Counting Sample Points

- In many cases it would be able to solve a probability problem by counting the number of points in the sample space. The fundamental principle of counting is stated in the following theorem.

Theorem 1.1 If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, Then the two operations can be performed together in $n_1 n_2$ ways

Example: How many sample points are in the sample space when a pair of dice are thrown once.

solution:

The pair of dice can land in $(6)(6) = 36$ ways

The First one can land in one of six ways, for each of these ways the second can also land in six ways.

In general

Theorem 1.2 If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the First two operation can be performed in n_3 ways, and so forth, then the sequence of K operations can be performed in $n_1 n_2 \cdots n_k$ ways.

Example: How many even three-digit numbers can be formed from the digits 1, 2, 5, 6 and 9 if each digit can be used only once?

Since the number must be even, there are two choices for the units position. For each of these there are four choices for the tens position, and then three choices for the hundreds position. Therefore a total of $(2)(4)(3) = 24$ even three-digits numbers.

Permutation:

A permutation is an arrangement of all or part of a set of objects

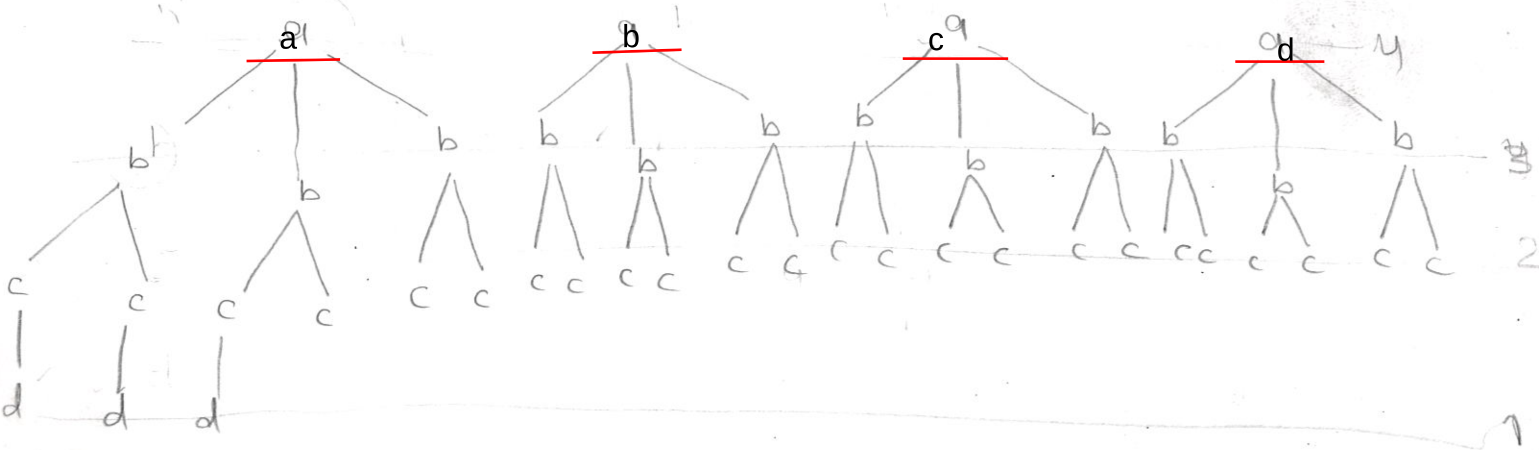
Theorem 1.3: The number of permutation of n distinct objects is $n!$

Example: Find the number of the four letters a, b, c, and d?
permutation of

solution:

The number of permutations is $4! = 24$

The first letter has 4 locations, for each of these there are 3
n $(n-1)$
locations for the second to choose and for each of above there
are $(n-2)$ 2



Definition: The number of permutations of n distict objects taken r at a time is

let r=2
n=4

$$_nP_r = \frac{n!}{(n-r)!}$$

Example: Two lottery tickets are drawn from 20 for the first and second prizes. Find the number of sample points in the space S.

solution: the total number of sample points is

$$_{20}P_2 = \frac{20!}{18!} = (20)(19) = 380$$

Theorem 1.5: The number of permutation of n distinct objects arranged in a circle is $(n-1)!$

Two circular permutations are not considered different unless corresponding objects in the two arrangements are preceded or followed by a different objects as proceeding in a clockwise direction. To arrange four objects in a circle, there is no new pemutation if they all move one position in a clockwise direction. By considering one object in a fixed position and arranging the other three in 3! ways, there will be six distict arrangement.

(4b)

Theorem 1.6 The number of distinct permutations of n things of which n_1 are of one kind, n_2 of second kind --- n_k of the kth kind is

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$

For the case of a b c d if b c d are equal to x then 3! permutation of these letters are reduced to 1

a b c d
a c b d
c a b d
c b a d
b a c d
b c a d

Example: How many different ways can three red, four yellow and two blue bulbs be arranged is a sitring of chrismas tree lights with nine sockets?

The total number of arrangements is

$$\frac{9!}{3!.\,4!.\,2!} = 1260$$

abc
acb
bca
bac
cab
cba

it is sometimes required to partition a set of n objects into r subsets called cells. A partition has been achieved if the intersection of every possible pair of the r subsets is the empty set $\emptyset$ and if the union of all subsets gives the original set. The order of the elements with in a cell is of no importance. Consider the set $\{a, e, i, o, u\}$. The possible partitions into two cells in which the first cell contains four elements and the second cell one element are. $\{(a,e,i,o),(u)\}$, $\{(a,i,o,u),(e)\}$, $\{(e,i,o,u),(a)\}$, $\{(a,e,o,u),(i)\}$, and $\{(a,e,i,u),(o)\}$. So there are five such ways to partition a set of five elements into two subsets or cells containing four elements in the first cell and one element in the second.

The number of partitions for this illustration is denoted by the symbol.

$$\binom{5}{4,1} = \frac{5!}{4!\,1!} = 5$$

where the top number represents the total number of elements and the bottom numbers represent the number of elements going into each cell.

Theorem 1.7 The number of ways of partitioning a set of n objects into r cells with n_1 elements in the first cell, n_2 elements in the second, and so forth, is

$$\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! n_2! \dots n_r!}$$

where $n_1 + n_2 + \dots + n_r = n$

Example How many ways can seven scientists be assigned to one triple and two double hotel rooms?

solution: the total number of possible partition

$$\binom{7}{3\ 2\ 2} = \frac{7!}{3!\ 2!\ 2!} = 210$$

In many problems it is required to know the number of ways of selecting r objects from n without regard to order. These selections are called Combinations. A combination is actually a partition with two cells, the one cell containing the r objects selected and the other cell containing the (n-r) objects that are left.

the number of such combinations, denoted by $\binom{n}{r, n-r}$ is usually shorted to $\binom{n}{r}$, since the number of elements in the second cell must be (n-r).

Theorem 1.8: The number of combinations of n distinct objects taken r at a time is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example: From four chemists and three physicists the number of committees of three that can be formed consisting of two chemists and one physicist.

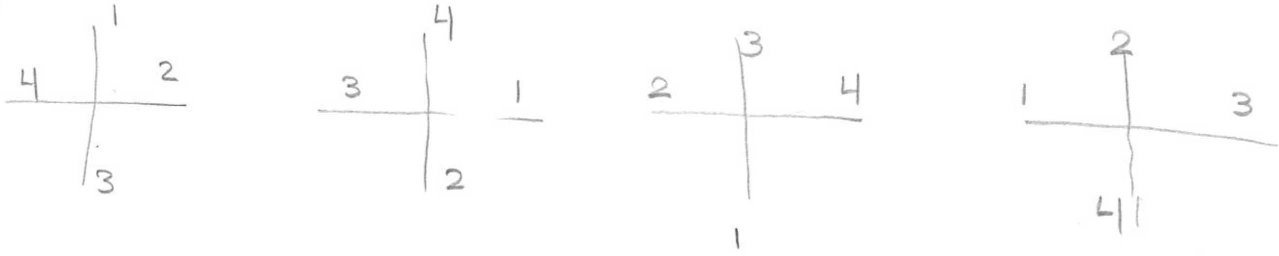
solution: The number of ways of selecting two chemists from four is

$\binom{4}{2} = \frac{4!}{2!\,2!} = 6$ the number of ways of selecting two chemists from four is

The number of selecting one physicist from three is

$$\binom{3}{1} = \frac{3!}{1!\,2!} = 3$$

using theorem 1.1 we found the number of committees that can be formed with two chemists and one physicis to be $(6)(3) = 18$



$$\frac{n!}{n} = (n-1)!$$

$$\underline{O_r\ (n-(r-1)}$$

$$n(n-1)(n-(n-1))$$