1. Matrix operations

1.1 Matrix multiplication

In matrix operations, addition and subtraction are identical to array element by element addition and subtraction. But, matirix multiplication, division, and exponentiation is not.

The matrix product of a row vector u with a column vector w is a scalar that is the sum of the product of the corresponding vector elements.

The number of columns in the 1st array must equal the number of rows in the 2nd array. The size of the resulted array is equal to the number of rows in the 1st array by the number of columns in the 2nd array.

$$\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = u_1w_1 + u_2w_2 + u_3w_3$$

The result is a scalar or 1x1 array

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} w_1u_1 & w_1u_2 & w_1u_3 \\ w_2u_1 & w_2u_2 & w_2u_3 \\ w_3u_1 & w_3u_2 & w_3u_3 \end{bmatrix}$$

The result is a 3x3 array

Example 1: (a) If $u = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ and $w = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$, find the product of uw and wu.

(b) If $A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$, find AB and BA.

MATLAB session:

```
>> u=[2 3 4];
>> w=[3;5;1];
>> u*w
ans =
    25
>> w*u
ans =
     6     9    12
    10    15    20
     2     3     4
```

MATLAB session:

```
>> A=[1;1;1];
>> B=[1:4];
>> A*B
ans =
     1     2     3     4
     1     2     3     4
     1     2     3     4
>> B*A
??? Error using ==> mtimes
Inner matrix dimensions must agree.
```

The error message in the solution of **B*A** appears because the number of columns in B is not the same as the number of rows in A

Now, you can think of matrix as being composed of row vectors. The result of eah row-column multiplication forms an element in the result, which is a column vector.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}$$

Finally, the row-column multiplications can be extended to other columns. The result is column vectors which forms a matrix.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

Example 2: If $A = \begin{bmatrix} 6 & -2 \\ 10 & 3 \\ 4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 8 \\ -5 & 12 \end{bmatrix}$. Find AB.

<u>MATLAB session:</u>

```
>> A=[6 -2;10 3;4 7];
>> B=[9 8;-5 12];
>> A*B
ans =
    64    24
    75   116
     1   116
```

Example 3: The following table shows the hourly cost of four types of manufacturing processes. It also shows the number of hours required by each process to produce three different products. (a) determine the cost of each process to produce one unit of product 1, (b) determine the cost to make one unit of each product, (c) compute the total cost if you produce 10 units of product 1, 5 units of product 2, and 7 units of product 3.

		Hours to produce one unit		
Process	*Hourly cost ($)*	*Product 1*	*Product 2*	*Product 3*
Lathe	10	6	5	4
Grinding	12	2	3	1
Milling	14	3	2	5
Welding	9	4	0	3

<u>MATLAB session:</u>

```
>> hourly_cost=[10 12 14 9]';
>> hours=[6 5 4;2 3 1;3 2 5;4 0 3];
>> a=hourly_cost.*hours(:,1) % take only the first column of the hours matrix
a =
    60
    24
    42
    36

>> b=hourly_cost'*hours
b =
   162   114   149
>> c=sum([10 5 7].*b)
c =
        3233
```

How you can get the same result using matrix multiplication?

1.2 Matrix division

Matrix division uses both the right and left division operators, / and \, to solve and give the solution of sets of linear algebric equations.

1.3 Matrix exponentiation

Raising a matrix to a power is equivalent to repeatedly multiplaying the matrix by its self, for example, $A^2 = AA$. This process requires the matrix to heve the same number of rows and columns (i.e. must be square). MATLAB uses the symbol ^ for matrix exponentiaion. To find A^2, type: **`A^A`** in MATLAB command window after enetring the elements of A.

2. Solution of linear algebric equations

In general, the set of m equations in n unknowns:

$$
\begin{array}{ccccc}
a_{11}x_1 + a_{12}x_2 + & \cdots & + a_{1n}x_n & = b_1 \\
a_{21}x_1 + a_{22}x_2 + & \cdots & + a_{1n}x_n & = b_2 \\
\vdots & \vdots & \vdots & \vdots \\
a_{n1}x_1 + a_{n2}x_2 + & \cdots & + a_{nn}x_n & = b_n
\end{array}
$$

This set of equations can be expressed in matrix notation as: $Ax = b$, where:

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

MATLAB provides the left division method which is based on the Gaussian elemination method for solving the equation set $Ax = b$. To solve such set in MATLAB, simply type: **`x=A\b`** and the result willl be a column vector containing the solution for unknowns x_1, x_2, … and x_n respectively.

Example 1: Solve the following set of equations:

$$
\begin{aligned}
3x_1 + 2x_2 - 9x_3 &= -65 \\
-9x_1 - 5x_2 + 2x_3 &= 16 \\
6x_1 + 7x_2 + 3x_3 &= 5
\end{aligned}
$$

<u>MATLAB session:</u>

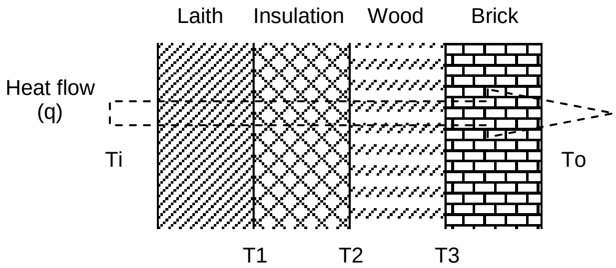
```
>> A=[3 2 -9;-9 -5 2;6 7 3];
>> b=[-65;16;5];
>> x=A\b
x =
    2.0000
   -4.0000
    7.0000
```

The solution is:
$x_1 = 2$
$x_2 = -4$
$x_3 = 7$

Example 2: The thermal resistance for a wall area of 1 m^2 are: R_1=0.036, R_2=4.01, R_3=0.408, and R_4=0.038 °K/W. Suppose that T_i=20 and T_o=-10°C. Find the other three temperatures and the heat loss rate (q). Knowing that:

$$q = \frac{1}{R_1}\left(T_i - T_1\right) = \frac{1}{R_2}\left(T_1 - T_2\right) = \frac{1}{R_3}\left(T_2 - T_3\right) = \frac{1}{R_4}\left(T_3 - T_o\right)$$

<u>Analysis:</u>



The solution gives a set of linear equations:

$$R_1 q + T_1 = T_i$$
$$R_2 q - T_1 + T_2 = 0$$
$$R_3 q - T_2 + T_3 = 0$$
$$R_4 q - T_3 = -T_o$$

Converting the set of equations to matrix notation gives:

$$\begin{bmatrix} R_1 & 1 & 0 & 0 \\ R_2 & -1 & 1 & 0 \\ R_3 & 0 & -1 & 1 \\ R_4 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} q \\ T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} T_i \\ 0 \\ 0 \\ -T_o \end{bmatrix}$$

It is seen that:

$$A = \begin{bmatrix} R_1 & 1 & 0 & 0 \\ R_2 & -1 & 1 & 0 \\ R_3 & 0 & -1 & 1 \\ R_4 & 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} T_i \\ 0 \\ 0 \\ -T_o \end{bmatrix}$$

<u>MATLAB session:</u>

```
>> R=[0.036 4.01 0.408 0.038];
>> Ti=20;To=-10;
>> A=[R(1) 1 0 0 ; R(2) -1 1 0 ; R(3) 0 -1 1 ; R(4) 0 0 -1];
>> b=[Ti;0;0;-To];
>> x=A\b
x =
    6.6785
   19.7596
   -7.0214
   -9.7462
```

The solution is:
q = 6.6785 W/m^2
T_1 = 19.7596°C
T_2 = -7.0214°C
T_3 = -9.7462°C

3. Singular set of equations

To solve any set of equations, the number of equations must equal to the number of unknowns. This is true when the determinant of the coefficient matrix is not equal to zero (i.e. $|A| \neq 0$). If $|A| = 0$, then the number of equations doesn't equal to the number of unknowns.

Example 1: Solve the following set of equations:

$$3x_1 - 4x_2 = 5$$
$$6x_1 - 8x_2 = 10$$

<u>MATLAB session:</u>

```
>> A=[3 -4;6 -8];
>> b=[5;10];
>> x=A\b
Warning: Matrix is singular to working precision.

x =

   NaN
   NaN
>> det(A)
ans =
     0
```

> Use the **det** function to calculate the determinant of a given matrix.

PROBLEMS

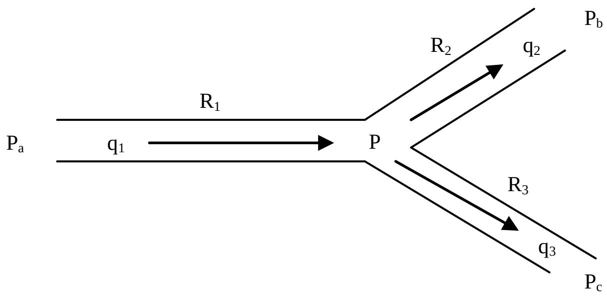
1. Use MATLAB to check the following set of linear equations for singularity. If they are not a singular set, solve them.

$$3x + 12y – 7z =5$$
$$5x – 6y -5z = -8$$
$$-2x + 7y + 9z = 5$$

2. A mixture of benzene and toluene with 50% benzene 50% toluene at 10°C is fed to a vessel in which the mixture is heated to 50°C. The liquid product is 40 mole percent benzene and the vapor product is 68.4 mole percent benzene. Use MATLAB to find the total quantity in the vapor and liquid phase respectively.

3. The following figure shows a network with three pipes. The volume flow rates in the pipes are q_1, q_2, and q_3. The pressures at the pipe ends are p_a, p_b, and p_c. The pressure at the injection is p_1. Thus for the three pipes, we have:

$$q_1 = \frac{1}{R_1}(p_a - p_1)\ ,\ q_2 = \frac{1}{R_2}(p_1 - p_b)\ ,\ q_3 = \frac{1}{R_3}(p_1 - p_c)$$



Where the R_i are the pipe resistances. From conservation of mass, $q_1 = q_2 + q_3$. Use MATLAB to solve the three flow rates q_1, q_2, q_3, and the pressure p_1, given the values of p_a=430, p_b=3600, and p_c=2880 lb/ft^2. The values of resistances are R_1=10,000, R_2=R_3=14,000 lb sec/ft^5.

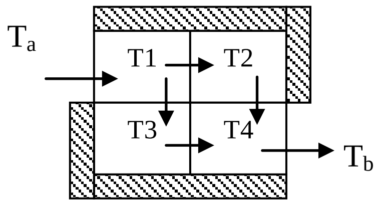
4. The following equation describes Baker's yeast ($C_6H_{10}NO_3$) fermentation:

$$a(C_6H_{12}O_6)+b(O_2)+c(NH_3)\rightarrow C_6H_{10}NO_3+d(H_2O)+e(CO_2)+f(C_2H_6O)$$

Determine the amount of ethanol *f* produced per each mole of yeast given: $R=CO_2/O_2=e/b=1.1$ and that the yeast yield is Y=0.5 (grams of yeast produced per gram of glucose consumed) is related to *a* as follows: $Y=144/180a$ where 144 is the molecular weight of yeast and 180 is the molecular weight of glucose. (*Hint*: apply atomic balance on C, O, N, and H atoms).

5. It is required to find the temperature distribution in the flat square plate shown below. The plate's edges are insulated so that no heat can escape, except at two points where the edge temperature is heated to T_a (150°C) and T_b (20°C) respectively. Let R be the thermal resistance of the material between the centers of adjacent squares and q_{ij} be the heat flow rate between the points whose temperatures are T_i and T_j. Apply the principles of conservation of energy to find the equations which describe the system then solve then using MATLAB to find the temperature distribution in the adjacent subsquares.

$$q_{ij}=\frac{\Delta T_{ij}}{R}$$



6. A company must purchase five kinds of material. The following table gives the price the company pays per ton of each material, along with the number of ton purchased in months of May, June, and July. Use MATLAB to calculate the total spent in each month.

		Quantity purchased (tons)		
Material	Price ($/ton)	May	June	July
A	300	5	4	6
B	550	3	2	4
C	400	6	5	3
D	250	3	5	4
E	500	2	4	3

7. The following table shows the composition of five commonly used aluminum alloys, which are known by their alloy numbers. Compute the amounts of raw materials needed to produce a given amount (in tons) of each alloy.

	Composition of alluminum alloys				
Alloy No.	%Cu	%Mg	%Mn	%Si	%Zn
4024	4.4	1.5	0.6	0	0
6061	0	1	0	0.6	0
7005	0	1.4	0	0	4.5
7075	1.6	2.5	0	0	5.6
356.0	0	0.3	0	7	0