piecewise continuous and of exponential order, then:

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

proof:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

Let $dv = f'(t)\,dt \Rightarrow v = f(t)$; $u = e^{-st} \Rightarrow du = -se^{-st}\,dt$

$$\therefore \int_0^\infty e^{-st} f'(t)\,dt = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty f(t)\,[-se^{-st}]\,dt$$

$$\therefore \mathcal{L}\{f'(t)\} = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty f(t)\,e^{-st}\,dt = [e^{-\infty} f(\infty) - e^{0} f(0)] + s\,\mathcal{L}\{f(t)\}$$

$$\therefore \mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

Theorem 2: $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

proof:

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0)$$

and we have: $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

$$\therefore \mathcal{L}\{f''(t)\} = s[s\mathcal{L}\{f(t)\} - f(0)] - f'(0)$$

$$= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

In general:

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$

This is the L.T. of the derivative $f^{(n)}$ of any order

ex. Find L.T. of $f(t) = t\sin\omega t$

Sol. $f'(t) = \omega t\cos\omega t + \sin\omega t$ and $f'(0) = 0$

$f''(t) = -\omega^2 t\sin\omega t + \omega\cos\omega t + \omega\cos\omega t$

$= -\omega^2 t\sin\omega t + 2\omega\cos\omega t$

$\therefore \mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

$\therefore \mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} = -\omega^2\mathcal{L}\{f(t)\} + 2\omega\frac{s}{s^2+\omega^2}$

$\therefore \mathcal{L}\{f(t)\} = \frac{2\omega s}{(s^2+\omega^2)^2}$

ex. Find L.T. of $f(t) = \cos\omega t$ using derivative theorem.

Sol. $f(t) = \cos\omega t$; $f'(t) = -\omega\sin\omega t$; $f''(t) = -\omega^2\cos\omega t$

$f(0) = 1$; $f'(0) = 0$

$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

$\mathcal{L}\{-\omega^2\cos\omega t\} = s^2\mathcal{L}\{\cos\omega t\} - s - 0$

$-\omega^2\mathcal{L}\{\cos\omega t\} = s^2\mathcal{L}\{\cos\omega t\} - s$

$\therefore \omega^2\mathcal{L}\{\cos\omega t\} + s^2\mathcal{L}\{\cos\omega t\} = s$

$\therefore \mathcal{L}\{\cos\omega t\} = F(s) = \frac{s}{s^2+\omega^2}$

Theorem 3: Laplace Transform of Integral

If $f(t)$ is piecewise continuous and of exponential order, then:

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(\tau)\} = \frac{1}{s}F(s)$$

and $\mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(\tau)\,d\tau$

↳ if a L.T. contains the factor $\frac{1}{s}$, the inverse of that transform can be found by
① Suppressing the factor $\frac{1}{s}$
② determining the inverse of the remaining portion of the transform
③ Integrating that inverse with respect to t from $0 \to t$

ex. Find I.L.T. of $\frac{\omega}{s(s^2+\omega^2)}$

sol.

we have $\frac{\omega}{s^2+\omega^2} = \mathcal{L}\{\sin\omega\tau\}$

$\therefore \frac{1}{s}\cdot\frac{\omega}{s^2+\omega^2} = \mathcal{L}\left\{\int_0^t \sin\omega\tau\,d\tau\right\}$

$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{\omega}{s^2+\omega^2}\right\} = \int_0^t \sin\omega\tau\,d\tau = -\frac{\cos\omega\tau}{\omega}\Big|_0^t$

$= \frac{-1}{\omega}(\cos\omega t - \cos 0) = \frac{-1}{\omega}(\cos\omega t - 1)$

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H.W. Find I.L.T. of $\frac{1}{s^2(s^2+\omega^2)}$ Sol: $\frac{1}{\omega^2}\cdot\frac{\sin\omega t}{\omega^3}$

Differential Equations and Initial value problems:-

To discuss how the L.T. method solves ODEs and initial value problems, consider the initial value problem:

$$y'' + ay' + by = r(t) \quad ; \quad y(0) = K_0 \; ; \; y'(0) = K_1$$

where a and b are constants.

- step1: Setting up the subsidiary equation: obtained by transforming the above eq. by means of the derivative theorem (theorem 2)

i.e. $[s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s)$

where $R(s) = \mathcal{L}\{r(t)\}$; $Y = \mathcal{L}\{y\}$

Collecting Y terms, then: $(s^2 + as + b)Y = (s+a)y(0) + y'(0) + R(s)$

- step2: Solution of the subsidiary equation:

$$Y(s) = \frac{(s+a)y(0) + y'(0)}{s^2+as+b} + \frac{R(s)}{s^2+as+b}$$

and apply the given values of $y(0)$ and $y'(0)$

- step3: Inversion of Y to obtain $y = \mathcal{L}^{-1}\{Y\}$ (usually by partial expansions method)

ex. Solve: $y'' - y = t \quad ; \; y(0) = 1 \; ; \; y'(0) = 1$

Sol. step 1: $s^2Y - sy(0) - y'(0) - Y = \frac{1}{s^2}$; where $\frac{1}{s^2} = \mathcal{L}\{t\}$

then: $(s^2-1)Y = s + 1 + \frac{1}{s^2}$

step2: $Y = \frac{s+1}{s^2-1} + \frac{1}{s^2}\cdot\frac{1}{s^2-1} = \frac{(s+1)}{(s+1)(s-1)} + \left[\frac{A}{s^2} + \frac{B}{s^2-1}\right]$

$$Y = \frac{1}{(s-1)} + \left[\frac{As^2 - A + Bs^2}{s^2(s^2-1)}\right]$$

$A + B = 0 \; ; \; -A = 1 \Rightarrow A = -1 \; ; \; B = 1$

$$\therefore Y = \frac{1}{s-1} + \frac{1}{s^2-1} - \frac{1}{s^2}$$

step3: $y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$

$$\therefore y(t) = e^t + \sinh t - t$$

ex. Find the particular solution of the D.E.:

$y'' - 3y' + 2y = 12e^{-2t}$ for which $y = 2$ and $y' = 6$ when $t = 0$ (i.e. $y_0 = 2$; $y_0' = 6$)

sol: Let $y = f(t)$; $f(0) = 2$, $f'(0) = 6$

Taking L.T. of both sides:

$$s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) - 3s\mathcal{L}\{f(t)\} + 3f(0) + 2\mathcal{L}\{f(t)\} = \mathcal{L}\{12e^{-2t}\}$$

$$s^2\mathcal{L}\{f(t)\} - 2s - 6 - 3s\mathcal{L}\{f(t)\} + 6 + 2\mathcal{L}\{f(t)\} = \frac{12}{s+2}$$

$$\mathcal{L}\{f(t)\}[s^2 - 3s + 2] = \frac{12}{s+2} + 2s = \frac{12 + 2s^2 + 4s}{s+2}$$

$$\mathcal{L}\{f(t)\}[(s-2)(s-1)] = \frac{12 + 2s^2 + 4s}{s+2}$$

$$\therefore \mathcal{L}\{f(t)\} = \frac{12 + 2s^2 + 4s}{(s+2)(s-2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-2} + \frac{C}{s-1}$$

solve to find A, B and C, then:

$A = 1$, $B = 7$, $C = -6$

$$\therefore \mathcal{L}\{f(t)\} = \frac{1}{s+2} + \frac{7}{s-2} - \frac{6}{s-1}$$

$$\therefore f(t) = e^{-2t} + 7e^{2t} - 6e^{t}$$

ex. Solve for $y(t)$ from the simultaneous equations:

$$y' + 2y + 6\int_0^t z\,dt = -2u(t) \quad ; y_0 = -5 \; ; \; z_0 = 6$$

$$y' + z' + z = 0$$

Sol. Taking L.T. of both equations:

$$s\mathcal{L}\{y\} - y_0 + 2\mathcal{L}\{y\} + \frac{6}{s}\mathcal{L}\{z\} = \frac{-2}{s}$$

$$s\mathcal{L}\{y\} - y_0 + s\mathcal{L}\{z\} - z_0 + \mathcal{L}\{z\} = 0$$

substitute y_0 and z_0, then: $s\mathcal{L}\{y\} + 5 + 2\mathcal{L}\{y\} + \frac{6}{s}\mathcal{L}\{z\} = \frac{-2}{s}$

$\frac{6}{s+1} * [\; s\mathcal{L}\{y\} + 5 + s\mathcal{L}\{z\} - 6 + \mathcal{L}\{z\} = 0 \;]$

$$(s^2+2s)\mathcal{L}\{y\} + 6\mathcal{L}\{z\} = -2 - 5s \quad (1)$$

$$= \frac{6s}{s+1}\mathcal{L}\{y\} + 6\mathcal{L}\{z\} = \frac{6}{s+1} \quad \text{substitute for } \mathcal{L}\{z\}$$

$$\therefore \left(s^2+2s - \frac{6s}{s+1}\right)\mathcal{L}\{y\} = -2 - 5s - \frac{6}{s+1}$$

$$\left(\frac{s^3+s^2+2s^2+2s-6s}{s+1}\right)\mathcal{L}\{y\} = \frac{-2s-2-5s^2-5s-6}{s+1}$$

$$\mathcal{L}\{y\} = \frac{-5s^2-7s-8}{s^3+3s^2-4s} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s-1}$$

$s^3+3s^2-4s = s(s^2+3s-4)$

Solve to find A, B, and C; then: $A = 2$, $B = -3$, $C = -4$

$$\therefore y(t) = 2u(t) - 3e^{-4t} - 4e^{t}$$

H.W. Solve the previous example with respect to $z(t)$.

H.W. Find a solution of each of the following D.E.:

1) $y'' + 4y' - 5y = 0 \quad ; y_0 = 1 , y_0' = 0$

2) $y'' + 4y = u(t) \quad ; y_0 = y_0' = 0$

3) $y''' - y'' + 4y' - 4y = t \quad ; y_0 = y_0' = 0 ; y_0'' = 1$

4) Solve for $y(t)$ and $z(t)$:

$(D+1)y + (2D+3)z = 0$

$(D-4)y + (3D-8)z = \sin t$

$y_0 = 2$, $z_0 = -1$ | where Dy, Dz and y' and z' respectively

Theorem 4:

If $\mathcal{L}\{f(t)\} = s\,\phi(s)$ then $f(t) = \frac{d}{dt}\mathcal{L}^{-1}\phi(s)$

if a L.T. of a function contains the factor s, the inverse of that transform can be found by:

① Suppressing the factor s

② determining the inverse of the remaining portion of the transform

③ differentiating that inverse with respect to t.

ex. find $\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$

Sol. Let $\phi(s) = \frac{1}{s^2+4} \Rightarrow \mathcal{L}^{-1}\{\phi(s)\} = \frac{\sin 2t}{2}$

$\therefore \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \frac{d}{dt}\mathcal{L}^{-1}\{\phi(s)\} = \frac{d}{dt}\frac{\sin 2t}{2} = \cos 2t$

Theorem 5:

If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{t f(t)\} = -\frac{d}{ds}F(s)$

and $f(t) = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}F(s)\right\}$

This formula is used if taking the I.L.T. of a derivative is easier than the I.L.T of the original function

ex. Find $\mathcal{L}\{t^2 \sin 2t\}$

sol. Let $f(t) = \sin 2t \Rightarrow F(s) = \frac{2}{s^2+4}$

$$\mathcal{L}\{t \sin 2t\} = -\frac{d}{ds} F(s) = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = \frac{2(2s)}{(s^2+4)^2} = \frac{4s}{(s^2+4)^2}$$

$$\mathcal{L}\{t^2 \sin 2t\} = -\frac{d}{ds}\left(\frac{4s}{(s^2+4)^2}\right) = -\left[\frac{(s^2+4)^2 \cdot 4 - (4s)2(s^2+4)(2s)}{(s^2+4)}\right]$$

$$= -\left[\frac{4s^2+16-16s^2}{(s^2+4)^3}\right] = \frac{12s^2-16}{(s^2+4)^3}$$

ex. If $\mathcal{L}\{y(t)\} = \ln\frac{s+1}{s-1}$; find $y(t)$

sol.

$$y(t) = \frac{-1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\ln\left(\frac{s+1}{s-1}\right)\right\} = \frac{-1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}[\ln(s+1) - \ln(s-1)]\right\}$$

$$= \frac{-1}{t}\mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{s-1}\right\} = -\frac{1}{t}\left(e^{-t} - e^{t}\right) = \frac{2\sinh t}{t}$$

Theorem 6:

If $\mathcal{L}\{f(t)\} = F(s)$ and if $\lim_{t\to 0^+} \frac{f(t)}{t}$ exists, then:

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$$

and $f(t) = t\,\mathcal{L}^{-1}\left\{\int_s^\infty F(s)\,ds\right\}$

proof: $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st}\,dt$

$$\int_s^\infty F(s)\,ds = \int_s^\infty \int_0^\infty f(t) e^{-st}\,dt\,ds = \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\,ds\right] dt$$

$$= \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt = \int_0^\infty \frac{f(t)}{t} e^{-st}\,dt = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

This formula is useful when taking I.L.T. of $\int_s^\infty F(s)\,ds$ is simpler than taking I.L.T. of $F(s)$

ex. Find L.T. of $\frac{\sin kt}{t}$

sol. Let $f(t) = \sin kt \Rightarrow F(s) = \frac{k}{s^2+k^2}$

$$\therefore \mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$$

$$\therefore \mathcal{L}\left\{\frac{\sin kt}{t}\right\} = \int_s^\infty \frac{k}{s^2+k^2}\,ds = \tan^{-1}\frac{s}{k}\Big|_s^\infty = \tan^{-1}\infty - \tan^{-1}\frac{s}{k}$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{k}$$

ex. Find $y(t)$ if $\mathcal{L}\{y(t)\} = \frac{s}{(s^2-1)^2}$

sol. Let $\Phi(s) = \frac{s}{(s^2-1)^2}$

$$\int_s^\infty \Phi(s)\,ds = \int_s^\infty \frac{s}{(s^2-1)^2}\,ds = \frac{-1}{2(s^2-1)}\Big|_s^\infty = \frac{-1}{\infty} - \frac{-1}{2(s^2-1)} = \frac{1}{2(s^2-1)}$$

$$\frac{1}{s^2-1} = \frac{A}{s+1} + \frac{B}{s-1} = \frac{As - A + Bs + B}{s^2-1} = \frac{(A+B)s + (-A+B)}{s^2-1}$$

$A+B=0$; $B-A=1 \Rightarrow A=-\frac{1}{2}$; $B=\frac{1}{2}$

$$\therefore \frac{1}{2}\left(\frac{-0.5}{s+1} + \frac{0.5}{s-1}\right) = \frac{1}{4}\left(\frac{-1}{s+1} + \frac{1}{s-1}\right)$$

we have: $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds \overset{\text{I.L.T.}}{\Rightarrow} f(t) = t\,\mathcal{L}^{-1}\left\{\int_s^\infty F(s)\,ds\right\}$

$$\therefore y(t) = t\,\mathcal{L}^{-1}\left\{\frac{1}{4}\left(\frac{-1}{s+1} + \frac{1}{s-1}\right)\right\} = \frac{t}{4}\left(-e^{-t} + e^{t}\right)$$

Theorem 7: Initial value theorem

$$f(0^+) = \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} s\,F(s) \qquad \text{where } F(s) = \mathcal{L}\{f(t)\}$$

ex. If $\mathcal{L}\{f(t)\} = \frac{s+1}{(s-1)(s+2)}$

then $f(0^+) = \lim_{s\to\infty} s\left(\frac{s+1}{(s-1)(s+2)}\right) = \lim_{t\to 0^+} f(t)$

Let $F(s) = \frac{s+1}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}$

$A = \frac{2}{3}$; $B = \frac{1}{3} \Rightarrow F(s) = \frac{2/3}{s-1} + \frac{1/3}{s+2}$

$\therefore f(t) = \frac{2}{3}e^{t} + \frac{1}{3}e^{-2t}$

$f(0^+) = \frac{2}{3} + \frac{1}{3} = 1$

Theorem 8: Final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \qquad \text{where } F(s) = \mathcal{L}\{f(t)\}$$

ex. If $f(t) = \sin t$ then $F(s) = \dfrac{1}{s^2+1}$

and $\lim_{t\to\infty} \overset{\sin t}{\uparrow} = \lim_{s\to 0} sF(s) = \lim_{s\to 0} s \cdot \dfrac{1}{s^2+1} = 0$

Theorem 9: First shifting theorem

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a) \qquad \text{where } F(s) = \mathcal{L}\{f(t)\}$$

proof: $F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt = \int_0^\infty e^{-st} \cdot e^{at} f(t)\,dt = \mathcal{L}\{e^{at} f(t)\}$

ex. Find L.T. of $e^{at}\cos\omega t$ and $e^{-at}\sin\omega t$.

sol. $\mathcal{L}\{\cos\omega t\} = \dfrac{s}{s^2+\omega^2} \Rightarrow \mathcal{L}\{e^{at}\cos\omega t\} = \mathcal{L}\{\cos\omega t\}_{s\to s-a}$

$\therefore \mathcal{L}\{e^{at}\cos\omega t\} = \dfrac{s-a}{(s-a)^2+\omega^2}$

$\mathcal{L}\{\sin\omega t\} = \dfrac{\omega}{s^2+\omega^2} \Rightarrow \mathcal{L}\{e^{-at}\sin\omega t\} = \mathcal{L}\{\sin\omega t\}_{s\to s+a}$

$\therefore \mathcal{L}\{e^{-at}\sin\omega t\} = \dfrac{\omega}{(s+a)^2+\omega^2}$

ex. Find I.L.T. of $F(s) = \dfrac{2s+5}{s^2+4s+13}$

sol.

$$F(s) = \frac{2s+5}{s^2+4s+13} = \frac{2s+5}{s^2+4s+4+9} = \frac{2s+5}{(s+2)^2+9} = \frac{2s+5}{(s+2)^2+3^2}$$

$$= \frac{2(s+2)}{(s+2)^2+3^2} + \frac{1}{3}\cdot\frac{3}{(s+2)^2+3^2}$$

$$\therefore f(t) = 2e^{-2t}\cos 3t + \frac{1}{3}e^{-2t}\sin 3t$$

ex. Solve the following D.E.:

$$y'' + 2y' + y = t e^{-t} \quad ; y_0 = 1 \; ; y_0' = -2$$

sol. By taking L.T. of both sides, we get:

$$[s^2 Y(s) - s\,y(0) - y'(0)] + 2[s\,Y(s) - y(0)] + Y(s) = \mathcal{L}\{t e^{-t}\}$$

we have $\mathcal{L}\{t\} = \frac{1}{s^2}$ and $\mathcal{L}\{f(t) e^{-at}\} = F(s)\big|_{s \to s+a}$

$$\therefore \mathcal{L}\{t e^{-t}\} = \frac{1}{(s+1)^2}$$

$$s^2 Y(s) - s + 2 + 2s\,Y(s) - 2 + Y(s) = \frac{1}{(s+1)^2}$$

$$Y(s)\,[\underbrace{s^2 + 2s + 1}_{(s+1)^2}] = \frac{1}{(s+1)^2} + s \Rightarrow Y(s) = \frac{1}{(s+1)^4} + \frac{s}{(s+1)^2}$$

$$\therefore y(t) = \frac{t^3}{3!} e^{-t} + \frac{d}{dt}\, t e^{-t} = \frac{t^3}{3!} e^{-t} - t e^{-t} + e^{-t}$$

$\mathcal{L}^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

بسبب وجود s في البسط فهو مشتقة

time domain بال

Theorem 10: Second shifting theorem

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as} F(s)$$

where: $F(s) = \mathcal{L}\{f(t)\}$; $f(t-a)\,u(t-a) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$

proof:

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = \int_0^\infty f(t-a)\,u(t-a)\,e^{-st}\,dt = \int_a^\infty f(t-a)\,e^{-st}\,dt$$

Let $t - a = T \Rightarrow dt = dT$

$$\int_0^\infty f(T)\,e^{-(T+a)s}\,dT = e^{-as} \int_0^\infty f(T)\,e^{-sT}\,dT = e^{-as} F(s)$$

note that: $\mathcal{L}\{f(t)\,u(t-a)\} = e^{-as}\,\mathcal{L}\{f(t+a)\}$

Laplace Transform

Theorem 11. Evaluation of Integrals

A number of integral problems having lower limit 0 and upper limit ∞ can be evaluated by the help of using L.T.

example. Evaluate $\int_0^\infty t\, e^{-3t} \sin t \, dt$

Sol.

$$\int_0^\infty t\, e^{-3t} \sin t \, dt = \lim_{s\to 0} \int_0^\infty t\, e^{-3t} \sin t \cdot e^{-st} \, dt$$

$$= \lim_{s\to 0} \mathcal{L}\{t\, e^{-3t} \sin t\}$$

we have: $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$

and $\mathcal{L}\{e^{-3t} \sin t\} = \frac{1}{(s+3)^2+1}$

because $\mathcal{L}\{e^{-at} f(t)\} = F(s+a)$ first shifting property

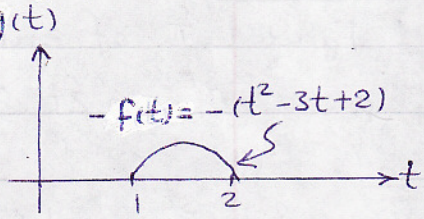
and $\mathcal{L}\{t e^{-3t} \sin t\} = -\frac{d}{ds}\left[\frac{1}{(s+3)^2+1}\right]$

$$= \frac{2(s+3)}{[(s+3)^2+1]^2}$$

because $\mathcal{L}\{t\, f(t)\} = -\frac{d}{ds} F(s)$

then: $\int_0^\infty t\, e^{-3t} \sin t \, dt = \lim_{s\to 0} \frac{2(s+3)}{[(s+3)^2+1]^2} = \frac{6}{100} = \frac{3}{50}$

ex. What is the L.T. of the function whose graph is shown.



sol. The eq. of this function is:

$$g(t) = -(t^2-3t+2)[u(t-1)-u(t-2)]$$
$$= -f(t)\,u(t-1) + f(t)\,u(t-2)$$
$$\mathcal{L}\{g(t)\} = -\mathcal{L}\{f(t)\,u(t-1)\} + \mathcal{L}\{f(t)\,u(t-2)\}$$

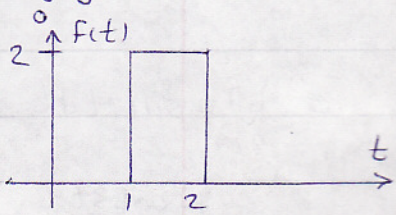
we have: $\mathcal{L}\{f(t)\,u(t-a)\} = e^{-as}\,\mathcal{L}\{f(t+a)\}$

and $f(t+1) = (t+1)^2 - 3(t+1) + 2 = t^2 - t$

$f(t+2) = (t+2)^2 - 3(t+2) + 2 = t^2 + t$

$$\therefore \mathcal{L}\{g(t)\} = -e^{-s}\,\mathcal{L}\{t^2-t\} + e^{-2s}\,\mathcal{L}\{t^2+t\}$$
$$= -e^{-s}\left(\frac{2}{s^3} - \frac{1}{s^2}\right) + e^{-2s}\left(\frac{2}{s^3} + \frac{1}{s^2}\right)$$

ex. Find the solution of the equation $y' + 3y + 2\int_0^t y\,dt = f(t)$ for which $y_0 = 1$ and $f(t)$ is as shown.



sol. $f(t) = 2[u(t-1) - u(t-2)]$

$$\therefore y' + 3y + 2\int_0^t y\,dt = 2u(t-1) - 2u(t-2)$$

by taking L.T. of both sides:

$$(sY(s) - 1) + 3Y(s) + 2\frac{Y(s)}{s} = 2\frac{e^{-t}}{s} - 2\frac{e^{-2t}}{s}$$

$$(s^2 + 3s + 2)\,Y(s) = 2e^{-s} - 2e^{-2s} + s$$

$$Y(s) = \frac{s}{(s+1)(s+2)} + \frac{2e^{-s}}{(s+1)(s+2)} - \frac{2e^{-2s}}{(s+1)(s+2)}$$

we have: $\dfrac{1}{(s+1)(s+2)} = \dfrac{1}{s+1} - \dfrac{1}{s+2}$

$$\therefore y(t) = \frac{d}{dt}\left(e^{-t} - e^{-2t}\right) + 2\left(e^{-(t-1)} - e^{-2(t-1)}\right)u(t-1)$$
$$- 2\left(e^{-(t-2)} - e^{-2(t-2)}\right)u(t-2)$$

Examples:

1. Find L.T. of each of the following functions:

a. $t e^{-3t} \sin 2t$; b. $e^{-3t} \int_0^t t \sin 2t \, dt$; c. $\frac{1-\cos 3t}{t}$

Sol.

a. $\mathcal{L}\{t e^{-3t} \sin 2t\} = -\frac{d}{ds} \mathcal{L}\{e^{-3t} \sin 2t\} = -\frac{d}{ds} \mathcal{L}\{\sin 2t\}_{s \to s+3}$

$= -\frac{d}{ds} \frac{2}{s^2+4}\Big|_{s \to s+3} = -\frac{d}{ds} \frac{2}{(s+3)^2+4} = \frac{4(s+3)}{[(s+3)^2+4]^2}$

b. $\mathcal{L}\{e^{-3t} \int_0^t t \sin 2t \, dt\}$

$t \sin 2t \xrightarrow{L.T.} -\frac{d}{ds} \mathcal{L}\{\sin 2t\} = -\frac{d}{ds} \frac{2}{s^2+4} = \frac{4s}{(s^2+4)^2}$

$\int_0^t t \sin 2t \, dt \xrightarrow{L.T.} \frac{1}{s} \mathcal{L}\{t \sin 2t\} = \frac{1}{s} \cdot \frac{4s}{(s^2+4)^2} = \frac{4}{(s^2+4)^2}$

$e^{-3t} \int_0^t t \sin 2t \, dt \xrightarrow{L.T.} \mathcal{L}\{\int_0^t t \sin 2t \, dt\}_{s \to s+3} = \frac{4}{[(s+3)^2+4]^2}$

c. $\mathcal{L}\{\frac{1-\cos 3t}{t}\} = \int_s^\infty \mathcal{L}\{1-\cos 3t\} ds = \int_s^\infty [\frac{1}{s} - \frac{s}{s^2+9}] ds$

$= \ln s - \frac{1}{2} \ln (s^2+9) \Big|_s^\infty = -\ln(s) + \frac{1}{2} \ln (s^2+9)$

$= \ln (s^2+9)^{1/2} - \ln (s) = \ln \frac{\sqrt{s^2+9}}{s}$

$[\ln \infty - \frac{1}{2} \ln \infty = 0]$

$\ln \infty = \infty$

2. Find I.L.T. of each of the following functions:

a. $\frac{1}{s^2(s+1)}$; b. $\ln\frac{s^2-1}{s^2}$; $s>0$

Sol.

a. $\mathcal{L}^{-1}\{\frac{1}{s^2(s+1)}\} = \int_0^t \mathcal{L}^{-1}\{\frac{1}{s(s+1)}\}dt = \int_0^t\int_0^t \mathcal{L}^{-1}\{\frac{1}{s+1}\}dt\,dt$

$= \int_0^t\int_0^t e^{-t}\,dt\,dt = -\int_0^t [e^{-t}]\Big|_0^t dt = -\int_0^t [e^{-t}-1]\,dt$

$= e^{-t}\Big|_0^t + t\Big|_0^t = e^{-t} - 1 + t$

b. $\ln\frac{s^2-1}{s^2} = \ln(s^2-1) - \ln(s^2) = \ln(s^2-1) - 2\ln(s)$

We have: $t f(t) = \mathcal{L}^{-1}\{-\frac{d}{ds}F(s)\}$

$\frac{d}{ds}\{\ln(s^2-1) - 2\ln(s)\} = \frac{2s}{s^2-1} - \frac{2}{s}$

$\therefore f(t) = \frac{-1}{t}\mathcal{L}^{-1}\{\frac{d}{ds}F(s)\} = \frac{-1}{t}\{\mathcal{L}^{-1}(\frac{2s}{s^2-1}) - \mathcal{L}^{-1}(\frac{2}{s})\}$

$= \frac{-1}{t}\left[2\frac{d}{dt}\mathcal{L}^{-1}\{\frac{1}{s^2-1}\} - 2\mathcal{L}^{-1}\{\frac{1}{s}\}\right] = \frac{-1}{t}\left(2\frac{d}{dt}(\frac{e^t}{2} - \frac{e^{-t}}{2}) - 2\right)$

$= \frac{-2}{t}\left(\frac{e^t+e^{-t}}{2} - 1\right) = \frac{2(1-\cosh t)}{t}$

3. What is $\int_0^\infty \frac{e^{-t} - e^{-2t}}{t}dt$ ← theorem 11 ... evaluation of integrals

Sol. We have: $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$

and $\lim_{s\to 0} e^{-st} = 1$

$\therefore$ Rewriting: $\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}\cdot\lim_{s\to 0}e^{-st}\,dt = \lim_{s\to 0}\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}e^{-st}dt$

$= \lim_{s\to 0}\left[\mathcal{L}\{\frac{e^{-t}-e^{-2t}}{t}\}\right] = \lim_{s\to 0}\left[\int_s^\infty \mathcal{L}\{e^{-t}-e^{-2t}\}ds\right]$

$= \lim_{s\to 0}\left[\int_s^\infty (\frac{1}{s+1} - \frac{1}{s+2})ds\right] = \lim_{s\to 0}\left[-\ln(s+1) + \ln(s+2)\right]$

$= -\ln 1 + \ln 2 = \ln 2$

4. Find a solution for the following D.E:

$$y'' + 3y' + 2y = u(t-1) \quad ; \quad y_0 = 0 \; , \; y_0' = 1$$

Sol: Taking L.T. of both sides:

$$[s^2 Y(s) - s\,y(0) - y'(0)] + 3[s\,Y(s) - y(0)] + 2\,Y(s) = \frac{e^{-s}}{s}$$

$$s^2 Y(s) - 1 + 3s\,Y(s) + 2\,Y(s) = \frac{e^{-s}}{s}$$

$$Y(s) = \frac{e^{-s}}{(s^2+3s+2)(s)} + \frac{1}{(s^2+3s+2)} = \frac{e^{-s}}{s(s+1)(s+2)} + \frac{1}{(s+1)(s+2)}$$

$$= \frac{e^{-s}}{s}\left[\frac{1}{(s+1)(s+2)}\right] + \frac{1}{(s+1)(s+2)}$$

Using partial fraction expansion: $Y(s) = \frac{e^{-s}}{s}\left[\frac{1}{s+1} - \frac{1}{s+2}\right] + \frac{1}{s+1} - \frac{1}{s-2}$

we have: $\mathcal{L}^{-1}\left\{\frac{1}{s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right)\right\} = \int_0^t \mathcal{L}^{-1}\left(\frac{1}{s+1} - \frac{1}{s+2}\right) dt$

$$= \int_0^t (e^{-t} - e^{-2t})\,dt = -e^{-t} + \frac{1}{2}e^{-2t} + \frac{1}{2}$$

and $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right)\right\}_{t \to t-1}$

$$\therefore y(t) = \left(-e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)} + \frac{1}{2}\right)u(t-1) + e^{-t} - e^{-2t}$$

Partial Fraction Expansion:

The general form of $F(s)$ can be written as: $F(s) = \frac{P(s)}{q(s)}$

Assuming that the degree of $q(s)$ is greater than the degree of $P(s)$, then we can write:

$$F(s) = \frac{P(s)}{q(s)} = \frac{P(s)}{(s-r_1)(s-r_2)\cdots(s-r_n)(s-r)^q[(s+a)^2+b^2]}$$

$$= \frac{k_1}{s-r_1} + \frac{k_2}{s-r_2} + \cdots + \frac{k_n}{s-r_n} + \frac{k_1'}{s-r} + \frac{k_2'}{(s-r)^2} + \cdots + \frac{k_q'}{(s-r)^q} + \frac{k_c}{s+a+jb} + \frac{k_c^*}{s+a-jb}$$

where k_i are constants.

The above equation of $F(s)$ has three types of roots for $q(s)$:

① Unrepeated simple roots:

$$K_i = \lim_{s \to r_i} (s-r_i)\frac{P(s)}{q(s)} \quad (i = 1, 2, 3, \cdots n)$$

② Repeated roots:

$$K_i' = \frac{1}{(q-i)!} \lim_{s \to r} \frac{d^{q-i}}{ds^{q-i}} \left[(s-r)^q \frac{P(s)}{q(s)} \right] \quad i = 1, 2, \cdots q$$

③ Complex roots:

$$K_c = \lim_{s \to -a-jb} (s+a+jb)\frac{P(s)}{q(s)}$$

Ex. If $F(s) = \frac{-s^2-s+5}{s(s^2+3s+2)}$; find $f(t)$

Sol. $F(s) = \frac{-s^2-s+5}{s(s^2+3s+2)} = \frac{-s^2-s+5}{s(s+1)(s+2)} = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2}$

~~where k_1, k_2, k_3 are unrepeated simple roots~~

$$k_1 = \lim_{s \to 0} s\left(\frac{-s^2-s+5}{s(s+1)(s+2)}\right) = \lim_{s \to 0} \frac{-s^2-s+5}{(s+1)(s+2)} = \frac{5}{2}$$

$$k_2 = \lim_{s \to -1} \frac{-s^2-s+5}{s(s+2)} = -5$$

$$k_3 = \lim_{s \to -2} \frac{-s^2-s+5}{s(s+1)} = \frac{3}{2}$$

$$\therefore F(s) = \frac{5/2}{s} - \frac{5}{s+1} + \frac{3/2}{s+2}$$

and $f(t) = \left(\frac{5}{2} - 5e^{-t} + \frac{3}{2}e^{-2t}\right)u(t)$

Example. If $F(s) = \frac{10s}{(s+2)^3(s+8)}$, find $f(t)$

sol. $F(s) = \frac{10s}{(s+2)^3(s+8)} = \frac{k_1}{s+8} + \frac{k_1'}{s+2} + \frac{k_2'}{(s+2)^2} + \frac{k_3'}{(s+3)^3}$

~~k_1 where k_1 is unrepeated simple root and k_1', k_2', k_3' are repeated roots~~

$$k_1 = \lim_{s\to -8} \frac{10s}{(s+2)^3} = \frac{10}{27}$$

$$k_1' = \frac{1}{(3-1)!} \lim_{s\to -2} \frac{d^{(3-1)}}{ds^{(3-1)}} \frac{10s}{s+8} = \frac{1}{2!} \lim_{s\to -2} \frac{d^2}{ds^2} \frac{10s}{s+8} = \frac{-10}{27}$$

$$k_2' = \frac{1}{(2-1)!} \lim_{s\to -2} \frac{d^{(2-1)}}{ds^{(2-1)}} \frac{10s}{s+8} = \frac{1}{1!} \lim_{s\to -2} \frac{d}{ds} \frac{10s}{s+8} = \frac{20}{9}$$

$$k_3' = \lim_{s\to -2} \frac{10s}{s+8} = \frac{-10}{3}$$

$$F(s) = \frac{10/27}{s+8} - \frac{10/27}{s+2} + \frac{20/9}{(s+2)^2} - \frac{10/3}{(s+2)^3}$$

$$\therefore f(t) = \left[\frac{10}{27}e^{-8t} - \frac{10}{27}e^{-2t} + \frac{20}{9}te^{-2t} - \frac{10}{3}*\frac{t^2}{2}e^{-2t}\right]u(t)$$

ex. If $F(s) = \frac{2s^2+6s+6}{(s+2)(s^2+2s+2)}$, find $f(t)$

sol

$$F(s) = \frac{2s^2+6s+6}{(s+2)(s^2+2s+2)} = \frac{2s^2+6s+6}{(s+2)[(s+1)^2+1]^2} = \frac{k}{s+2} + \frac{k_c}{s+1+j} + \frac{k_c^*}{s+1-j}$$

$$K = \lim_{s\to -2} \frac{2s^2+6s+6}{s^2+2s+2} = 1$$

$$k_c = \lim_{s\to -1-j} \frac{2s^2+6s+6}{(s+2)(s+1-j)} = \frac{1}{2} + j\frac{1}{2}$$

$$k_c^* = \frac{1}{2} - j\frac{1}{2}$$

$$F(s) = \frac{1}{s+2} + \frac{\frac{1}{2}(1+j)}{s+1+j} + \frac{\frac{1}{2}(1-j)}{s+1-j}$$

$$= \frac{1}{s+2} + \frac{1/2}{s+1+j} + \frac{j\,1/2}{s+1+j} + \frac{1/2}{s+1-j} - \frac{j\,1/2}{s+1-j}$$

$$\therefore f(t) = \left[e^{-2t} + \frac{1}{2} e^{-(1+j)t} + \frac{j}{2} e^{-(1+j)t} + \frac{1}{2} e^{-(1-j)t} - \frac{j}{2} e^{-(1-j)t} \right] u(t)$$

$$= \left[e^{-2t} + \cos t\, e^{-t} + \sin t\, e^{-t} \right] u(t)$$

The Heaviside Expansion Theorems:

Theorem: If $f(t) = \mathcal{L}^{-1}\{ P(s)/q(s) \}$, where $P(s)$ and $q(s)$ are polynomials and the degree of $q(s)$ is greater than the degree of $P(s)$, then the terms in $f(t)$ which correspond to an unrepeated, irreducible quadratic factor $(s+a)^2 + b^2$ of $q(s)$ are $\frac{e^{-at}}{b}(\phi_i \cos bt + \phi_r \sin bt)$ where ϕ_r and ϕ_i are real and imaginary parts of $\phi(-a+jb)$

ex. If $F(s) = \frac{2s^2+6s+6}{(s+2)(s^2+2s+2)}$, Find $f(t)$

Sol.

$$F(s) = \frac{2s^2+6s+6}{(s+2)(s^2+2s+2)} = \frac{K}{s+2} + \frac{K_c}{s^2+2s+2} = \frac{k}{s+2} + \frac{k_c}{(s+1)^2+1^2}$$

$\therefore a = 1 \; ; \; b = 1 \; ; \; \phi(s) = \frac{2s^2+6s+6}{s+2}$

$$k = \lim_{s \to -2} \frac{2s^2+6s+6}{s^2+2s+2} = 1$$

$$\phi(-a+jb) = \phi(-1+j) = \frac{2(-1+j)^2+6(-1+j)+6}{(-1+j)+2} = 1+j = \phi_r + \phi_i$$

$\therefore \phi_r = 1 \; ; \; \phi_i = 1$

and $f(t) = \left[e^{-2t} + \left(\frac{e^{-t}}{1} (\cos t + \sin t) \right) \right] u(t)$

e^{-2t} from the term $\frac{k}{s+2} = \frac{1}{s+2}$

ex. If $F(s) = \dfrac{s}{(s+2)^2(s^2+2s+10)}$, find $f(t)$

Sol. $F(s) = \dfrac{s}{(s+2)^2(s^2+2s+10)} = \dfrac{s}{(s+2)^2[(s+1)^2+3^2]}$

$$= \frac{k_1'}{s+2} + \frac{k_2'}{(s+2)^2} + \frac{k_c}{[(s+1)^2+3^2]}$$

$$k_1' = \frac{1}{1!}\lim_{s\to -2}\frac{d}{ds}\frac{s}{s^2+2s+10} = \lim_{s\to -2}\frac{-s^2+10}{(s^2+2s+10)^2} = \frac{3}{50}$$

$$k_2' = \lim_{s\to -2}\frac{s}{s^2+2s+10} = \frac{-1}{5}$$

$$\mathcal{L}^{-1}\left\{\frac{3/50}{s+2} - \frac{1/5}{(s+2)^2}\right\} = \frac{3}{50}e^{-2t} - \frac{1}{5}e^{-2t}\cdot\frac{1}{1!}t = \frac{(3-10t)}{50}e^{-2t}$$

Now, for the term: $(s+1)^2+3^2 \Rightarrow a=1, b=3$

$$\phi(-a+jb) = \phi(-1+j3) = \frac{-1+j3}{[(-1+j3)+2]^2} = \frac{-1+j3}{-8+j6} = \frac{13}{50} - j\frac{9}{50}$$

$$\therefore f(t) = \left[\frac{(3-10t)}{50}e^{-2t} + \frac{e^{-t}}{3}\left[\frac{-9}{50}\cos 3t + \frac{13}{50}\sin 3t\right]\right]u(t)$$

Laplace Transform of periodic Functions:

Theorem: If $f(t)$ is a periodic function with period T on $[0,\infty)$ and piecewise regular of exponential order, then:

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T f(t)\, e^{-st}\, dt}{1 - e^{-Ts}} \qquad ; s>0$$

proof:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt = \int_0^T f(t)\, e^{-st}\, dt + \int_T^{2T} f(t)\, e^{-st}\, dt + \int_{2T}^{3T} f(t) e^{-st}\, dt + \cdots$$

Let $t = \tau + nT$ → for the $(n+1)$st integral, then:

$$\mathcal{L}\{f(t)\} = \int_0^T f(\tau)\, e^{-s\tau}\, d\tau + \int_0^T f(\tau+T)\, e^{-(\tau+T)s}\, d\tau + \int_0^T f(\tau+2T)\, e^{-(\tau+2T)s}\, d\tau + \cdots$$

$$= \int_0^T f(\tau)\, e^{-s\tau}\, d\tau + e^{-sT}\int_0^T f(\tau)\, e^{-s\tau}\, d\tau + e^{-2sT}\int_0^T f(\tau)\, e^{-s\tau}\, d\tau + \cdots$$

$$= \int_0^T f(\tau) e^{-s\tau}\, d\tau \left[1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \cdots\right]$$

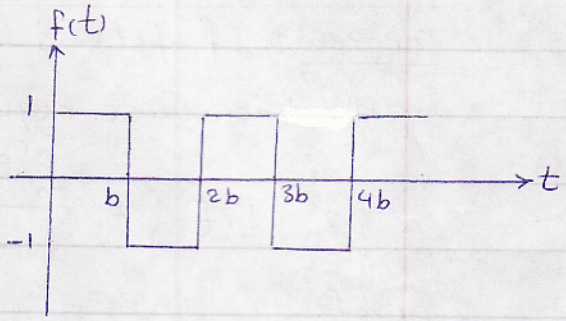
if $s>0 \Rightarrow e^{-sT} < 1$ and the series is a geometric series

if $e^{-sT} = r$, then $1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$ for $r<1$

or $1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \cdots = \frac{1}{1-e^{-sT}}$

$$\therefore \mathcal{L}\{f(t)\} = \frac{\int_0^T f(t) e^{-st}\, dt}{1 - e^{-sT}}$$

ex. Find L.T. of the function shown.



sol. $T = 2b$

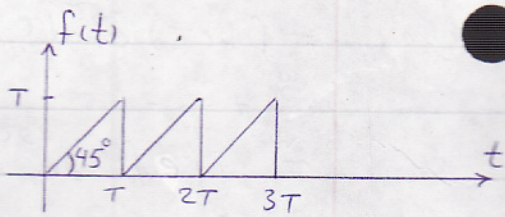
$$\mathcal{L}\{f(t)\} = \frac{\int_0^T f(t)\, e^{-st}\, dt}{1 - e^{-sT}}$$

$$= \frac{1}{1 - e^{-2bs}} \int_0^{2b} f(t)\, e^{-st}\, dt$$

$$= \frac{1}{1 - e^{-2bs}} \left[\int_0^b e^{-st}\, dt + \int_b^{2b} (-1)\, e^{-st}\, dt \right] = \frac{1}{1 - e^{-2bs}} \left[-\frac{e^{-st}}{s} \Big|_0^b + \frac{e^{-st}}{s} \Big|_b^{2b} \right]$$

$$= \frac{1}{1 - e^{-2bs}} \cdot \frac{1}{s} \left[-e^{-sb} + 1 + e^{-2bs} - e^{-bs} \right] = \frac{1 - 2e^{-bs} + e^{-2bs}}{s(1 - e^{-2bs})}$$

ex. Find L.T. of the function shown.



sol. $f(t) = t$

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T f(t)\, e^{-st}\, dt}{1 - e^{-sT}}$$

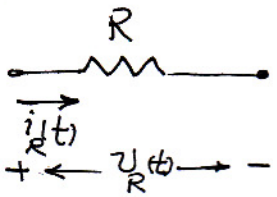
$$= \frac{1}{1 - e^{-sT}} \int_0^T t e^{-st}\, dt = \frac{1}{1 - e^{-sT}} \cdot \frac{e^{-st}}{s^2} [-st - 1] \Big|_0^T = \frac{1 - (1 + Ts)e^{-Ts}}{s^2(1 - e^{-Ts})}$$

The Laplace Transform in Circuit Analysis

1. Circuit elements in the s-domain:

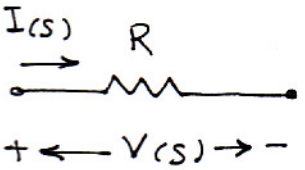
used to transforma ect. into the s-domain using Laplace transform and representing the initial conditions on energy-storage elements in the s-domain.

a. Resistor:



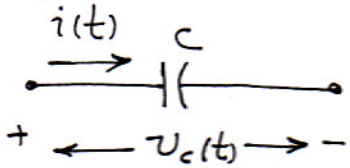
according to Ohm's law:

$$v(t) = R\,i(t) \xrightarrow{L.T.} V(s) = R\,I(s)$$



b. capacitor:

$$v_c(t) = \frac{q(t)}{C}$$



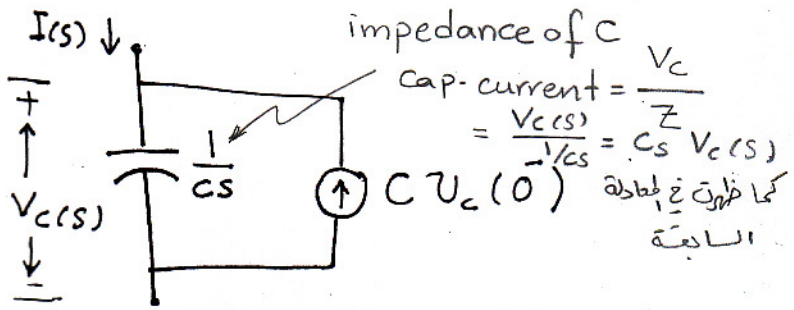
where $q(t)$ = total capacitor charge (coloumb)

and $v_c(0^-) = \frac{q(0^-)}{C}$ initial charge (initial condition) (constant)

voltage due to initial

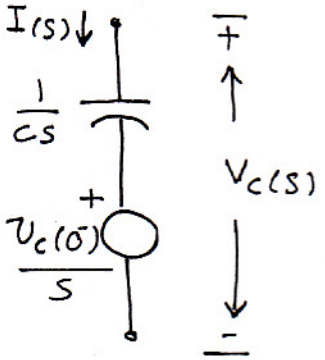
also $v_c(t) = \frac{1}{C}\int_0^t i(t)\,dt + v_c(0^-) \xrightarrow{L.T.} V_c(s) = \frac{1}{Cs} I(s) + \frac{v_c(0^-)}{s}$

or $I(s) = Cs\,V_c(s) - C\,v_c(0^-)$



capacitor + current source

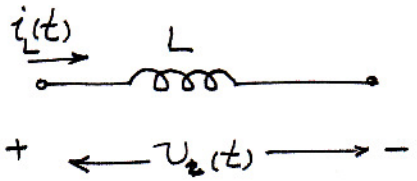
(mostly used in node voltage analysis)



Capacitor + voltage source

(mostly used in loop current analysis)

c. Inductor:



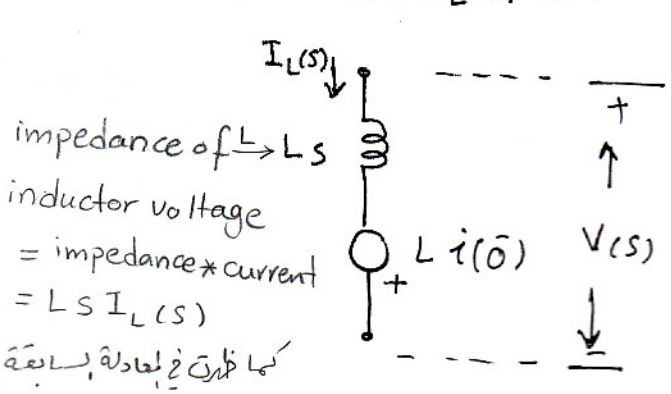
$$i_L(t) = \frac{\Psi(t)}{L}$$

where $\Psi(t)$ = flux (weber)

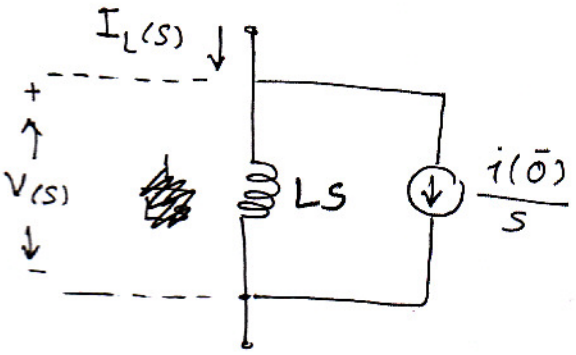
and $i(\bar{0}) = \frac{\Psi(\bar{0})}{L}$ initial current due to initial flux (initial condition) (constant)

also: $i_L(t) = \frac{1}{L}\int_{\bar{0}}^{t} v(t)\,dt + i(\bar{0}) \xrightarrow{L.T.} I_L(S) = \frac{1}{LS}V(S) + \frac{i(\bar{0})}{S}$

or $V(S) = LS\,I_L(S) - L\,i(\bar{0})$



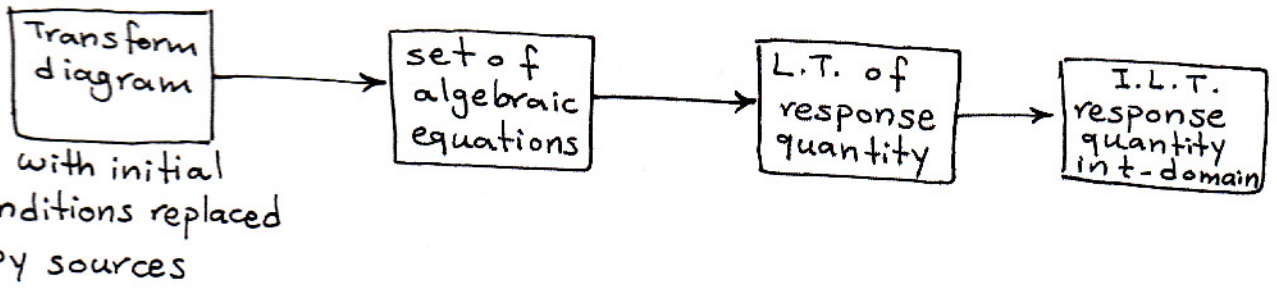
inductor + voltage source



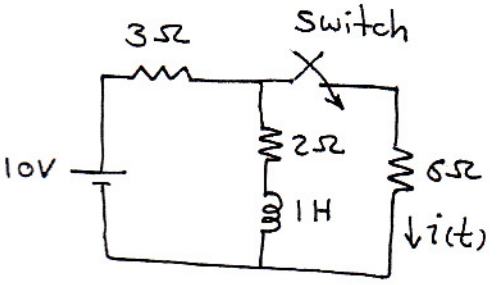
inductor + current source

2. Solution for network transients with L.T. :

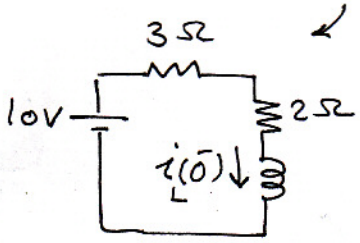
illustrates how to use the Laplace transform to determine the transient behaviour of electric ccts.



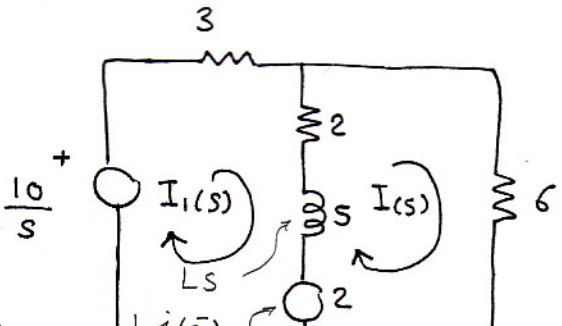
ex. for the cct. shown, find $i(t)$ for $t \geq 0$ if the the switch is kept open for a long time and is closed at $t=0$



sol. The last paragraph in the question means that L had enough time to charge before the switch is closed ; $i_L(0^-) = \frac{10V}{3+2} = 2A$



at 0^- the switch is open



cct. diagram in s-domain

$$\frac{10}{s} = 3I_1(s) + 2[I_1(s) - I(s)] + s[I_1(s) - I(s)] \quad \div 2$$

$$\frac{10}{s} + 2 = (5+s)\,I_1(s) - (2+s)\,I(s) \quad \text{——①}$$

$$6\,I(s) + 2 + 2[I(s) - I_1(s)] + s[I(s) - I_1(s)] = 0$$

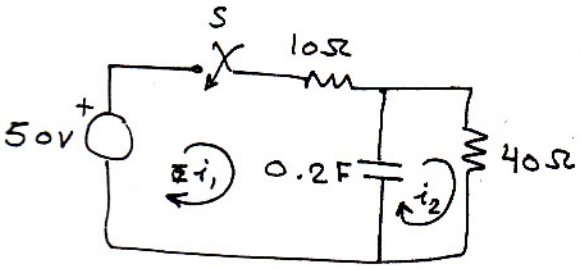
$$(8+s)\,I(s) - (2+s)\,I_1(s) = -2 \quad \text{——②}$$

substitute eq.② in eq.① to eliminate $I_1(s)$, then:

$$I(s) = \frac{\frac{20}{s} + 4}{9s(s+4)} = \frac{20+4s}{9s(s+4)} = \frac{5/9}{s} - \frac{1/9}{s+4}$$

Taking I.L.T., then: $\quad i(t) = \left(\frac{5}{9} - \frac{1}{9}e^{-4t}\right) u(t)\ A$

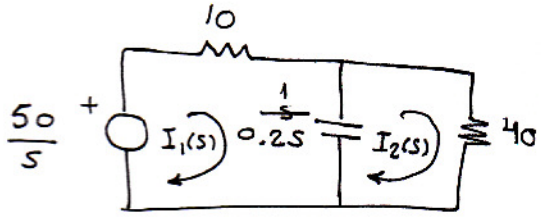
ex. In the cct. shown, find the loop currents $i_1(t)$ and $i_2(t)$ which result when the switch is closed at $t=0$



sol. for $t=0^-$ the switch is open and $v_c(0^-)=0$

$$\frac{50}{s} = 10\,I_1(s) + \frac{1}{0.2s}\left[I_1(s)-I_2(s)\right] \quad \text{——①}$$

$$40\,I_2(s) + \frac{1}{0.2s}\left[I_2(s)-I_1(s)\right]=0 \quad \text{——②}$$



substitute eq.② in eq.① to eliminate $I_1(s)$, then:

$$I_2(s) = \frac{5/s}{8s+5} = \frac{5}{s(8s+5)} = \frac{1}{s} - \frac{8}{8s+5} = \frac{1}{s} - \frac{1}{s+\frac{5}{8}}$$

$$\therefore i_2(t) = \left(1 - e^{-\frac{5}{8}t}\right)u(t) \quad A$$

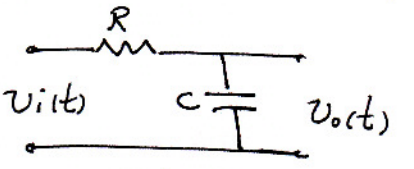
In the same manner:

$$I_1(s) = \frac{5(8s+1)}{5(8s+5)} = \frac{1}{s} + \frac{4}{s+\frac{5}{8}} \Rightarrow i_1(t) = \left(1+4e^{-\frac{5}{8}t}\right)u(t)$$

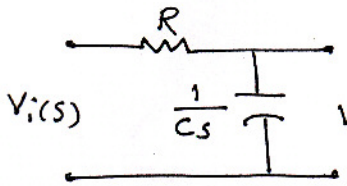
3. Transfer function:

of an electrical system is the ratio of the L.T. of the o/p to the i/p at zero initial condition.

ex. Derive the transfer function for the cct. shown:

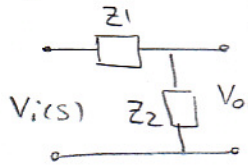


L.T. →



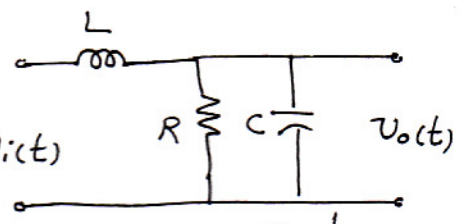
$$\frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{Cs}}{R+\frac{1}{Cs}} = \frac{1}{RCs+1}$$

in general:

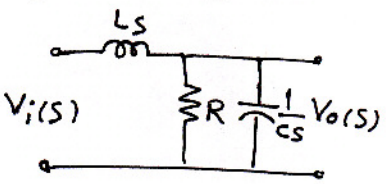


$$\frac{V_o(s)}{V_i(s)} = \frac{Z_2}{Z_1+Z_2}$$

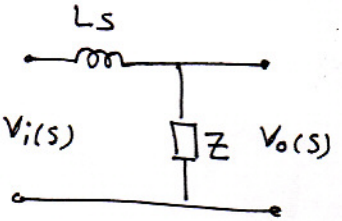
ex. Derive the transfer function for the cct. shown.



L.T. →



⇒



$$Z = R \,//\, \frac{1}{Cs} = \frac{R\cdot\frac{1}{Cs}}{R+\frac{1}{Cs}} = \frac{R}{RCs+1}$$

$$\therefore \frac{V_o(s)}{V_i(s)} = \frac{R/(RCs+1)}{Ls + R/(RCs+1)} = \frac{R}{RLCs^2+Ls+R}$$

The Z-Transform:

The z-transform is a useful tool in the analysis of discrete-time signals and systems. It may be used to solve constant coefficient difference equations and evaluate the response of LTI systems to a given input and design linear filters.

Definition:

The z-transform of a discrete-time signal $f(nT)$ is defined as:

$$F(z) = \mathcal{Z}\{f(nT)\} = \sum_{n=-\infty}^{\infty} f(nT)\, z^{-n}$$

where z is a complex variable.

Since the z-transform in an infinite power series, it exists only for the values of z that makes the series converges. These values are called region of convergence (ROC) of $F(z)$.

ex. Determine the z-transform of the function: $f(n) = a^n u(n)$

Sol.

$$a^n u(n) = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$F(z) = \sum_{n=-\infty}^{\infty} f(n)\, z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

we recall that: $1 + A + A^2 + A^3 + \cdots = \frac{1}{1-A}$; if $|A| < 1$

then if $|az^{-1}| < 1$ (or $|z| > |a|$) then the series converges to $1/(1-az^{-1})$

$$\therefore F(z) = \frac{1}{1-az^{-1}} \quad ; \quad \text{ROC}: |z| > |a|$$

ex. Determine the z-transform of the signal: $f(n) = -a^n u(-n-1)$

Sol. $-a^n u(-n-1) = \begin{cases} 0 & n \geq 0 \\ -a^n & n \leq -1 \end{cases}$

$$F(z) = \sum_{n=-\infty}^{n=-1} -a^n z^{-n} = -\sum_{n=-\infty}^{-1} (a z^{-1})^n$$ Let $-n = i$, then:

$$F(z) = -\sum_{i=1}^{\infty} (a z^{-1})^{-i} = -\sum_{i=1}^{\infty} (a^{-1} z)^i$$

we recall that: $A + A^2 + A^3 + \dots = A(1 + A + A^2 + \dots) = \frac{A}{1-A}$; $|A| < 1$

$$\therefore F(z) = \frac{-a^{-1} z}{1 - a^{-1} z} = \frac{1}{1 - a z^{-1}}$$; ROC: $|z| < |a|$ تستخرج من العلاقة الأصلية

ex. Determine the z-transform of the signal:

$f(n) = a^n u(n) + b^n u(-n-1)$

Sol.

$$F(z) = \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n + \sum_{i=1}^{\infty} (b^{-1} z)^i$$

where $i = -n$

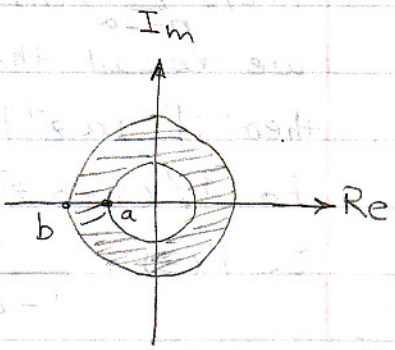
The first power series ($\sum_{n=0}^{\infty} (a z^{-1})^n$) converges to $\frac{1}{1 - a z^{-1}}$ if $|a z^{-1}| > 1$ (or $|z| > |a|$).

The second power series ($\sum_{i=1}^{\infty} (b^{-1} z)^i$) converges to $\frac{b^{-1} z}{-b^{-1} z + 1}$ if $|b^{-1} z| < 1$ (or $|z| < |b|$)

If $|b| < |a|$ then $F(z)$ does not exist, while if $|b| > |a|$ then:

$$F(z) = \frac{1}{1 - a^{-1} z^{-1}} + \frac{b^{-1} z}{1 - b^{-1} z}$$

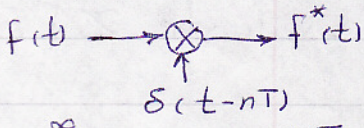
ROC: $|a| < |z| < |b|$



The relationship between F(z) and F(s):

The samples of a signal $f(t)$ can be represented by the amplitude modulated impulse train:

$$f^*(t) = \sum_{n=-\infty}^{\infty} f(nT)\,\delta(t-nT)$$



with the corresponding L.T. : $F^*(s) = \sum_{n=-\infty}^{\infty} F(nT)\, e^{-snT}$

and F.T. : $F^*(\omega) = \sum_{n=-\infty}^{\infty} f(nT)\, e^{-j\omega nT}$

where [*] means a sampled signal.

If we substitute $z = e^{st}$ then we have : $F^*(s) = F(z)\Big|_{z=e^{st}}$

If the region of convergence of $F(z)$ includes the unit circle $|z|=1$, then: $F^*(\omega) = F(z)\Big|_{z=e^{j\omega T}}$

The existence of the z-transform is defined by the existence of L.T. :

$$f^*(t) \xrightarrow{\mathcal{L}} F^*(s) \xrightarrow{s=\frac{1}{T}\ln z} F(z)$$

example: Find the z-transform of the impulse function $\delta(t)$

Sol. $f^*(t) = f(t) = \delta(t)$ through ideal sampler

$$\mathcal{L}\{f^*(t)\} = \mathcal{L}\{\delta(t)\} = 1 \leftarrow \int_{-\infty}^{\infty} \delta(t)\, e^{-st}\, dt = e^0 = 1$$

$$F(z) = F^*(s)\Big|_{s=\frac{1}{T}\ln z} = 1$$

example: Find the z-transform of the train of pulses $\delta(t-nT)$

Sol. $f^*(t) = \delta_T(t) = \delta(t-nT)$ through ideal sampler

$$\mathcal{L}\{f^*(t)\} = \mathcal{L}\{\delta(t-nT)\} = e^{-nTs} \leftarrow \mathcal{L}\{\delta(t \mp nT)\} = e^{\mp nTs}$$

$$\therefore F(z) = F(s)\Big|_{s=\frac{1}{T}\ln z} = \sum_{n=0}^{\infty} e^{-nT\left(\frac{1}{T}\ln z\right)} = \sum_{n=0}^{\infty} e^{-n\ln z} = \sum_{n=0}^{\infty} e^{\ln z^{-n}} = \sum_{n=0}^{\infty} z^{-n}$$

we have: $1+A+A^2+\cdots = \frac{1}{1-A}$, $|A|<1$

let $A = z^{-1} \Rightarrow 1+z^{-1}+z^{-2}+\cdots = \frac{1}{1-z^{-1}} = \frac{1}{\frac{z-1}{z}} = \frac{z}{z-1}$

ROC: $|A|<1 \Rightarrow |z^{-1}|<1 \Rightarrow |z|>1$

$$F(z) = 1+z^{-1}+z^{-2}+z^{-3}+\cdots = \frac{z}{z-1} \quad , \quad |z|>1$$

H.W. (1) Find the Z.T. of the unit step function $u(t)$.

(2) Evaluate $\mathcal{Z}\{e^{-at}\}$

Determination of Z.T. using residue method:

The use of z-transform definition to determine $F(z)$ is not always suitable as in the previous examples.

A most general method is to use the following formula:

$$F(z) = \sum_{\text{poles}}^{n} \text{Res}\left[F(s) \frac{z}{z-e^{sT}} \right]$$

where n is the number of poles in $F(s)$

Res is the residual computed at each pole

Theorem: If $F(s)$ has a pole of order m at $s=a$ then the residue of $F(s)$ at $s=a$ is:

$$\underset{s=a}{\text{Res}} = \frac{1}{(m-1)!} \lim_{s\to a} \frac{d^{m-1}}{ds^{m-1}} \left[(s-a)^m F(s) \frac{z}{z-e^{sT}} \right]$$

example: Evaluate $\mathcal{Z}\{\cos at\}$

Sol.

$$\cos at \xrightarrow{L.T.} \frac{s}{s^2+a^2}$$

F(s) has two different poles $-ja$ and ja

$$\operatorname*{Res}_{s=ja}\left[\frac{s}{s^2+a^2}\cdot\frac{z}{z-e^{sT}}\right] = \lim_{s\to ja}\left[(s-ja)\frac{s}{(s+ja)(s-ja)}\cdot\frac{z}{z-e^{sT}}\right]$$

$$= \frac{ja}{2ja}\cdot\frac{z}{z-e^{jaT}} = \frac{1}{2}\cdot\frac{z}{z-e^{jaT}}$$

$$\operatorname*{Res}_{s=-ja}\left[\frac{s}{s^2+a^2}\cdot\frac{z}{z-e^{sT}}\right] = \lim_{s\to -ja}\left[(s+ja)\frac{s}{(s+ja)(s-ja)}\cdot\frac{z}{z-e^{sT}}\right]$$

$$= \frac{-ja}{-2ja}\cdot\frac{z}{z-e^{-jaT}} = \frac{1}{2}\cdot\frac{z}{z-e^{-jaT}}$$

$$\therefore F(z) = \sum_{n}^{2} \operatorname{Res} = \sum^{2}\operatorname{Res} = \frac{1}{2}\cdot\frac{z}{z-e^{jaT}} + \frac{1}{2}\cdot\frac{z}{z-e^{-jaT}}$$

$$= \frac{z(z-\cos aT)}{z^2-2z\cos aT+1}$$

ROC: $|z|>1$ ← prove?

$$\mathcal{Z}\{\cos\omega_o t\} = \mathcal{Z}\left\{\frac{1}{2}e^{j\omega_o t}+\frac{1}{2}e^{-j\omega_o t}\right\}$$

$$= \frac{1}{2}\mathcal{Z}\{e^{j\omega_o t}\}+\frac{1}{2}\mathcal{Z}\{e^{-j\omega_o t}\}$$

$$\downarrow$$

$$\frac{1}{1-e^{j\omega_o T}z^{-1}},\ \text{ROC: } |e^{j\omega_o T}z^{-1}|<1$$

$$|z|>|e^{j\omega_o T}|$$

H.W. Determine $\mathcal{Z}\{\sin at\}$

ex. Evaluate $\mathcal{Z}[t]$

sol. $t \xrightarrow{L.T.} \frac{1}{s^2}$

F(s) has one pole ($s=0$) of order 2

$$\operatorname*{Res}_{s=0} = \frac{1}{(2-1)!}\lim_{s\to 0}\frac{d}{ds}\left[s^2\cdot\frac{1}{s^2}\cdot\frac{z}{z-e^{sT}}\right]$$

$$= \lim_{s\to 0}\frac{-z(-Te^{sT})}{(z-e^{sT})^2} = \frac{zT}{(z-1)^2}$$

$e^{j\omega_o T} = \cos\omega_o T + j\sin\omega_o T$

$|e^{j\omega_o T}| = \sqrt{\cos^2\omega_o T+\sin^2\omega_o T}$

$= \sqrt{1} = 1$

$\therefore$ ROC: $|z|>1$

H.W. Evaluate $\mathcal{Z}\{t^2\}$

ex. Evaluate $\mathcal{Z}\{te^{-at}\}$

Sol.

$$\mathcal{L}\{te^{-at}\} = \frac{1}{s^2}\Big|_{s\to s+a} = \frac{1}{(s+a)^2}$$

F(s) has a pole ($s=-a$) of order 2

$$\operatorname*{Res}_{s=-a} = \frac{1}{(2-1)!}\lim_{s\to -a}\frac{d}{ds}\left[(s+a)^2\frac{1}{(s+a)^2}\frac{z}{z-e^{sT}}\right]$$

$$= \lim_{s\to -a}\left[\frac{-z(-Te^{sT})}{(z-e^{sT})^2}\right] = \frac{zTe^{-aT}}{(z-e^{-aT})^2}$$

ex. given $\frac{1}{s(s+1)}$ evaluate F(z)

Sol. F(s) has two different poles at $s=0$ and $s=-1$

$$\operatorname*{Res}_{s=0} = \lim_{s\to 0}\left[\frac{1}{(s+1)}\cdot\frac{z}{z-e^{sT}}\right] = \frac{z}{z-1}$$

$$\operatorname*{Res}_{s=-1} = \lim_{s\to -1}\left[\frac{1}{s}\cdot\frac{z}{z-e^{sT}}\right] = \frac{-z}{z-e^{-T}}$$

$$\therefore F(z) = \frac{z}{z-1} - \frac{z}{z-e^{-T}} = \frac{z(1-e^{-T})}{(z-1)(z-e^{-T})}$$

H.W. Find F(z): for:

a) $\frac{1}{(s+a)(s+b)}$

b) $\frac{s+a}{(s+\alpha)(s+\beta)}$

c) $\frac{1}{s(s+a)(s+b)}$

d) $\frac{s+1}{(s+2)^2+1}$

e) $\frac{s^2+2s+5}{s^2(s+1)}$

f) $\frac{s^2+2s+5}{s(s+1)^2}$

g) $\frac{s+1}{(s+2)(s+3)^2}$

Properties of the Z-transform:

1. Linearity:

If $Z\{f_1(n)\} = F_1(z)$ and $Z\{f_2(n)\} = F_2(z)$ then:

$Z\{a f_1(n) + b f_2(n)\} = a F_1(z) + b F_2(z)$

where a and b are constants.

2. Shifting property:

If $Z\{f(n)\} = F(z)$, then:

$Z\{f(n-n_0)\} = z^{-n_0} F(z)$ right shift

$$Z\{f(n+n_0)\} = \begin{cases} z^{n_0} F(z) \\ z^{n_0}\left[F(z) - \sum_{n=0}^{n_0-1} f(n) z^{-n}\right] \end{cases}$$ if $f(n)$ is causal (i.e. $f(n)=0$ $n<0$)

ex. Evaluate Z.T. of $u(n)$ delayed by one sampling period. ($n_0 = 1$)

sol. $Z\{u(n-1)\} = z^{-1} Z\{u(n)\} = z^{-1} \frac{z}{z-1} = \frac{1}{z-1}$

3. Scaling:

If $Z\{f(n)\} = F(z)$, then: $Z\{a^n f(n)\} = F(\frac{z}{a})$

ex. Determine the Z.T. of the signal $f(n) = a^n \cos\omega_0 n\, u(n)$

sol.

knowing that $Z\{\cos\omega_0 n\, u(n)\} = \frac{z(z-\cos\omega_0 T)}{z^2 - 2z\cos\omega_0 T + 1}$, then:

$$Z\{a^n \cos\omega_0 n\, u(n)\} = \frac{\frac{z}{a}(\frac{z}{a} - \cos\omega_0 T)}{(\frac{z}{a})^2 - 2\frac{z}{a}\cos\omega_0 T + 1} = \frac{z(z - a\cos\omega_0 T)}{z^2 - 2z\underset{\downarrow a}{\cos\omega_0 T} + a^2}$$

4. Multiplication by n^k:

If $Z\{f(n)\} = F(z)$ then: $Z\{n^k f(n)\} = -z^k \frac{d^k}{dz^k} F(z)$

ex. Find the Z.T. of $n a^n u(n)$

sol. knowing that $Z\{a^n u(n)\} = \frac{1}{1 - a z^{-1}}$, then:

$$Z\{n a^n u(n)\} = -z \frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right) = \frac{az^{-1}}{(1-az^{-1})^2}$$

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5. Time Reversal:

If $\mathcal{Z}\{x(n)\} = X(z)$ → ROC: $r_1 < |z| < r_2$, then:

$\mathcal{Z}\{x(-n)\} = X(z^{-1})$ ROC: $\frac{1}{r_2} < |z| < \frac{1}{r_1}$

ex. Evaluate $\mathcal{Z}\{u(-n)\}$

sol. $\mathcal{Z}\{x(-n)\} = X(z^{-1})$

and $\mathcal{Z}\{u(n)\} = \frac{1}{1-z^{-1}}$, ROC: $|z| > 1$

$\therefore \mathcal{Z}\{u(-n)\} = \frac{1}{1-(z^{-1})^{-1}} = \frac{1}{1-z}$, ROC: $|z| < 1$

6. Initial Value Theorem:

If $\lim_{z\to\infty} F(z)$ exists, then: $\lim_{t\to 0} f^*(t) = \lim_{z\to\infty} F(z)$

7. Final Value Theorem:

If $\mathcal{Z}\{f(n)\} = F(z)$, and if the function $(1-z^{-1})F(z)$ does not have poles on or outside the unit circle $|z|=1$ in the z-plane, then:

$\lim_{t\to\infty} f^*(t) = \lim_{n\to\infty} f(n) = \lim_{z\to 1} (1-z^{-1})F(z)$

ex. given $F(z) = \dfrac{0.792\,z^2}{(z-1)(z^2-0.416z+0.208)}$, find the final value of $f(n)$

sol. since $(1-z^{-1})F(z) = \dfrac{0.792\,z}{z^2-0.416z+0.208}$ does not have poles on or outside $|z|=1$, then:

$\lim_{n\to\infty} f(n) = \lim_{z\to 1} \dfrac{0.792\,z}{z^2-0.416z+0.208} = 1$

The Inverse Z-Transform:

The I.Z.T. of F(z) is not unique and hence

$F(z) \xrightarrow{\text{I.Z.T.}} f^*(t) = f(n) \neq f(t)$

There are three approaches to obtain the I.Z.T.

1. The power series method:

by definition of Z.T.:

$$F(z) = \sum_{n=-\infty}^{\infty} f(n) z^{-n} = \cdots + f(-1) z^{1} + f(0) + f(1) z^{-1} + \cdots$$

where the series values $f(n)$ are the coefficients of z^{-n} in the expansion. Therefore, if we can find the power series expansion of F(z), then the series values of $f(n)$ are found by removing z^{-n}.

ex. Given $F(z) = \dfrac{(1-e^{-aT})z}{z^2-(1+e^{-aT})z+e^{-aT}}$, find $f(n)$

sol.

dividing F(z) by z^2 to get: $F(z) = \dfrac{(1-e^{-aT})z^{-1}}{1-(1+e^{-aT})z^{-1}+e^{-aT}z^{-2}}$

dividing the denominator into the numerator yields:

$$F(z) = (1-e^{-aT})z^{-1} + (1-e^{-2aT})z^{-2} + (1-e^{-3aT})z^{-3} + \cdots$$

since we have: $\mathcal{Z}\{\delta(t-nT)\} = z^{-n}$

$$\therefore f^*(t) = f(n) = (1-e^{-aT})\delta(t-T) + (1-e^{-2aT})\delta(t-2T) + \cdots = \sum_{n=0}^{\infty} (1-e^{-anT})\delta(t-nT)$$

$$= \sum_{n=0}^{\infty} (1-e^{-anT}) z^{-n}$$

ex. Given $F(z) = \dfrac{(0.368z + 0.264)z}{z^3 - 2z^2 + 1.632z - 0.632}$

sol. $F(z) = \dfrac{(0.368z + 0.264)z^{-2}}{1 - 2z^{-1} + 1.632z^{-2} - 0.632z^{-3}}$

$$F(z) = 0.368z^{-1} + z^{-2} + 1.4z^{-3} + 1.4z^{-4} + 1.147z^{-5} + 0.896z^{-6} + 0.802z^{-7}$$

$$\therefore f^*(t) = f(n) = 0.368\,\delta(t-T) + \delta(t-2T) + 1.4\,\delta(t-3T) + 1.4\,\delta(t-4T) + 1.147\,\delta(t-5T) + 0.896\,\delta(t-6T) + 0.802\,\delta(t-7T)$$

2. The partial fraction method:

ex. Given $F(z) = \dfrac{2z(z^2-1)}{(z^2+1)^2}$, find $f(n)$

Sol. Expanding $\dfrac{F(z)}{z}$ into partial fraction, then:

$$\therefore \frac{F(z)}{z} = \frac{A}{(z+j)^2} + \frac{A'}{(z+j)} + \frac{B}{(z-j)^2} + \frac{B'}{(z-j)}$$

$$A = \lim_{z\to -j} \frac{d}{dz}\frac{2(z^2-1)}{(z-j)^2} = 1$$

$$A' = \frac{1}{(2-1)!}\lim_{z\to -j}\frac{d}{dz}\frac{2(z^2-1)}{(z-j)^2} = 0$$

$$B = \lim_{z\to j}\frac{2(z^2-1)}{(z+j)^2} = 1$$

$$B' = \frac{1}{(2-1)!}\lim_{z\to j}\frac{d}{dz}\frac{2(z^2-1)}{(z+j)^2} = 1$$

$$\therefore F(z) = \frac{z}{(z+j)^2} + \frac{z}{(z-j)^2}$$

knowing that $-j = e^{-j\frac{\pi}{2}}$, then we can write:

$$F(z) = e^{j\pi/2}\frac{z e^{-j\pi/2}}{(z - e^{-j\pi/2})^2} + e^{-j\pi/2}\frac{z\, e^{j\pi/2}}{(z - e^{j\pi/2})^2}$$

we have $Z\{te^{-at}\} = \dfrac{zTe^{aT}}{(z-e^{-aT})^2}$, then: if $a = -j\frac{\pi}{2}$ and $a = j\frac{\pi}{2}$, $T=1$

$$f(n) = e^{j\frac{\pi}{2}}\left(n e^{-j\frac{\pi}{2}n}\right) + e^{-j\frac{\pi}{2}}\left(n\, e^{j\frac{\pi}{2}n}\right)$$

$$= jn\left[\cos\tfrac{\pi}{2}n - j\sin\tfrac{\pi}{2}n\right] - jn\left[\cos\tfrac{\pi}{2}n + j\sin\tfrac{\pi}{2}n\right] = 2n\sin\tfrac{\pi}{2}n$$

3. Contour integration:

$$f(n)=\frac{1}{2\pi j}\oint_C F(z)\,z^{n-1}\,dz = \sum(\text{residues of } F(z)z^{n-1} \text{ at the poles inside C})$$

where the path C is the circle encloses the poles of F(z).

ex. Given $F(z)=\frac{10z}{(z-1)(z-2)}$, find f(n).

Sol.

$$f(n)=\frac{1}{2\pi j}\oint_C F(z)\,z^{n-1}\,dz=\frac{1}{2\pi j}\oint_C \frac{10z^n}{(z-1)(z-2)}\,dz$$

$$f(n)=\operatorname*{Res}_{z=1}\frac{10z^n}{(z-1)(z-2)}+\operatorname*{Res}_{z=2}\frac{10z^n}{(z-1)(z-2)}$$

$$f(n)=\lim_{z\to 1}\frac{10z^n}{z-2}+\lim_{z\to 2}\frac{10z^n}{(z-1)}=-10+10(2)^n=-10(1-2^n)$$

ex. Use all the three methods (power series, partial fraction and contour integration) to find I.Z.T. of

$$F(z)=\frac{0.632z^2}{(z-1)(z^2-0.736z+0.368)}$$

Sol. a) by using contour integration:

$$f(n)=\frac{1}{2\pi j}\oint_C \frac{0.632z^{n+1}}{(z-1)[z-(0.368+j0.48)][z-(0.368-j0.48)]}$$

$$=\lim_{z\to 1}\frac{0.632z^{n+1}}{z^2-0.736z+0.368}+\lim_{z\to 0.368+j0.48}\frac{0.632z^{n+1}}{(z-1)(z-(0.368-j0.48))}$$

$$+\lim_{z\to 0.368-j0.48}\frac{0.632z^{n+1}}{(z-1)(z-(0.368+j0.48))}$$

$$=1-0.5(0.606)^n e^{j0.92n}-0.5(0.606)^n e^{-j0.92n}$$

$$=1-(0.606)^n\frac{e^{j0.92n}+e^{-j0.92n}}{2}=1-(0.606)^n\cos(0.92n)$$

b) by using partial fraction method:

$$\frac{F(z)}{z} = \frac{A}{z-1} + \frac{B_c}{z-0.368+j0.482} + \frac{B_c^*}{z-0.368-j0.482}$$

$$A = \lim_{z\to 1} \frac{0.632z}{z^2-0.736z+0.368} = 1$$

$$B_c = \lim_{z\to 0.368-j0.482} \frac{0.632z}{(z-1)(z-0.368-j0.482)} = -0.5$$

$$B_c^* = -0.5$$

$$\therefore F(z) = \frac{z}{z-1} - \frac{0.5z}{z-e^{-0.5-j0.92}} - \frac{0.5z}{z-e^{-0.5+j0.92}}$$

knowing that $Z\{e^{-at}\} = \frac{z}{z-e^{-aT}}$

$$T=1 \Rightarrow f(n) = 1 - 0.5\,e^{(-0.5-j0.92)n} - 0.5\,e^{(-0.5+j0.92)n}$$

$$= 1 - e^{-0.5n}\cos(0.92n)$$

This expression is not the same as in part (a), but it gives exactly the same answer for $n=0,1,2,\ldots$

(c) by using power series method:

$$F(z) = \frac{0.632z^2}{z^3-1.736z^2+1.104z-0.368} = \frac{0.632z^{-1}}{1-1.736z^{-1}+1.104z^{-2}-0.368z^{-3}}$$

$$= 0.632z^{-1} + 1.092z^{-2} + 1.207z^{-3} + 1.116z^{-4} + \ldots$$

$$\therefore f(n) = 0.632\delta(t-T) + 1.092\delta(t-2T) + 1.207\delta(t-3T) + 1.116\delta(t-4T) + \ldots$$

which gives the same results of parts (a) and (b).