6 Sequences and series

Contents

6.1 INTRODUCTION

Much of the material in this chapter is of a fundamental nature and is applicable to many different areas of engineering. For example, if continuous signals or waveforms, such as those described in Chapter 2, are sampled at periodic intervals we obtain a sequence of measured values. Sequences also arise when we attempt to obtain approximate solutions of equations which model physical phenomena. Such approximations are necessary if a solution is to be obtained using a digital computer. For many problems of practical interest to engineers a computer solution is the only possibility. The z transform is an example of an infinite series which is particularly important in the field of digital signal processing. Signal processing is concerned with modifying signals in order to improve them in some way. For example, the signals received from space satellites have to undergo extensive processing in order to counteract the effects of noise, and to filter out unwanted frequencies, before they can provide, say, acceptable visual images. Digital signal processing is signal processing carried out using a computer. So, skill in manipulating sequences and series is crucial. Later chapters will develop these concepts and show examples of their use in solving real engineering problems.

6.2 SEQUENCES

> A **sequence** is a set of numbers or terms, not necessarily distinct, written down in a definite order.

For example,

$$1, 3, 5, 7, 9 \qquad \text{and} \qquad 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$$

are both sequences. Sometimes we use the notation '...' to indicate that the sequence continues. For example, the sequence $1, 2, 3, \ldots, 20$ is the sequence of integers from 1 to 20 inclusive. These sequences have a finite number of terms but we shall frequently deal with ones involving an infinite number of terms. To indicate that a sequence might go on for ever we can use the ... notation. Thus

$$2, 4, 6, 8, \ldots$$

and

$$1, -1, 1, -1, \ldots$$

can be assumed to continue indefinitely.

In general situations we shall write a sequence as

$$x[1], x[2], x[3], \ldots$$

or more compactly,

$$x[k] \qquad k = 1, 2, 3, \ldots$$

An alternative notation is

$$x_1, x_2, x_3, \ldots$$

The former notation is usually used in signal processing where the terms in the sequence represent the values of the signal. The latter notation arises in the numerical solution of equations. Hence both forms will be required. Often $x[1]$ will be the first term of the sequence although this is not always the case. The sequence

$$\ldots, x[-3], x[-2], x[-1], x[0], x[1], x[2], x[3], \ldots$$

is usually written as

$$x[k] \qquad k = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

A complete sequence, as opposed to a specific term of a sequence, is often written using braces, for example

$$\{x[k]\} = x[1], x[2], \ldots$$

although it is common to write $x[k]$ for both the complete sequence and a general term in the sequence when there is no confusion, and this is the convention we shall adopt in this book.

A sequence can also be regarded as a function whose domain is a subset of the set of integers. For example, the function defined by

$$x: \mathbb{N} \rightarrow \mathbb{R} \qquad x: k \rightarrow \frac{3k}{2}$$

is the sequence

$$x[0] = 0 \qquad x[1] = \frac{3}{2} \qquad x[2] = 3 \qquad x[3] = \frac{9}{2} \quad \dots$$

The values in the range of the function are the terms of the sequence. The independent variable is k. Functions of this sort differ from those described in Chapter 2 because the independent variable is not selected from a continuous interval but rather is **discrete**. It is, nevertheless, possible to represent $x[k]$ graphically as illustrated in Examples 6.1–6.3, but instead of a piecewise continuous curve, we now have a collection of isolated points.

Example 6.1 Graph the sequences given by

(a) $x[k] = \begin{cases} 0 & k < 0 \\ 1 & k \geqslant 0 \end{cases} \qquad k = \dots, -3, -2, -1, 0, 1, 2, \dots$, that is $k \in \mathbb{Z}$

(b) $x[k] = \begin{cases} 1 & k \text{ even} \\ -1 & k \text{ odd} \end{cases} \qquad k \in \mathbb{Z}$

Solution (a) From the definition of this sequence, the term $x[k]$ is zero if $k < 0$ and 1 if $k \geqslant 0$. The graph is obtained by plotting the terms of the sequence against k (see Figure 6.1). This sequence is known as the **unit step sequence**. We shall denote this by $u[k]$.

(b) The sequence $x[k]$ is shown in Figure 6.2.

Example 6.2 Graph the sequence defined by

$$x[k] = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} \qquad k = \dots, -3, -2, -1, 0, 1, 2, 3, \dots$$

Solution From the definition, if $k = 0$ then $x[k] = 1$. If k is not equal to zero the corresponding term in the sequence equals zero. Figure 6.3 shows the graph of this sequence which is commonly called the **Kronecker delta sequence**.

Example 6.3 The sequence $x[k]$ is obtained by measuring or **sampling** the continuous function $f(t) = \sin t$, $t \in \mathbb{R}$, at $t = -2\pi, -3\pi/2, -\pi, -\pi/2, 0, \pi/2, \pi, 3\pi/2$ and 2π. Write down the terms of this sequence and show them on a graph.

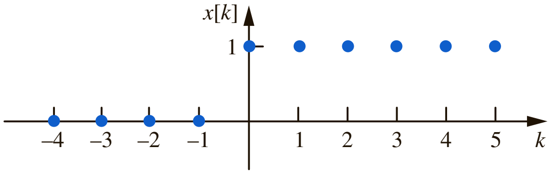

Figure 6.1
The unit step sequence.

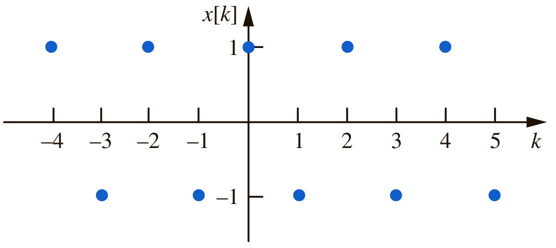

Figure 6.2
The sequence $x[k] = \begin{cases} 1 & k \text{ even} \\ -1 & k \text{ odd.} \end{cases}$

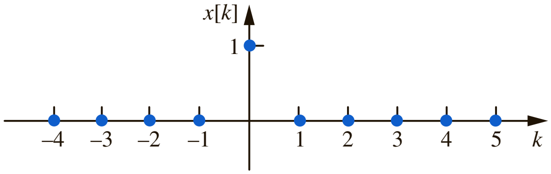

Figure 6.3
The Kronecker delta sequence.

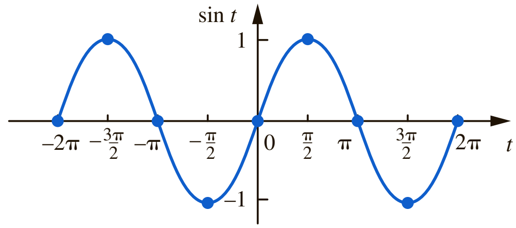

Figure 6.4
The function $f(t) = \sin t$ with sampled points shown.

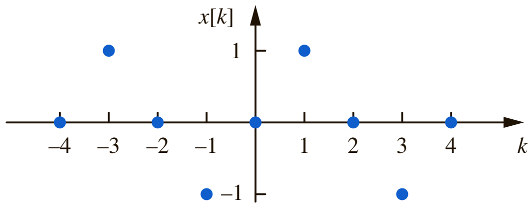

Figure 6.5
Sequence formed from sampling $f(t) = \sin t$.

Solution The function $f(t) = \sin t$, for $-2\pi \leqslant t \leqslant 2\pi$, is shown in Figure 6.4. We sample the continuous function at the required points. The sample values are shown as •. From the graph we see that

$$x[k] = 0, 1, 0, -1, 0, 1, 0, -1, 0 \qquad k = -4, -3, \ldots, 3, 4$$

The graph of $x[k]$ is shown in Figure 6.5.

Sometimes it is possible to describe a sequence by a rule giving the kth term. For example, the sequence for which $x[k] = 2^k$, $k = 0, 1, 2, \ldots$, is given by 1, 2, 4, 8 . . . . On occasions, a rule gives $x[k]$ in terms of earlier members of the sequence. For example, the previous sequence could have been defined by $x[k] = 2x[k-1]$, $x[0] = 1$. The sequence is then said to be defined **recursively** and the defining formula is called a **recurrence relation** or **difference equation**. Difference equations are particularly important in digital signal processing and are dealt with in Chapter 22.

Example 6.4 Write down the terms $x[k]$ for $k = 0, \ldots, 7$ of the sequence defined recursively as

$$x[k] = x[k-2] + x[k-1]$$

where $x[0] = 1$ and $x[1] = 1$.

Solution The values of $x[0]$ and $x[1]$ are given. Using the given recurrence relation we find

$$x[2] = x[0] + x[1] = 2$$
$$x[3] = x[1] + x[2] = 3$$

Continuing in this fashion we find the first eight terms of the sequence are

$$1, 1, 2, 3, 5, 8, 13, 21$$

This sequence is known as the Fibonacci sequence.

6.2.1 Arithmetic progressions

An arithmetic progression is a sequence where each term is found by adding a fixed quantity, called the **common difference**, to the previous term.

Example 6.5 Write down the first five terms of the arithmetic progression where the first term is 1 and the common difference is 3.

Solution The second term is found by adding the common difference, 3, to the first term, 1, and so the second term is 4. Continuing in this way we can construct the sequence

$$1, 4, 7, 10, 13, \ldots$$

A more general arithmetic progression has first term a and common difference d, that is

$$a, a+d, a+2d, a+3d, \ldots$$

It is easy to see that the kth term is

$$a + (k-1)d$$

All arithmetic progressions can be written recursively as $x[k] = x[k-1] + d$.

> Arithmetic progression: $a, a+d, a+2d, \ldots$
>
> $a =$ first term, $d =$ common difference, kth term $= a + (k-1)d$

Example 6.6 Find the 10th and 20th terms of the arithmetic progression with a first term 5 and common difference -4.

Solution Here $a = 5$ and $d = -4$. The kth term is $5 - 4(k-1)$. Therefore the 10th term is $5 - 4(9) = -31$ and the 20th term is $5 - 4(19) = -71$.

6.2.2 Geometric progressions

A geometric progression is a sequence where each term is found by multiplying the previous term by a fixed quantity called the **common ratio**.

Example 6.7 Write down the geometric progression whose first term is 1 and whose common ratio is $\frac{1}{2}$.

Solution The second term is found by multiplying the first by the common ratio, $\frac{1}{2}$, that is $\frac{1}{2} \times 1 = \frac{1}{2}$. Continuing in this way we obtain the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

A general geometric progression has first term a and common ratio r and can therefore be written as

$$a, ar, ar^2, ar^3, \ldots$$

and it is easy to see that the kth term is ar^{k-1}. All geometric progressions can be written recursively as $x[k] = rx[k-1]$.

> Geometric progression: $a, ar, ar^2, \ldots$
>
> $a =$ first term, $r =$ common ratio, kth term $= ar^{k-1}$

6.2.3 More general sequences

We have already met a number of infinite sequences. For example,

(1) $x[k] = 2, 4, 6, 8, \ldots$

(2) $x[k] = 1, \frac{1}{2}, \frac{1}{4}, \ldots$

In case (1) the terms of the sequence go on increasing without bound. We say the sequence is **unbounded**. On the other hand, in case (2) it is clear that successive terms get smaller and smaller and as $k \to \infty$, $x[k] \to 0$. The notion of getting closer and closer to a fixed value is very important in mathematics and gives rise to the concept of a **limit**. In case (2) we say 'the limit of $x[k]$ as k tends to infinity is 0' and we write this concisely as

$$\lim_{k\to\infty} x[k] = 0$$

We say that the sequence converges to 0, and because its terms do not increase without bound we say it is **bounded**.

More formally, we say that a sequence $x[k]$ **converges** to a limit l if, by proceeding far enough along the sequence, all subsequent terms can be made to lie as close to l as we wish. Whenever a sequence is not convergent it is said to be **divergent**.

It is possible to have sequences which are bounded but nevertheless do not converge to a limit. The sequence

$$x[k] = -1, 1, -1, 1, -1, 1, \ldots$$

clearly fails to have a limit as $k \to \infty$ although it is bounded, that is its values all lie within a given range. This particular sequence is said to **oscillate**.

It is possible to evaluate the limit of a sequence, when such a limit exists, from knowledge of its general term. To be able to do this we can make use of certain rules, the proofs of which are beyond the scope of this book, but which we now state:

> If $x[k]$ and $y[k]$ are two sequences such that $\lim_{k\to\infty} x[k] = l_1$, and $\lim_{k\to\infty} y[k] = l_2$, where l_1 and l_2 are finite, then:
>
> (1) The sequence given by $x[k] \pm y[k]$ has limit $l_1 \pm l_2$.
>
> (2) The sequence given by $cx[k]$, where c is a constant, has limit cl_1.
>
> (3) The sequence $x[k]y[k]$ has limit $l_1 l_2$.
>
> (4) The sequence $\frac{x[k]}{y[k]}$ has limit $\frac{l_1}{l_2}$ provided $l_2 \neq 0$.

Furthermore, we can always assume that

$$\lim_{k\to\infty} \frac{1}{k^m} = 0 \qquad \text{for any constant } m > 0$$

Example 6.8 Find, if possible, the limit of each of the following sequences, $x[k]$.

(a) $x[k] = \dfrac{1}{k}$ $\qquad k = 1, 2, 3, 4, \ldots$

(b) $x[k] = 5$ $\qquad k = 1, 2, 3, 4, \ldots$

(c) $x[k] = 3 + \dfrac{1}{k}$ $\qquad k = 1, 2, 3, 4, \ldots$

(d) $x[k] = \dfrac{1}{k+1}$ $\qquad k = 1, 2, 3, 4, \ldots$

(e) $x[k] = k^2$ $\qquad k = 1, 2, 3, 4, \ldots$

Solution (a) The sequence $x[k]$ is given by

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

Successive terms get smaller and smaller, and as $k \to \infty$, $x[k] \to 0$. By proceeding far enough along the sequence we can get as close to the limit 0 as we wish. Hence

$$\lim_{k\to\infty} x[k] = \lim_{k\to\infty} \frac{1}{k} = 0$$

(b) The sequence $x[k]$ is given by $5, 5, 5, 5, \ldots$. This sequence has limit 5.

(c) The sequence $3, 3, 3, 3, \ldots$ has limit 3. The sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$ has limit 0. Therefore, using rule (1) we have

$$\lim_{k\to\infty} 3 + \frac{1}{k} = 3 + 0 = 3$$

The terms of the sequence $x[k] = 3 + \frac{1}{k}$ are given by $4, 3\frac{1}{2}, 3\frac{1}{3}, \ldots$, and by proceeding far enough along we can make all subsequent terms lie as close to the limit 3 as we wish.

(d) The sequence $x[k] = \dfrac{1}{k+1}$, $k = 1, 2, 3, 4, \ldots$, is given by

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

and has limit 0.

(e) The sequence $x[k] = k^2, k = 1, 2, 3, 4, \ldots$, is given by $1, 4, 9, 16, \ldots$, and increases without bound. This sequence has no limit – it is divergent.

Example 6.9 Given a sequence with general term $x[k] = \dfrac{k-1}{k+1}$, find $\lim_{k\to\infty} x[k]$.

Solution It is meaningless simply to write $k = \infty$ to obtain $\lim_{k\to\infty} x[k] = \dfrac{\infty - 1}{\infty + 1}$, since such a quantity is undefined. What we should do is try to rewrite $x[k]$ in a form in which we can sensibly let $k \to \infty$. Dividing both numerator and denominator by k, we write

$$\frac{k-1}{k+1} = \frac{1-(1/k)}{1+(1/k)}$$

Then, as $k \to \infty$, $1/k \to 0$ so that

$$\lim_{k\to\infty}\left(\frac{1-(1/k)}{1+(1/k)}\right) = \frac{\lim_{k\to\infty}(1-(1/k))}{\lim_{k\to\infty}(1+(1/k))} \qquad \text{by rule (4)}$$

$$= \frac{1}{1}$$

$$= 1$$

Example 6.10 Given a sequence with general term

$$x[k] = \frac{3k^2-5k+6}{k^2+2k+1}$$

find $\lim_{k\to\infty} x[k]$.

Solution Dividing the numerator and denominator by k^2 introduces terms which tend to zero as $k \to \infty$, that is

$$\frac{3k^2-5k+6}{k^2+2k+1} = \frac{3-(5/k)+(6/k^2)}{1+(2/k)+(1/k^2)}$$

Then as $k \to \infty$, we find

$$\lim_{k\to\infty} x[k] = \frac{3}{1} = 3$$

Example 6.11 Examine the behaviour of $\dfrac{k^2}{3k+1}$ as $k \to \infty$.

Solution

$$\frac{k^2}{3k+1} = \frac{k}{3+(1/k)}$$

As $k \to \infty$, $1/k \to 0$ so that the denominator approaches 3. On the other hand, as $k \to \infty$ the numerator tends to infinity so that this sequence diverges to infinity.

EXERCISES 6.2

1 Graph the sequences given by

(a) $x[k] = k$, $k = 0, 1, 2, 3, \ldots$

(b) $x[k] = \begin{cases} 3 & k = 2 \\ 0 & \text{otherwise} \end{cases} \qquad k = 0, 1, 2, 3, \ldots$

(c) $x[k] = e^{-k}$, $k = 0, 1, 2, 3, \ldots$

2 The sequence $x[k]$ is obtained by sampling $f(t) = \cos(t+2)$, $t \in \mathbb{R}$. The sampling begins at $t = 0$ and thereafter at $t = 1, 2, 3, \ldots$. Write down the first six terms of the sequence.

3 A sequence, $x[k]$, is defined by

$$x[k] = \frac{k^2}{2} + k, \; k = 0, 1, 2, 3, \ldots$$

State the first five terms of the sequence.

4 Write down the first five terms, and plot graphs, of the sequences given recursively by

(a) $x[k] = \dfrac{x[k-1]}{2}, \qquad x[0] = 1$

(b) $x[k] = 3x[k-1] - 2x[k-2]$,
$x[0] = 2, \quad x[1] = 1$

5 A recurrence relation is defined by
$x[n+1] = x[n] + 10, \quad x[0] = 1,$
$n = 0, 1, 2, 3, \ldots$

Find $x[1]$, $x[2]$, $x[3]$ and $x[4]$.

6 A sequence is defined by means of the recurrence relation
$x[n+1] = x[n] + n^2, \quad x[0] = 1,$
$n = 0, 1, 2, 3, \ldots$
Write down the first five terms.

7 Consider the difference equation
$x[n+2] - x[n+1] = 3x[n],$
$n = 0, 1, 2, 3, \ldots$
If $x[0] = 1$ and $x[1] = 2$, find the terms $x[2], x[3], \ldots, x[6]$.

8 Write down the 10th and 19th terms of the arithmetic progressions

(a) 8, 11, 14, ...

(b) 8, 5, 2, ...

9 An arithmetic progression is given by

$$b, \frac{2b}{3}, \frac{b}{3}, 0, \ldots$$

(a) State the sixth term.

(b) State the kth term.

(c) If the 20th term has a value of 15, find b.

10 Write down the 5th and 10th terms of the geometric progression 8, 4, 2, ...

11 Find the 10th and 20th terms of the geometric progression with first term 3 and common ratio 2.

12 A geometric progression is given by

$$a, ar, ar^2, ar^3, \ldots$$

If $|(k+1)\text{th term}| > |k\text{th term}|$ and $(k+1)\text{th term} \times k\text{th term} < 0$, which of the following, if any, must be true?

(a) $r > 1$ (b) $a > 1$
(c) $r < -1$ (d) a is negative
(e) $-1 < r < 1$

13 A geometric progression has first term $a = 1$. The ninth term exceeds the fifth term by 240. Find possible values for the eighth term.

14 If $x[k] = \frac{3k+2}{k}$ find $\lim_{k\to\infty} x[k]$.

15 Find $\lim_{k\to\infty} \frac{3k+2}{k^2+7}$.

16 Find the limits as k tends to infinity, if they exist, of the following sequences:

(a) $x[k] = k^3$

(b) $x[k] = \frac{2k+3}{4k+2}$

(c) $x[k] = \frac{k^2+k}{k^2+k+1}$

17 Find $\lim_{k\to\infty} \left(\frac{6k+7}{3k-2}\right)^4$.

18 Find $\lim_{k\to\infty} x[k]$, if it exists, when

(a) $x[k] = (-1)^k$

(b) $x[k] = 2 - \frac{k}{10}$

(c) $x[k] = \left(\frac{1}{3}\right)^k$

(d) $x[k] = \frac{3k^3 - 2k^2 + 4}{5k^3 + 2k^2 + 4}$

(e) $x[k] = \left(\frac{1}{5}\right)^{2k}$

Solutions

2 cos 2, cos 3, cos 4, cos 5, cos 6, cos 7

3 $0, \frac{3}{2}, 4, \frac{15}{2}, 12$

4 (a) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ (b) 2, 1, −1, −5, −13

5 11, 21, 31, 41

6 1, 1, 2, 6, 15

7 5, 11, 26, 59, 137

8 (a) 10th term = 35, 19th term = 62
(b) 10th term = −19, 19th term = −46

9 (a) $-\frac{2b}{3}$ (b) $\frac{b(4-k)}{3}$ (c) $-\frac{45}{16}$

10 $\frac{1}{2}, \frac{1}{64}$

11 1536, 1 572 864

12 Only (c) must be true

13 ± 128

14 3

15 0

16 (a) Limit does not exist (b) $\frac{1}{2}$ (c) 1

17 16

18 (a) Limit does not exist
(b) Limit does not exist
(c) 0 (d) $\frac{3}{5}$ (e) 0

6.3 SERIES

Whenever the terms of a sequence are added together we obtain what is known as a **series**. For example, if we add the terms of the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, we obtain the series S, where

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$

This series ends after the fourth term and is said to be a **finite series**. Other series we shall meet continue indefinitely and are said to be **infinite series**.

Given an arbitrary sequence $x[k]$, we use the **sigma** notation

$$S_n = \sum_{k=1}^{n} x[k]$$

to mean the sum $x[1] + x[2] + \cdots + x[n]$, the first and last values of k being shown below and above the Greek letter Σ, which is pronounced 'sigma'. If the first term of the sequence is $x[0]$ rather than $x[1]$ we would write $\sum_{k=0}^{n} x[k]$.

6.3.1 Sum of a finite arithmetic series

An arithmetic series is the sum of an arithmetic progression. Consider the sum

$$S = 1 + 2 + 3 + 4 + 5$$

Clearly this sums to 15. When there are many more terms it is necessary to find a more efficient way of adding them up. The equation for S can be written in two ways:

$$S = 1 + 2 + 3 + 4 + 5$$

and

$$S = 5 + 4 + 3 + 2 + 1$$

If we add these two equations together we get

$$2S = 6 + 6 + 6 + 6 + 6$$

There are five terms so that

$$2S = 5 \times 6 = 30$$

that is

$$S = 15$$

Now a general arithmetic series with k terms can be written as

$$S_k = a + (a + d) + (a + 2d) + \cdots + (a + (k - 1)d)$$

but rewriting this back to front, we have

$$S_k = (a + (k - 1)d) + (a + (k - 2)d) + \cdots + (a + d) + a$$

Adding together the first term in each series produces $2a + (k-1)d$. Adding the second terms together produces $2a + (k-1)d$. Indeed adding together the ith terms yields $2a + (k-1)d$. Hence,

$$2S_k = \underbrace{(2a + (k-1)d) + (2a + (k-1)d) + \cdots + (2a + (k-1)d)}_{k \text{ times}}$$

that is

$$2S_k = k(2a + (k-1)d)$$

so that

$$S_k = \frac{k}{2}(2a + (k-1)d)$$

This formula tells us the sum to k terms of the arithmetic series with first term a and common difference d.

> Sum of an arithmetic series: $S_k = \frac{k}{2}(2a + (k-1)d)$

Example 6.12 Find the sum of the arithmetic series containing 30 terms, with first term 1 and common difference 4.

Solution We wish to find S_k:

$$S_k = \underbrace{1 + 5 + 9 + \cdots}_{30 \text{ terms}}$$

Using $S_k = \frac{k}{2}(2a + (k-1)d)$ we find $S_{30} = \frac{30}{2}(2 + 29 \times 4) = 1770$.

Example 6.13 Find the sum of the arithmetic series with first term 1, common difference 3 and with last term 100.

Solution We already know that the kth term of an arithmetic progression is given by $a + (k-1)d$. In this case the last term is 100. We can use this fact to find the number of terms. Thus,

$$100 = 1 + 3(k-1)$$

that is

$$\begin{aligned} 3(k-1) &= 99 \\ k-1 &= 33 \\ k &= 34 \end{aligned}$$

So there are 34 terms in this series. Therefore the sum, S_{34}, is given by

$$\begin{aligned} S_{34} &= \frac{34}{2}\{2(1) + (33)(3)\} \\ &= 17(101) \\ &= 1717 \end{aligned}$$

6.3.2 Sum of a finite geometric series

A geometric series is the sum of the terms of a geometric progression. If we sum the geometric progression $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ we find

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \tag{6.1}$$

If there had been a large number of terms it would have been impractical to add them all directly. However, let us multiply Equation (6.1) by the common ratio, $\frac{1}{2}$:

$$\frac{1}{2}S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \tag{6.2}$$

so that, subtracting Equation (6.2) from Equation (6.1), we find

$$S - \frac{1}{2}S = 1 - \frac{1}{32}$$

since most terms cancel out. Therefore $\frac{1}{2}S = \frac{31}{32}$ and so $S = \frac{31}{16} = 1\frac{15}{16}$.

We can apply this approach more generally: when we have a geometric progression with first term a and common ratio r, the sum to k terms is

$$S_k = a + ar + ar^2 + ar^3 + \cdots + ar^{k-1}$$

Multiplying by r gives

$$rS_k = ar + ar^2 + ar^3 + \cdots + ar^{k-1} + ar^k$$

Subtraction gives $S_k - rS_k = a - ar^k$, so that

$$S_k = \frac{a(1 - r^k)}{1 - r} \qquad \text{provided } r \neq 1$$

This formula gives the sum to k terms of the geometric series with first term a and common ratio r.

> Sum of a geometric series: $S_k = \dfrac{a(1 - r^k)}{1 - r} \qquad r \neq 1$

6.3.3 Sum of an infinite series

When dealing with infinite series the situation is more complicated. Nevertheless, it is frequently the case that the answer to many problems can be expressed as an infinite series. In certain circumstances, the sum of a series tends to a finite answer as more and more terms are included and we say the series has **converged**. To illustrate this idea, consider the graphical interpretation of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, as given in Figure 6.6.

Start at 0 and move a length 1: total distance moved $= 1$

Move further, a length $\frac{1}{2}$: total distance moved $= 1\frac{1}{2}$

Move further, a length $\frac{1}{4}$: total distance moved $= 1\frac{3}{4}$

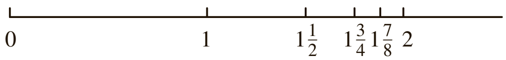

Figure 6.6
Graphical interpretation of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.

At each stage the extra distance moved is half the distance remaining up to the point $x = 2$. It is obvious that the total distance we move cannot exceed 2 although we can get as close to 2 as we like by adding on more and more terms. We say that the series $1 + \frac{1}{2} + \frac{1}{4} + \cdots$ converges to 2. The sequence of total distances moved, given previously,

$$1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, \ldots$$

is called the **sequence of partial sums** of the series.

For any given infinite series $\sum_{k=1}^{\infty} x[k]$, we can form the sequence of partial sums,

$$S_1 = \sum_{k=1}^{1} x[k] = x[1]$$

$$S_2 = \sum_{k=1}^{2} x[k] = x[1] + x[2]$$

$$S_3 = \sum_{k=1}^{3} x[k] = x[1] + x[2] + x[3]$$

$$\vdots$$

If the sequence $\{S_n\}$ converges to a limit S, we say that the infinite series has a sum S or that it has converged to S. Clearly not all infinite series will converge. For example, consider the series

$$1 + 2 + 3 + 4 + 5 + \cdots$$

The sequence of partial sums is $1, 3, 6, 10, 15, \ldots$. This sequence diverges to infinity and so the series $1 + 2 + 3 + 4 + 5 + \cdots$ is divergent.

It is possible to establish tests or **convergence criteria** to help us to decide whether or not a given series converges or diverges, but for these you must refer to a more advanced text.

6.3.4 Sum of an infinite geometric series

In the case of an infinite geometric series, it is possible to derive a simple formula for its sum when convergence takes place. We have already seen that the sum to k terms is given by

$$S_k = \frac{a(1 - r^k)}{1 - r} \qquad r \neq 1$$

What we must do is allow k to become large so that more and more terms are included in the sum. Provided that $-1 < r < 1$, then $r^k \to 0$ as $k \to \infty$. Then $S_k \to \frac{a}{1 - r}$. When this happens we write

$$S_\infty = \frac{a}{1 - r}$$

where S_∞ is known as the 'sum to infinity'. If $r > 1$ or $r < -1$, r^k fails to approach a finite limit as $k \to \infty$ and the geometric series diverges.

Sum of an infinite geometric series: $S_\infty = \dfrac{a}{1-r}$ $\quad -1 < r < 1$

Example 6.14 Find the sum to k terms of the following series and deduce their sums to infinity:

(a) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$ (b) $12 + 6 + 3 + 1\frac{1}{2} + \cdots$

Solution (a) This is a geometric series with first term 1 and common ratio 1/3. Therefore,

$$S_k = \frac{a(1-r^k)}{1-r} = \frac{1(1-(1/3)^k)}{2/3} = \frac{3}{2}\left(1-\left(\frac{1}{3}\right)^k\right)$$

As $k \to \infty$, $(1/3)^k \to 0$ so that $S_\infty = 3/2$.

(b) This is a geometric series with first term 12 and common ratio $\frac{1}{2}$. Therefore,

$$S_k = 24(1-(1/2)^k)$$

As $k \to \infty$, $(1/2)^k \to 0$ so that $S_\infty = 24$. This could, of course, have been obtained directly from the formula for the sum to infinity.

EXERCISES 6.3

1. An arithmetic series has a first term of 4 and its 30th term is 1000. Find the sum to 30 terms.
2. Find the sum to 20 terms of the arithmetic series with first term a, and common difference d, given by
   (a) $a = 4, d = 3$ (b) $a = 4, d = -3$
3. If the sum to 10 terms of an arithmetic series is 100 and its common difference, d, is -3, find its first term.
4. The sum to 20 terms of an arithmetic series is identical to the sum to 22 terms. If the common difference is -2, find the first term.
5. Find the sum to five terms of the geometric series with first term 1 and common ratio 1/3. Find the sum to infinity.
6. Find the sum of the first 20 terms of the geometric series with first term 3 and common ratio 1.5.
7. Find the sum of the arithmetic series with first term 2, common difference 2, and last term 50.
8. The sum to infinity of a geometric series is four times the first term. Find the common ratio.
9. The sum to infinity of a geometric series is twice the sum of the first two terms. Find possible values of the common ratio.
10. Express the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ in sigma notation.
11. Write down the first six terms of the series $\sum_{k=0}^{\infty} z^{-k}$.
12. Explain why $\sum_{k=1}^{\infty} x[k]$ is the same as $\sum_{n=1}^{\infty} x[n]$. Further, explain why both can be written as $\sum_{k=0}^{\infty} x[k+1]$.

Solutions

1. 15 060
2. (a) 650 (b) -490
3. 23.5
4. 41

5 1.494, $S_\infty = \frac{3}{2}$

6 19 946

7 650

8 $\frac{3}{4}$

9 $\pm\frac{1}{\sqrt{2}}$

10 $\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n}$

11 $1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{z^5}$

Technical Computing Exercises 6.3

Most technical computing languages have built-in functions for generating geometric series. MATLAB® has the function `cumprod` which calculates the **cumulative product** of the numbers passed to its input. It takes the first term and then successively multiplies the succeeding arguments in turn by the result.

For example, in MATLAB® we could type:

```
S=cumprod([1 0.5 0.5 0.5 0.5])
```

to produce the geometric series and store it in a row vector `S`.

```
1.000000   0.500000   0.250000
           0.125000   0.062500
```

1 Calculate the sum of all of the elements in the finite series given above.

2 Increase the number of elements in the series to 10 and note the difference in your answer.

3 Compare the answers to the previous two exercises to the exact equation given at the end of Section 6.3.2 with $a = 1$ and $r = 0.5$. What would you expect to happen if there were 100 elements in the series?

6.4 THE BINOMIAL THEOREM

It is straightforward to show that the expression $(a + b)^2$ can be written as $a^2 + 2ab + b^2$. It is slightly more complicated to expand the expression $(a + b)^3$ to $a^3 + 3a^2b + 3ab^2 + b^3$. However, it is often necessary to expand quantities such as $(a + b)^6$ or $(a + b)^{10}$, say, and then the algebra becomes extremely lengthy. A simple technique for expanding expressions of the form $(a + b)^n$, where n is a positive integer, is given by Pascal's triangle.

Pascal's triangle is the triangle of numbers shown in Figure 6.7, where it is observed that every entry is obtained by adding the two entries on either side in the preceding row, always starting and finishing a row with a '1'. You will note that the third row down, 1 2 1, gives the coefficients in the expansion of $(a + b)^2 = 1a^2 + 2ab + 1b^2$, while the fourth row, 1 3 3 1, gives the coefficients in the expansion of $(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$. Furthermore, the terms in these expansions are composed of decreasing powers of a and increasing powers of b. When we come to expand the quantity $(a + b)^4$ the row beginning '1 4' in the triangle will provide us with the necessary coefficients in the expansion and we must simply take care to put in place the appropriate powers of a and b. Thus $(a + b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$.

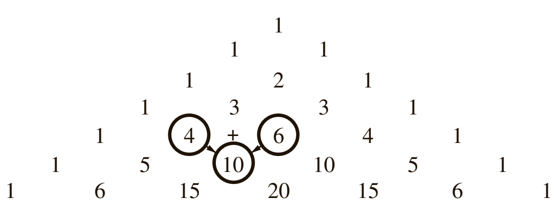

Figure 6.7
Pascal's triangle.

Example 6.15 Use Pascal's triangle to expand $(a+b)^6$.

Solution We look to the row commencing '1 6', that is 1 6 15 20 15 6 1, because $a+b$ is raised to the power 6. This row provides the necessary coefficients. Thus,

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

Example 6.16 Expand $(1+x)^7$ using Pascal's triangle.

Solution Forming the row commencing '1 7' we select the coefficients

1 7 21 35 35 21 7 1

In this example, $a = 1$ and $b = x$ so that

$$(1+x)^7 = 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$$

When it is necessary to expand the quantity $(a+b)^n$ for large n, it is clearly inappropriate to use Pascal's triangle since an extremely large triangle would have to be constructed. However, it is frequently the case that in such situations only the first few terms in the expansion are required. This is where the **binomial theorem** is useful.

The binomial theorem states that when n is a positive integer

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + b^n$$

It is also frequently quoted for the case when $a = 1$ and $b = x$, so that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n \qquad (6.3)$$

Example 6.17 Expand $(1+x)^{10}$ up to the term in x^3.

Solution We could use Pascal's triangle to answer this question and look to the row commencing '1 10' but to find this row considerable calculations would be needed. We shall use the binomial theorem in the form of Equation (6.3). Taking $n = 10$, we find

$$\begin{aligned}(1+x)^{10} &= 1 + 10x + \frac{10(9)}{2!}x^2 + \frac{(10)(9)(8)}{3!}x^3 + \cdots \\ &= 1 + 10x + 45x^2 + 120x^3 + \cdots\end{aligned}$$

so that, up to and including x^3, the expansion is

$$1 + 10x + 45x^2 + 120x^3$$

We have assumed in the foregoing discussion that n is a positive integer in which case the expansion given by Equation (6.3) will eventually terminate. In Example 6.17 this would occur when we reached the term in x^{10}. It can be shown, however, that when n is not a positive integer the function $(1+x)^n$ and the expansion given by

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \tag{6.4}$$

have the same value provided $-1 < x < 1$. However, when n is not a positive integer the series does not terminate and we must deal with an infinite series. This series converges when $-1 < x < 1$ and the expansion is then said to be valid. When x lies outside this interval the infinite series diverges and so bears no relation to the value of $(1+x)^n$. The expansion is then said to be invalid.

The binomial theorem:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \qquad -1 < x < 1$$

Example 6.18 Use the binomial theorem to expand $\frac{1}{1+x}$ in ascending powers of x up to and including the term in x^3.

Solution $\frac{1}{1+x}$ can be written as $(1+x)^{-1}$. Using the binomial theorem given by Equation (6.4) with $n = -1$, we find

$$\begin{aligned}(1+x)^{-1} &= 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \cdots \\ &= 1 - x + x^2 - x^3 + \cdots\end{aligned}$$

provided $-1 < x < 1$. Consequently, if in future applications we come across the series $1 - x + x^2 - x^3 + \ldots$, we shall be able to write it in the form $(1+x)^{-1}$. This **closed form** avoids the use of an infinite series and so it is easier to handle. We shall make use of this technique in Chapter 22 when we meet the z transform.

Example 6.19 Obtain a quadratic approximation to $(1-2x)^{1/2}$ using the binomial theorem.

Solution Using Equation (6.4) with x replaced by $-2x$ and $n = \frac{1}{2}$ we have

$$\begin{aligned}(1-2x)^{1/2} &= 1 + \left(\frac{1}{2}\right)(-2x) + \frac{(1/2)(-1/2)}{2!}(-2x)^2 + \cdots \\ &= 1 - x - \frac{1}{2}x^2 + \cdots\end{aligned}$$

provided that $-1 < -2x < 1$, that is $-\frac{1}{2} < x < \frac{1}{2}$. A quadratic approximation is therefore $1 - x - \frac{1}{2}x^2$.

EXERCISES 6.4

1 Use the binomial theorem to expand

(a) $(1+x)^3$ (b) $(1+x)^4$ (c) $\left(1+\frac{x}{3}\right)^4$

(d) $\left(1-\frac{x}{2}\right)^5$ (e) $\left(2+\frac{x}{2}\right)^5$ (f) $\left(3-\frac{x}{4}\right)^4$

2 Use Pascal's triangle to expand $(a+b)^8$.

3 Use Pascal's triangle to expand $(2x+3y)^4$.

4 Expand $(a-2b)^5$.

5 Use the binomial theorem to find the expansion of $(3-2x)^6$ up to and including the term in x^3.

6 Obtain the first four terms in the expansion of $\left(1+\frac{1}{2}x\right)^{10}$.

7 Obtain the first five terms in the expansion of $(1+2x)^{1/2}$. State the range of values of x for which the expansion is valid. Choose a value of x within the range of validity and compute values of your expansion for comparison with the true function values.

8 Expand $\left(1+\frac{1}{2}x\right)^{-4}$ in ascending powers of x up to the term in x^4, stating the range of values of x for which the expansion is valid.

9 Expand $\left(1+\frac{1}{x}\right)^{-1/2}$ in descending powers up to the fourth term.

10 (a) Expand $(1+x^2)^4$.

(b) Expand $(1+1/x^2)^4$.

11 A function, $f(x)$, is given by

$$f(x)=\left(1+\frac{1}{x}\right)^{1/2}$$

(a) Obtain the first four terms in the expansion of $f(x)$ in descending powers of x. State the range of values of x for which the expansion is valid.

(b) By writing $f(x)$ in the form

$$f(x)=x^{-1/2}(1+x)^{1/2}$$

obtain the first four terms in the expansion of $f(x)$ in ascending powers of x. State the range of values of x for which the expansion is valid.

12 The function, $g(x)$, is defined by

$$g(x)=\frac{1}{(1+x^2)^4}$$

(a) Obtain the first four terms in the expansion of $g(x)$ in ascending powers of x. State the range of values of x for which the expansion is valid.

(b) By rewriting $g(x)$ in an appropriate form, obtain the first four terms in the expansion of $g(x)$ in descending powers of x. State the range of values of x for which the expansion is valid.

Solutions

1 (a) $1+3x+3x^2+x^3$

(b) $1+4x+6x^2+4x^3+x^4$

(c) $1+\frac{4x}{3}+\frac{2x^2}{3}+\frac{4x^3}{27}+\frac{x^4}{81}$

(d) $1-\frac{5x}{2}+\frac{5x^2}{2}-\frac{5x^3}{4}+\frac{5x^4}{16}-\frac{x^5}{32}$

(e) $32+40x+20x^2+5x^3+\frac{5x^4}{8}+\frac{x^5}{32}$

(f) $81-27x+\frac{27x^2}{8}-\frac{3x^3}{16}+\frac{x^4}{256}$

2 $a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+28a^2b^6+8ab^7+b^8$

3 $16x^4+96x^3y+216x^2y^2+216xy^3+81y^4$

4 $a^5-10a^4b+40a^3b^2-80a^2b^3+80ab^4-32b^5$

5 $729-2916x+4860x^2-4320x^3$

6 $1+5x+\frac{45x^2}{4}+15x^3$

7 $1+x-\frac{x^2}{2}+\frac{x^3}{2}-\frac{5x^4}{8}$ valid for $-\frac{1}{2}<x<\frac{1}{2}$

8 $1-2x+\frac{5x^2}{2}-\frac{5x^3}{2}+\frac{35x^4}{16}$ valid for $-2<x<2$

9 $1-\frac{1}{2x}+\frac{3}{8x^2}-\frac{5}{16x^3}$

10 (a) $1 + 4x^2 + 6x^4 + 4x^6 + x^8$

(b) $1 + \frac{4}{x^2} + \frac{6}{x^4} + \frac{4}{x^6} + \frac{1}{x^8}$

11 (a) $1 + \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{16x^3}$ valid for $|x| > 1$

(b) $x^{-1/2}\left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}\right)$ valid for $|x| < 1$

12 (a) $1 - 4x^2 + 10x^4 - 20x^6$ valid for $|x| < 1$

(b) $x^{-8}\left(1 - \frac{4}{x^2} + \frac{10}{x^4} - \frac{20}{x^6} \cdots\right)$ valid for $|x| > 1$

6.5 POWER SERIES

A particularly important class of series are known as **power series** and these are infinite series involving integer powers of the variable x. For example,

$$1 + x + x^2 + x^3 + \cdots$$

and

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

are both power series. Note that a power series can be regarded as an infinite polynomial. Many common functions can be expressed in terms of a power series, for example

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad x \text{ in radians}$$

which converges for any value of x. For example,

$$\sin(0.5) = 0.5 - \frac{(0.5)^3}{6} + \frac{(0.5)^5}{120} - \cdots$$

Taking just the first three terms, we find

$$\sin(0.5) \approx 0.5 - 0.020\,833\,3 + 0.000\,260\,4 = 0.479\,427\,1$$

as compared with the true value, $\sin 0.5 = 0.479\,425\,5$.

More generally, a power series is only meaningful if the series converges for the particular value of x chosen. We define an important quantity known as the **radius of convergence**, R, as the largest value for which an x chosen in the interval $-R < x < R$ causes the series to converge.

The open interval $(-R, R)$ is known as the **interval of convergence**. Tests for convergence of a power series are the subject of more advanced texts. Further consideration will be given to power series in Chapter 18, but for future reference we give some common expansions now:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad x \text{ in radians}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad x \text{ in radians}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

all of which converge for any value of x.

Each of these series converges rapidly when x is small, and so can be used to obtain useful approximations. In particular, we note that

If x is small and measured in radians

$$\sin x \approx x \qquad \text{and} \qquad \cos x \approx 1 - \frac{x^2}{2!}$$

These formulae are known as the **small-angle approximations**.

EXERCISES 6.5

1 The power series expansion of e^x is given by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and is valid for any x. Take four terms of the series when $x = 0$, 0.1, 0.5 and 1, to compare the sum to four terms with the value of e^x obtained from your calculator. Comment upon the result.

2 Using the power series expansion for $\cos x$:

(a) Write down the power series expansion for $\cos 2x$.

(b) Write down the power series expansion for $\cos(x/2)$.

By considering the power series expansion for $\cos(-x)$ show that $\cos x = \cos(-x)$.

3 By considering the power series expansion of e^x find $\sum_{k=0}^{\infty} 1/k!$.

4 Obtain a cubic approximation to $e^x \sin x$.

5 (a) State the power series expansion for e^{-x}.

(b) By using your solution to (a) and the expansion for e^x, deduce the power series expansions of $\cosh x$ and $\sinh x$.

Solutions

1

x	e^x	Sum to 4 terms
0	1	1
0.1	1.1052	1.1052
0.5	1.6487	1.6458
1	2.7183	2.6667

Values are in close agreement when x is small.

2 (a) $1 - 2x^2 + \frac{2x^4}{3} - \cdots$ (b) $1 - \frac{x^2}{8} + \frac{x^4}{384} - \cdots$

3 e

4 $x + x^2 + \frac{x^3}{3}$

5 (a) $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$

(b) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots,$

$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

6.6 SEQUENCES ARISING FROM THE ITERATIVE SOLUTION OF NON-LINEAR EQUATIONS

It is often necessary to solve equations of the form $f(x) = 0$. For example,

$$f(x) = x^3 - 3x^2 + 7 = 0, \qquad f(x) = \ln x - \frac{1}{x} = 0$$

To **solve** means to find values of x which satisfy the given equation. These values are known as **roots**. For example, the roots of $x^2 - 3x + 2 = 0$ are $x = 1$ and $x = 2$ because when these values are substituted into the equation both sides are equal. Equations where the unknown quantity, x, occurs only to the first power are called **linear equations**. Otherwise an equation is **non-linear**. A simple way of finding the roots of an equation $f(x) = 0$ is to sketch a graph of $y = f(x)$ as shown in Figure 6.8.

The roots are those values of x where the graph cuts or touches the x axis. Generally, there is no analytical way of solving the equation $f(x) = 0$ and so it is often necessary to resort to approximate or **numerical** techniques of solution. An **iterative** technique is one which produces a sequence of approximate solutions which may converge to a root. Iterative techniques can fail in that the sequence produced can diverge. Whether or not this happens depends upon the equation to be solved and the availability of a good estimate of the root. Such an estimate could be obtained by sketching a graph. The technique we shall describe here is known as **simple iteration**. It requires that the equation be rewritten in the form $x = g(x)$. An estimate of the root is made, say x_0, and this value is substituted into the r.h.s. of $x = g(x)$. This yields another estimate, x_1. The process is then repeated. Formally we express this as

$$x_{n+1} = g(x_n)$$

This is a recurrence relation which produces a sequence of estimates $x_0, x_1, x_2, \ldots$. Under certain circumstances the sequence will converge to a root of the equation. It is particularly simple to program this technique on a computer. A check would be built into the program to test whether or not successive estimates are converging.

Example 6.20 Solve the equation $f(x) = \mathrm{e}^{-x} - x = 0$ by simple iteration.

Solution The equation must first be arranged into the form $x = g(x)$, and so we write $\mathrm{e}^{-x} - x = 0$ as

$$x = \mathrm{e}^{-x}$$

In this example we see that $g(x) = \mathrm{e}^{-x}$. The recurrence relation which will produce estimates of the root is

$$x_{n+1} = \mathrm{e}^{-x_n}$$

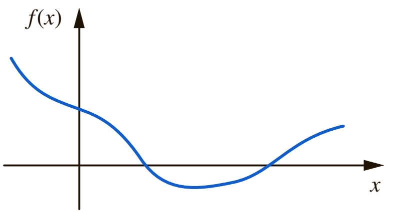

Figure 6.8
A root of $f(x) = 0$ occurs where the graph touches or crosses the x axis.

Table 6.1
Iterative solution of $\mathrm{e}^{-x} - x = 0$.

n	x_n
0	0
1	1
2	0.368
3	0.692
4	0.501
5	0.606
6	0.546
⋮	⋮
⋮	0.567

Suppose we estimate $x_0 = 0$. Then

$$x_1 = e^{-x_0} = e^{-0} = 1$$

Then

$$x_2 = e^{-x_1} = e^{-1} = 0.368$$

The process is continued. The calculation is shown in Table 6.1 from which we see that the sequence eventually converges to 0.567 (3 d.p.). We conclude that $x = 0.567$ is a root of $e^{-x} - x = 0$.

Note that if the equation to be solved involves trigonometric functions, angles will usually be measured in radians and not degrees.

It is possible to devise a test to check whether any given rearrangement will converge. For details of this you should refer to a textbook on numerical analysis. There are other more sophisticated iterative methods for the solution of non-linear equations. One such method, the Newton–Raphson method, is discussed in Chapter 12.

EXERCISES 6.6

1 Show that the quadratic equation $x^2 - 5x - 7 = 0$ can be written in the form $x = \sqrt{7 + 5x}$. With $x_0 = 6$ locate a root of this equation.

2 For the quadratic equation of Question 1 show that an alternative rearrangement is $x = \dfrac{x^2 - 7}{5}$. With $x_0 = 0.6$ find the second solution of this equation.

3 For the quadratic equation of Question 1 show that another rearrangement is $x = \dfrac{7}{x} + 5$. Try to solve the equation using various initial estimates. Investigate further alternative arrangements of the original equation.

4 Show that one recurrence relation for the solution of the equation

$$e^x + 10x - 3 = 0$$

is

$$x_{n+1} = \frac{3 - e^{x_n}}{10}$$

With $x_0 = 0$ locate a root of the given equation.

5 (a) Show that the cubic equation $x^3 + 3x - 5 = 0$ can be written as

(i) $x = \dfrac{5 - x^3}{3}$,

(ii) $x = \dfrac{5}{x^2 + 3}$.

(b) By sketching a graph for values of x between 0 and 3 obtain a rough estimate of the root of the equation given in part (a).

(c) Determine which, if either, of the arrangements in part (a) converges more rapidly to the root.

Solutions

1 6.14

2 −1.14

4 0.18

5 (b) 1.15

(c) Arrangement (ii) converges more rapidly

Technical Computing Exercises 6.6

Write a computer program to implement the simple iteration method. By comparing successive estimates the program should check whether convergence is taking place.

REVIEW EXERCISES 6

1 Write down and graph the first five terms of the sequences $x[k]$ defined by

(a) $x[k] = (-1)^k, \qquad k = 0, 1, 2, 3, \dots$

(b) $x[k] = \dfrac{(-1)^k}{2k+1}, \qquad k = 0, 1, 2, 3, \dots$

2 Find expressions for the kth terms of the sequences whose first five terms are

(a) 1, 9, 17, 25, 33

(b) −1, 1, −1, 1, −1

3 For the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, \dots$$

show that

$$\lim_{k\to\infty} \frac{x[k+1]}{x[k]} = \frac{1}{2}\left(1+\sqrt{5}\right)$$

[*Hint*: write $x[k+1] = x[k] + x[k-1]$, form $x[k+1]/x[k]$ and take limits.]

4 Use the binomial theorem to expand $(1 + x + x^2)^5$ as far as the term in x^3.

5 Use the binomial theorem to expand $\dfrac{1}{(3+x)^3}$ in ascending powers of x as far as the term in x^3.

6 The power series expansion for $\ln(1 + x)$ is given by

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

and is valid for $-1 < x \leqslant 1$. Take a number of values of x in this interval and obtain an approximate value of $\ln(1 + x)$ by means of this series. Compare your answers with the values obtained from your calculator.

7 By multiplying both numerator and denominator of $\dfrac{\sqrt{k+1}-\sqrt{k}}{2}$ by $\sqrt{k+1}+\sqrt{k}$ find

$$\lim_{k\to\infty} \frac{\sqrt{k+1}-\sqrt{k}}{2}$$

8 Find $\displaystyle\lim_{k\to\infty}\left(\frac{3k-1}{2k+7}\right)^3$.

9 Find $\displaystyle\lim_{k\to\infty}\frac{2k^5-3k^2}{7k^7+2k}$.

10 Write down the first eight terms of the series $\sum_{k=1}^{n} k$. By noting that this is an arithmetic series show that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

11 Write down the first six terms of the sequence defined by the recurrence relation

$$x[n+3] = x[n+2] - 2x[n]$$
$$x[0] = 0 \qquad x[1] = 2 \qquad x[2] = 3$$

12 Find the limit, if it exists, as $k \to \infty$ of the geometric progression

$$a, ar, ar^2, \dots, ar^{k-1}, \dots$$

when

(a) $-1 < r < 1$

(b) $r > 1$

(c) $r < -1$

(d) $r = 1$

(e) $r = -1$

13 An arithmetic series has a first term of 4 and the 10th term is 0.

(a) Find S_{20}.

(b) If $S_n = 0$, find n.

14 A geometric series has

$$S_3 = \frac{37}{8} \qquad S_6 = \frac{3367}{512}$$

Find the first term and the common ratio.

15 (a) Write down the series given by $\sum_{r=1}^{5} r^2$.

(b) The sum of the squares of the first n whole numbers can be found from the formula

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

Use this formula to find

(i) $\sum_{r=1}^{5} r^2$, (ii) $\sum_{r=1}^{10} r^2$, (iii) $\sum_{r=6}^{10} r^2$.

16 (a) Write down the series given by $\sum_{r=1}^{6} r^3$.

(b) The sum of the cubes of the first n whole numbers can be found from the formula

$$\sum_{r=1}^{n} r^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Use this formula to find

(i) $\sum_{r=1}^{6} r^3$, (ii) $\sum_{r=1}^{12} r^3$, (iii) $\sum_{r=7}^{12} r^3$.

17 The third term of an arithmetic progression is 18. The fifth term is 28. Find the sum of 20 terms.

18 (a) Find an expression for the general term in the sequence

$$2, 5, 10, 17, \ldots$$

(b) Define the sequence in terms of a recurrence relation.

19 (a) Show that the equation $x^3 + 2x - 14 = 0$ can be rearranged into the form $x = \sqrt[3]{14 - 2x}$. With $x_0 = 2$ use simple iteration to find a root of the equation.

(b) Rearrange the equation $0.8 \sin x - 0.5x = 0$ into the form $x = g(x)$. With $x_0 = 2$ use simple iteration to find a root of the equation.

(c) Rearrange the equation $x^3 = 2e^{-x}$ into the form $x = g(x)$. With $x_0 = 0$ use simple iteration to find a root of the equation.

20 Write out explicitly the first four terms of the series

$$\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}.$$

21 Write out explicitly the first four terms of the series

$$\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m!(m+1)!}.$$

Solutions

1 (a) 1, −1, 1, −1, 1 (b) $1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, \frac{1}{9}$

2 (a) $8k - 7 \quad k \geqslant 1$ (b) $(-1)^k \quad k \geqslant 1$

4 $1 + 5x + 15x^2 + 30x^3$

5 $\frac{1}{27}\left(1 - x + \frac{2x^2}{3} - \frac{10x^3}{27} + \cdots\right)$

7 0

8 $\frac{27}{8}$

9 0

11 0, 2, 3, 3, −1, −7

12 (a) 0 (b) no limit (c) no limit
(d) a (e) no limit

13 (a) $-\frac{40}{9}$ (b) 19

14 $2, \frac{3}{4}$

15 (a) $1 + 4 + 9 + 16 + 25$
(b) (i) 55 (ii) 385 (iii) 330

16 (a) $1 + 8 + 27 + 64 + 125 + 216$
(b) (i) 441 (ii) 6084 (iii) 5643

17 1110

18 (a) $x[k] = k^2 + 1, k = 1, 2, 3, \ldots$
(b) $x[k+1] = x[k] + 2k + 1$

19 (a) 2.13
(b) $x = 1.6 \sin x$, 1.6
(c) $x = \sqrt[3]{2e^{-x}}$, 0.93

20 $1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + \cdots$

21 $\frac{1}{2}x - \frac{1}{16}x^3 + \frac{1}{384}x^5 - \frac{1}{18432}x^7 + \cdots$