STRENGTH OF MATERIALS

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Reference

Mechanics of Materials   **By: E. J. HEARN**

CH.1/ SIMPLE STRESS AND STRAIN

1. Load (P) (N)

In any engineering structure or mechanism the individual components will be subjected to external forces arising from the service conditions or environment in which the component works.

$$\Sigma P_X = 0 \qquad ,\Sigma P_Y = 0 \qquad ,\Sigma M_o = 0$$

If a cylindrical bar is subjected to a direct pull or push along its axis as shown in Figure (1), then it is said to be subjected to *tension* or *compression.*
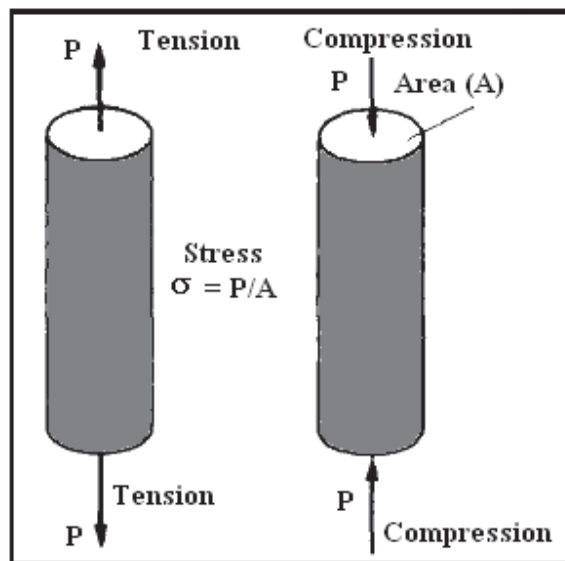

Figure (1) Types of direct stress (Tension or Compression)

In the SI system of units load is measured in newtons, loads appear in SI multiples, i.e. kilonewtons (kN) or meganewtons (MN). There are a number of different ways in which load can be applied to a member. Typical loading types are:

(a) Static or dead loads, i.e. non-fluctuating loads, generally caused by gravity effects.
(b) Liue loads, as produced by, for example, lorries crossing a bridge.
(c) Impact or shock loads caused by sudden blows.
(d) Fatigue, fluctuating or alternating loads.

2. Direct or normal stress (σ), (N/m^2)

A bar is subjected to a uniform tension or compression, i.e. a direct force, which is uniformly or equally applied across the cross section, then the internal forces set up are also distributed uniformly and the bar is said to be subjected to a uniform **direct or normal stress,** the stress being defined as

$$Stress = \frac{Load}{area} = \frac{P}{A}$$

Stress (σ) may thus be (i) compressive stress or (ii) tensile stress depending on the nature of the load and will be measured in units of (N/m^2).

3. Direct strain *(ε)*

Figure (2) show a bar is subjected to a direct load, and hence a stress, the bar will change in length. If the bar has an original length L and changes in length by an amount *δL,* the *strain* produced is defined as follows:

$$\text{strain } (\varepsilon) = \frac{\text{change in length}}{\text{original length}} = \frac{\delta L}{L}$$

Strain is thus a measure of the deformation of the material and is non-dimensional,
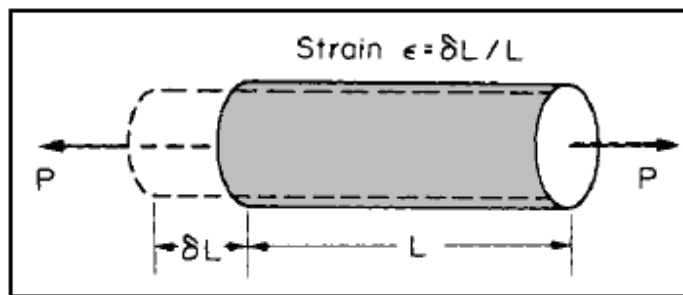

Figure (2)

Alternatively, strain can be expressed as a percentage strain

$$\text{strain } (\varepsilon) = \frac{\delta L}{L} \times 100\%$$

4. Sign convention for direct stress and strain

Tensile stresses and strains are considered **POSITIVE** in sense producing an *increase* in length. Compressive stresses and strains are considered **NEGATIVE** in sense producing a *decrease* in length.

5. Elastic materials - Hooke's law (E), (N/m^2)

A material is said to be *elastic* if it returns to its original, unloaded dimensions when load is removed. A particular form of elasticity which applies to a large range of engineering materials, at least over part of their load range, produces deformations which are proportional to the loads producing them. stress is proportional to strain. *Hooke's law,* in its simplest form*, therefore states that

$$\text{stress } (\sigma) \propto \text{strain } (\varepsilon)$$

$$\frac{\text{stress}}{\text{strain}} = \text{constant*}$$

Other classifications of materials with which the reader should be acquainted are as follows:

A material which has a uniform structure throughout without any flaws or discontinuities is termed a ***homogeneous*** material. ***Non-homogeneous*** or ***inhomogeneous*** materials such as concrete and poor-quality cast iron will thus have a structure which varies from point to point depending on its constituents and the presence of casting flaws or impurities.

If a material exhibits uniform properties throughout in all directions it is said to be ***isotropic;*** conversely one which does not exhibit this uniform behaviour is said to be ***nonisotropic*** or ***anisotropic.***

An ***orthotropic*** material is one which has different properties in different planes. A typical example of such a material is wood, although some composites which contain systematically orientated "***inhomogeneities***" may also be considered to fall into this category.

6. Modulus of elasticity - Young's modulus (E), (N/m^2)

Within the elastic limits of materials, i.e. within the limits in which Hooke's law applies, it has been shown that:

$$\frac{stress}{strain} = constant$$

This constant is given the symbol E and termed the ***modulus of elasticity*** or ***Young's modulus***, Thus

$$E = \frac{Stress}{Strain} = \frac{\sigma}{\varepsilon} \qquad .....(1)$$

$$E = \frac{P.L}{A.\delta L} \qquad .....(2)$$

Young's modulus E is generally assumed to be the same in tension or compression and for most engineering materials has a high numerical value. Typically, E **=** 200 **x** l0^9 N/m^2 for steel.

$$\varepsilon = \frac{\sigma}{E} \qquad ......(3)$$

In most common engineering applications strains do not often exceed 0.003 or 0.3 % so that the assumption used later in the text that deformations are small in relation to original dimensions is generally well founded.

7. Tensile test

The standard tensile test in which a circular bar of uniform cross-section is subjected to a gradually increasing tensile load until failure occurs. Measurements of the change in length of a selected *gauge length* of the bar are recorded throughout the loading operation by means of extensometers and a graph of load against extension or stress against strain is produced as shown in Fig. (3); this shows a typical result for a test on a mild (low carbon) steel bar; other materials will exhibit different graphs but of a similar general form see Figures (5) to(7).
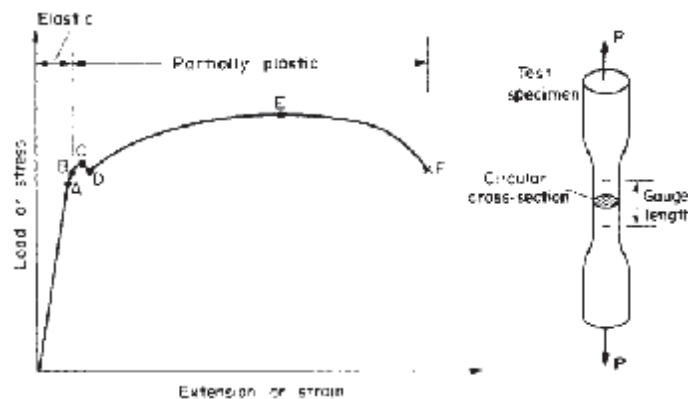

Figure (3) Typical tensile test curve for mild steel.

For the first part of the test it will be observed that Hooke's law is obeyed, the material behaves elastically and stress is proportional to strain, giving the straight-line graph indicated. Some point A is eventually reached, however, when the linear nature of the graph ceases and this point is termed the *limit* of *proportionality.*

C, termed the *upper yield point*

D, the *lower yield point*

That stress which, when removed, produces a permanent strain or "set" of 0.1 % of the original gauge length-see Fig. (4a).
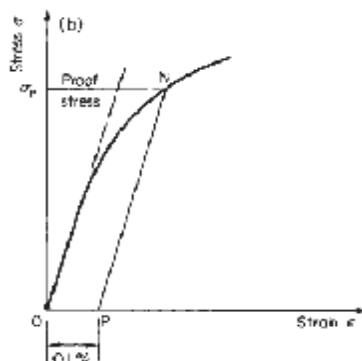

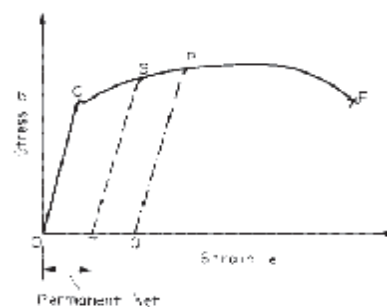**Figure (4a) Determination of 0.1 % proof stress.**

Figure (4b) Permanent deformation or "set" after straining beyond the yield point.

Typical stress-strain curves resulting from tensile tests on other engineering materials are shown in Figs. (5) to (7).
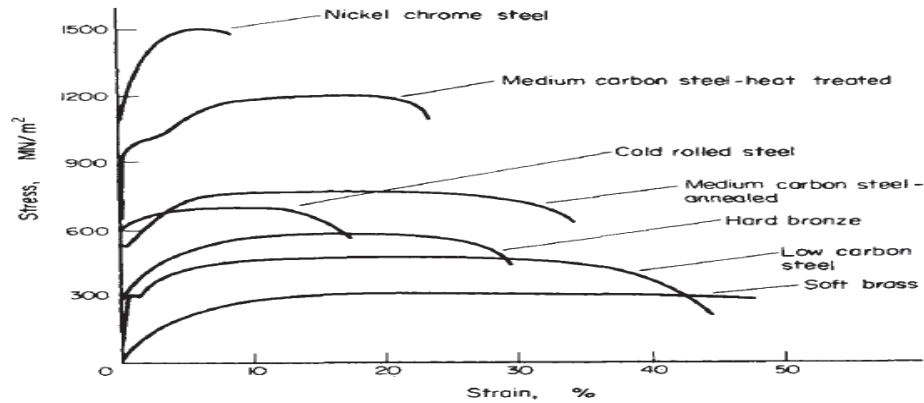

Figure (5)Tensile test curves for various metals.
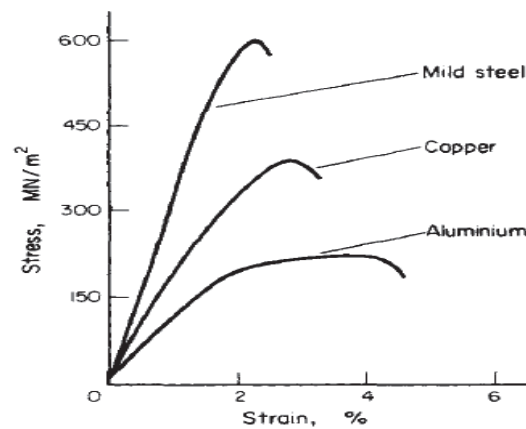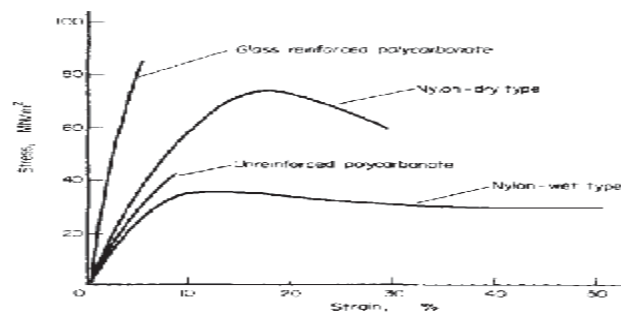

Figure (6) Typical stress - strain curves for hard drawn wire material-note large reduction in strain values from those of Figure (5)

Figure(7) Typical tension test results for various types of nylon and polycarbonate.

8. Ductile materials

It has been observed above that the partially plastic range of the graph of Figure (3) covers a much wider part of the strain axis than does the elastic range. Thus the extension of the material over this range is considerably in excess of that associated with elastic loading. The capacity of a material to allow these large extensions, i.e. the ability to be drawn out plastically, is termed its *ductility.* Materials with high ductility are termed *ductile* materials, members with low ductility are termed *brittle* materials. **A** quantitative value of the ductility is obtained by measurements of the *percentage elongation* or *percentage reduction in area,* both
being defined below.

$$\text{Percentage elongation} = \frac{\text{increase in gauge length to fracture}}{\text{original gauge length}} \times 100$$

$$\text{Percentage reduction in area} = \frac{\text{reduction in cross-sectional area of necked portion}}{\text{original area}} \times 100$$

9. Brittle materials

A brittle material is one which exhibits relatively small extensions to fracture so that the partially plastic region of the tensile test graph is much reduced (Fig. **8).** Whilst Fig. (3) referred to a low carbon steel, Fig. **(8)** could well refer to a much higher strength steel with a higher carbon content. There is little or no necking at fracture for brittle materials.
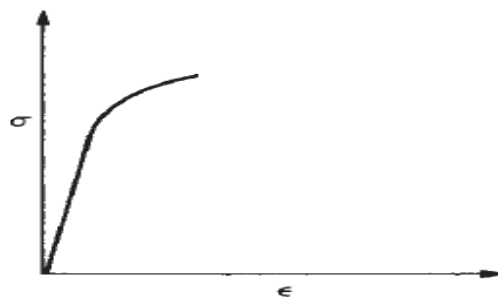
Figure(8) Typical tensile test curve for a brittle material

10. Poisson's ratio (ν)

Consider the rectangular bar of Figure (9) subjected to a tensile load. Under the action of this load the bar will increase in length by an amount δL giving a longitudinal strain in the bar .
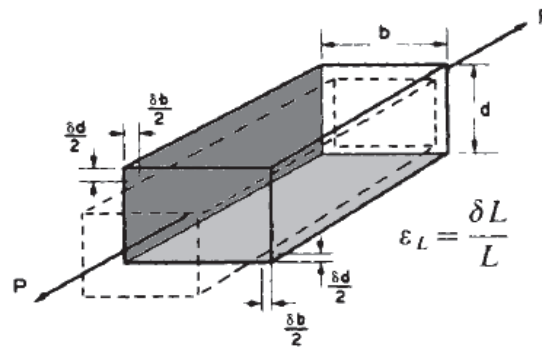
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$$\varepsilon_L = \frac{\delta L}{L}$$

Figure (9)

The bar will also exhibit, however, a ***reduction*** in dimensions laterally, i.e. its breadth and depth will both reduce. The associated lateral strains will both be equal, will be of opposite sense to the longitudinal strain, and will be given by

$$\varepsilon_{lat} = -\frac{\delta b}{b} = -\frac{\delta d}{d}$$

Provided the load on the material is retained within the elastic range the ratio of the lateral and longitudinal strains will always be constant. This ratio is termed ***Poisson's ratio.***

$$\text{Poisson's ratio } (\nu) = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{(-\delta d/d)}{\delta L/L} \quad \dots (4)$$

The negative sign of the lateral strain is normally ignored to leave Poisson's ratio simply as a ratio of strain magnitudes. It must be remembered, however, that the longitudinal strain induces a lateral strain of opposite sign. For most engineering materials the value of v lies between 0.25 and 0.33.
Since

$$\text{longitudinal strain} = \frac{\text{longitudinal stress}}{\text{Young's modulus}} = \frac{\sigma}{E} \quad \dots (4a)$$

$$\text{lateral strain} = \nu \frac{\sigma}{E} \quad \dots (4b)$$

11. Application of Poisson's ratio to a two-dimensional stress system

A two-dimensional stress system is one in which all the stresses lie within one plane such as the X-Y plane as shown in figure (10).
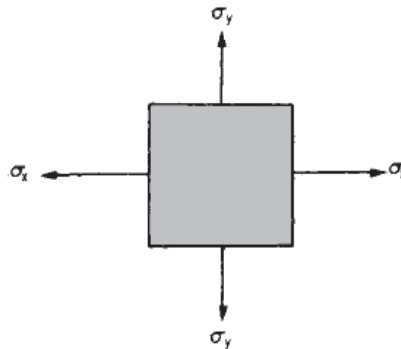
Figure (10) Simple two-dimensional system of direct stresses.

The following strains will be produced

(a) in the X direction resulting from $\varepsilon_x = \sigma_x /E$

(b) in the Y direction resulting from $\varepsilon_y = \sigma_y/E$.

(c) in the X direction resulting from $\varepsilon_y = - v(\sigma_y /E)$,

(d) in the Y direction resulting from $\varepsilon_x = - v(\sigma_x /E)$.

strains (c) and (d) being the so-called ***Poisson's ratio strain,*** opposite in sign to the applied strains, i.e. compressive.

The total strain in the X direction will therefore be given by:

$$\varepsilon_x = \frac{\sigma_x}{E} - v\frac{\sigma_y}{E} = \frac{1}{E}(\sigma_x - v\sigma_y) \qquad \qquad ..... (5)$$

and the total strain in the Y direction will be:

$$\varepsilon_y = \frac{\sigma_y}{E} - v\frac{\sigma_x}{E} = \frac{1}{E}(\sigma_y - v\sigma_x) \qquad \qquad ..... (6)$$

If any stress is, in fact, compressive its value must be substituted in the above equations together with a negative sign following the normal sign convention.

12. Shear stress(τ) , (N/m^2)

Consider a block or portion of material as shown in Figure (11) subjected to a set of **equal** and opposite forces Q. (Such a system could be realised in a bicycle brake block when contacted with the wheel.) then a ***shear stress τ*** is set up, defined as follows:

$$\text{shear stress}(\tau) = \frac{\text{shear load}}{\text{area resisting shear}} = \frac{Q}{A}$$

This shear stress will always be ***tangential*** to the area on which it acts; direct stresses, however, are always ***normal*** to the area on which they act.
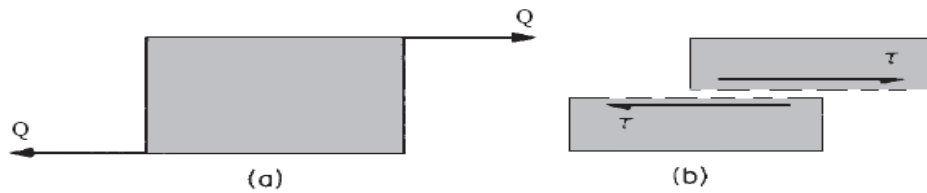
Figure(11) Shear force and resulting shear stress system showing typical form of failure by relative sliding of planes.

13. Shear strain (γ)

If one again considers the block of Figure (11a)to be a bicycle brake block it is clear that the rectangular shape of the block will not be retained as the brake is applied and the shear forces introduced. The block will in fact change shape or "strain" into the form shown in Figure (12) The angle of deformation y is then termed the ***shear strain.***
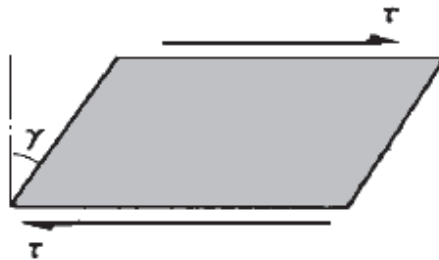Shear strain is measured in radians and hence is non-dimensional, i.e. it has no units.

Figure (12) Deformation (shear strain) produced by shear stresses.

14. Modulus of rigidity (G), (N/m^2)

For materials within the elastic range the shear strain is proportional to the shear stress producing it, The constant G is termed the ***modulus of rigidity*** or ***shear modulus*** and is directly comparable to the modulus of elasticity used in the direct stress application.

$$\frac{\text{shear stress}}{\text{shear strain}} = \frac{\tau}{\gamma} = \text{constant} = G \qquad \text{..... (7)}$$

15. Double shear

Consider the simple riveted lap joint shown in Figure (13a) When load is applied to the plates the rivet is subjected to shear forces tending to shear it on one plane as indicated. In the butt joint with two cover plates of Figure (13b), however, each rivet is subjected to possible shearing on two faces, i.e. ***double shear.*** In such cases twice the area of metal is resisting the applied forces so that the shear stress set up is given by

$$\text{shear stress } \tau \text{ (in double shear)} = \frac{P}{2A} \qquad \text{..... (8)}$$
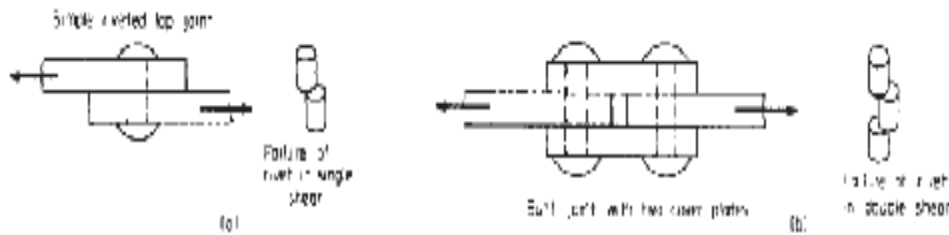
Figure (13) (a) Single shear. (b) Double shear.

16. Allowable working stress-factor of safety

The most suitable strength or stiffness criterion for any structural element or component is normally some maximum stress or deformation which must not be exceeded. In the case of stresses the value is generally known as the *maximum allowable working stress.* Because of uncertainties of loading conditions, design procedures, production methods, etc., designers generally introduce a *factor of safety* into their designs, defined as follows

:

$$\text{factor of safety} = \frac{\text{maximum stress}}{\text{allowable working stress}} \quad \dots \dots (9)$$

$$\text{factor of safety} = \frac{\text{yield stress (or proof stress)}}{\text{allowable working stress}}$$

$$\text{load factor} = \frac{\text{load at failure}}{\text{allowable working load}} \quad \dots \dots (10)$$

18. Temperature stresses

When the temperature of a component is increased or decreased the material respectively expands or contracts. If this expansion or contraction is not resisted in any way then the processes take place free of stress. If, however, the changes in dimensions are restricted then stresses termed *temperature stresses* will be set up within the material.

Consider a bar of material with a linear coefficient of expansion α. Let the original length of the bar be L and let the temperature *increase* be t. If the bar is free to expand the change in length would be given by

$$\Delta L = L\alpha t \quad \dots \dots (11)$$

and the new length

$$L' = L + L\alpha t = L(1 + \alpha t)$$

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$$\varepsilon = \frac{\Delta L}{L} = \frac{L\alpha t}{L(1 + \alpha t)}$$

$$\varepsilon = \frac{L\alpha t}{L} = \alpha t$$

But $$\frac{\sigma}{\varepsilon} = E$$

$$\therefore \quad \text{stress } \sigma = E\varepsilon = E\alpha t$$

Examples

Example 1

Determine the stress in each section of the bar shown in Figure (14) when subjected to an axial tensile load of 20 kN. The central section is 30 mm square cross-section; the other portions are of circular section, their diameters being indicated. What will be the total extension of the bar? For the bar material E = 210GN/m².
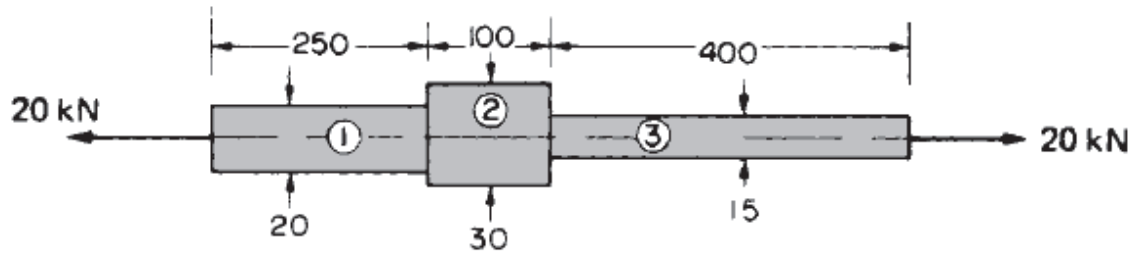

Figure (14) all dimensions mm

Solution

$$\text{Stress} = \frac{\text{force}}{\text{area}} = \frac{P}{A}$$

$$\text{Stress in section (1)} = \frac{20 \times 10^3}{\frac{\pi(20 \times 10^{-3})^2}{4}} = \frac{80 \times 10^3}{\pi \times 400 \times 10^{-6}} = 63.66 \text{ MN/m}^2$$

$$\text{Stress in section (2)} = \frac{20 \times 10^3}{30 \times 30 \times 10^{-6}} = 22.2 \text{ MN/m}^2$$

$$\text{Stress in section (3)} = \frac{20 \times 10^3}{\frac{\pi(15 \times 10^{-3})^2}{4}} = \frac{80 \times 10^3}{\pi \times 225 \times 10^{-6}} = 113.2 \text{ MN/m}^2$$

Now the extension of a bar can always be written in terms of the stress in the bar since

$$E = \frac{\text{stress}}{\text{strain}} = \frac{\sigma}{\delta/L}$$

i.e.
$$\delta = \frac{\sigma L}{E}$$

$$\therefore \quad \text{extension of section (1)} = 63.66 \times 10^6 \times \frac{250 \times 10^{-3}}{210 \times 10^9} = 75.8 \times 10^{-6} \text{ m}$$

$$\text{extension of section (2)} = 22.2 \times 10^6 \times \frac{100 \times 10^{-3}}{210 \times 10^9} = 10.6 \times 10^{-6} \text{ m}$$

$$\text{extension of section (3)} = 113.2 \times 10^6 \times \frac{400 \times 10^{-3}}{210 \times 10^9} = 215.6 \times 10^{-6} \text{ m}$$

$$\therefore \quad \text{total extension} = (75.8 + 10.6 + 215.6)10^{-6}$$
$$= 302 \times 10^{-6} \text{ m}$$
$$= 0.302 \text{ mm}$$

Example 2

(a) A 25 mm diameter bar is subjected to an axial tensile load of 100 kN. Under the action of this load a **200mm** gauge length is found to extend 0.19 **x** 10^{-3} mm. Determine the modulus of elasticity for the bar material.

(b) If, in order to reduce weight whilst keeping the external diameter constant, the bar is bored axially to produce a cylinder of uniform thickness, what is the maximum diameter of bore possible given that the maximum allowable stress is 240MN/m^2? The load can be assumed to remain constant at 100 kN.

(c) What will be the change in the outside diameter of the bar under the limiting stress quoted in (b)? (E = **210GN/m2** and **v** = 0.3).

Solution

(a) From eqn. (1.2),

$$\text{Young's modulus } E = \frac{PL}{A\,\delta L}$$

$$\simeq \frac{100 \times 10^3 \times 200 \times 10^{-3}}{\frac{\pi(25 \times 10^{-3})^2}{4} \times 0.19 \times 10^{-3}}$$

$$= 214\,\text{GN/m}^2$$

(b) Let the required bore diameter be d mm; the cross-sectional area of the bar will then be reduced to

$$A = \left[\frac{\pi \times 25^2}{4} - \frac{\pi d^2}{4}\right]10^{-6} = \frac{\pi}{4}(25^2 - d^2)10^{-6}\,\text{m}^2$$

$\therefore$ $\quad$ stress in bar $= \dfrac{P}{A} = \dfrac{4 \times 100 \times 10^3}{\pi(25^2 - d^2)10^{-6}}$

But this stress is restricted to a maximum allowable value of $240\,\text{MN/m}^2$.

$\therefore$ $\qquad 240 \times 10^6 = \dfrac{4 \times 100 \times 10^3}{\pi(25^2 - d^2)10^{-6}}$

$\therefore$ $\qquad 25^2 - d^2 = \dfrac{4 \times 100 \times 10^3}{240 \times 10^6 \times \pi \times 10^{-6}} = 530.5$

$\therefore$ $\qquad d^2 = 94.48 \quad \text{and} \quad d = 9.72\,\text{mm}$

The maximum bore possible is thus **9.72 mm**.

(c) The change in the outside diameter of the bar will be obtained from the lateral strain,

i.e. $\qquad\qquad$ lateral strain $= \dfrac{\delta d}{d}$

But $\qquad$ Poisson's ratio $v = \dfrac{\text{lateral strain}}{\text{longitudinal strain}}$

and $\qquad$ longitudinal strain $= \dfrac{\sigma}{E} = \dfrac{240 \times 10^6}{210 \times 10^9}$

$\therefore \qquad\qquad \dfrac{\delta d}{d} = -v\dfrac{\sigma}{E} = -\dfrac{0.3 \times 240 \times 10^6}{210 \times 10^9}$

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$\therefore$ change in outside diameter $= -\dfrac{0.3 \times 240 \times 10^6}{210 \times 10^9} \times 25 \times 10^{-3}$

$$= -8.57 \times 10^{-6}\,\text{m} \ (\text{a reduction})$$

Example 3

The coupling shown in Figure (15) is constructed from steel of rectangular cross-section and is designed to transmit a tensile force of 50 kN. If the bolt is of **15** mm diameter calculate:

(a) the shear stress in the bolt;
(b) the direct stress in the plate;
(c) the direct stress in the forked end of the coupling.
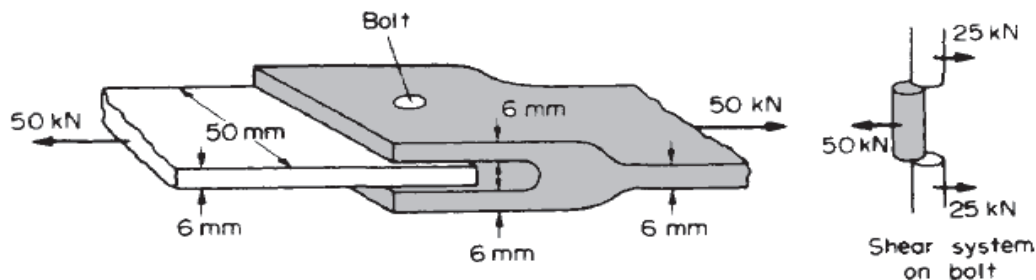
Figure (15)

Solution

(a) The bolt is subjected to double shear, tending to shear it as shown in Fig. 1.14b. There is thus twice the area of the bolt resisting the shear and from eqn. (1.8)

$$\text{shear stress in bolt} = \frac{P}{2A} = \frac{50 \times 10^3 \times 4}{2 \times \pi (15 \times 10^{-3})^2}$$

$$= \frac{100 \times 10^3}{\pi (15 \times 10^{-3})^2} = 141.5\,\text{MN/m}^2$$

(b) The plate will be subjected to a direct tensile stress given by

$$\sigma = \frac{P}{A} = \frac{50 \times 10^3}{50 \times 6 \times 10^{-6}} = 166.7\,\text{MN/m}^2$$

(c) The force in the coupling is shared by the forked end pieces, each being subjected to a direct stress

$$\sigma = \frac{P}{A} = \frac{25 \times 10^3}{50 \times 6 \times 10^{-6}} = 83.3\,\text{MN/m}^2$$

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Example 4

Derive an expression for the total extension of the tapered bar of circular cross-section shown in Figure (16) when it is subjected to an axial tensile load W.
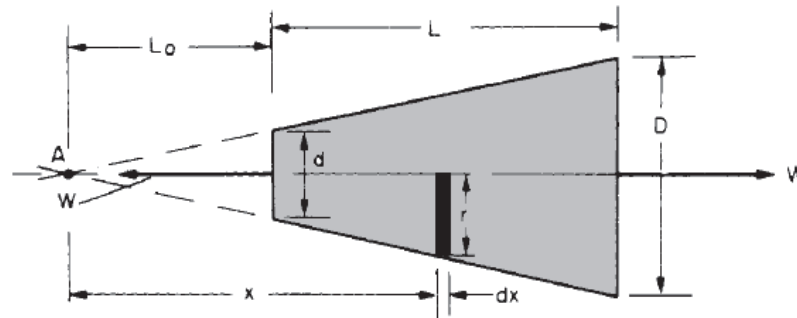
Figure (16)

Solution

From the proportions of Fig. 1.21,

$$\frac{d/2}{L_0} = \frac{(D-d)/2}{L}$$

$$\therefore \qquad L_0 = \frac{d}{(D-d)} L$$

Consider an element of thickness dx and radius r, distance x from the point of taper A.

$$\text{Stress on the element} = \frac{W}{\pi r^2}$$

But

$$\frac{r}{x} = \frac{d}{2L_0}$$

$$\therefore \qquad r = d \left(\frac{D-d}{2dL} \right) x = \frac{x(D-d)}{2L}$$

$$\therefore \qquad \text{stress on the element} = \frac{4WL^2}{\pi(D-d)^2 x^2}$$

$$\therefore \qquad \text{strain on the element} = \frac{\sigma}{E}$$

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and extension of the element $= \dfrac{\sigma \, dx}{E}$

$$= \dfrac{4WL^2}{\pi(D-d)^2 x^2 E} \, dx$$

$\therefore$ total extension of bar $= \displaystyle\int_{L_0}^{L_0+L} \dfrac{4WL^2}{\pi(D-d)^2 E} \dfrac{dx}{x^2}$

$$= \dfrac{4WL^2}{\pi(D-d)^2 E} \left[-\dfrac{1}{x} \right]_{L_0}^{L_0+L}$$

$$= \dfrac{4WL^2}{\pi(D-d)^2 E} \left[-\dfrac{1}{(L_0+L)} - \left(-\dfrac{1}{L_0} \right) \right]$$

But $\qquad\qquad L_0 = \dfrac{d}{(D-d)} L$

$\therefore \qquad L_0 + L = \dfrac{d}{(D-d)} L + L = \dfrac{(d+D-d)}{D-d} L = \dfrac{DL}{(D-d)}$

$\therefore$ total extension

$$= \dfrac{4WL^2}{\pi(D-d)^2 E} \left[-\dfrac{(D-d)}{DL} + \dfrac{(D-d)}{dL} \right] = \dfrac{4WL}{\pi(D-d)E} \left[\dfrac{(-d+D)}{Dd} \right]$$

$$= \dfrac{4WL}{\pi DdE}$$

Example 5
The following figures were obtained in a standard tensile test on a specimen of low carbon steel:

       diameter of specimen, 11.28 mm;

       gauge length, 56mm;

       minimum diameter after fracture, **6.45** mm.

Using the above information and the table of results below, produce:

(1) a load/extension graph over the complete test range;

(2) a load/extension graph to an enlarged scale over the elastic range of the specimen.

Load (kN)	2.47	4.97	7.4	9.86	12.33	14.8	17.27	19.74	22.2	24.7
Extension (m × 10⁻⁶)	5.6	11.9	18.2	24.5	31.5	38.5	45.5	52.5	59.5	66.5
Load (kN)	27.13	29.6	32.1	33.3	31.2	32	31.5	32	32.2	34.5
Extension (m × 10⁻⁶)	73.5	81.2	89.6	112	224	448	672	840	1120	1680
Load (kN)	35.8	37	38.7	39.5	40	39.6	35.7	28		
Extension (m × 10⁻⁶)	1960	2520	3640	5600	7840	11200	13440	14560		

Using the two graphs and other information supplied, determine the values of
(a) Young's modulus of elasticity;
(b) the ultimate tensile stress;
(c) the stress at the upper and lower yield points;
(d) the percentage reduction of **area;**
(e) the percentage elongation;
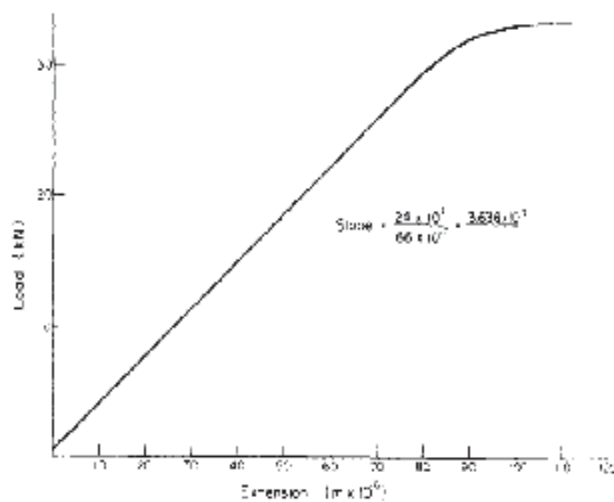(f) the nominal and actual stress at fracture.

Solution

Figure (17) Load-extension graph for elastic range.

(a)

$$\text{Young's modulus } E = \frac{\sigma}{\varepsilon} = \frac{\text{load}}{\text{area}} \times \frac{\text{gauge length}}{\text{extension}}$$

$$= \frac{\text{load}}{\text{extension}} \times \frac{\text{gauge length}}{\text{area}}$$

$$E = \text{slope of graph} \times \frac{L}{A} = 3.636 \times 10^8 \times \frac{56 \times 10^{-3}}{100 \times 10^{-6}}$$

$$= 203.6 \times 10^9 \text{ N/m}^2$$

$$\therefore \qquad E = 203.6 \text{ GN/m}^2$$

(b) $$\text{Ultimate tensile stress} = \frac{\text{maximum load}}{\text{cross-section area}} = \frac{40.2 \times 10^3}{100 \times 10^{-6}} = 402 \text{ MN/m}^2$$

(c) $$\text{Upper yield stress} = \frac{33.3 \times 10^3}{100 \times 10^{-6}} = 333 \text{ MN/m}^2$$

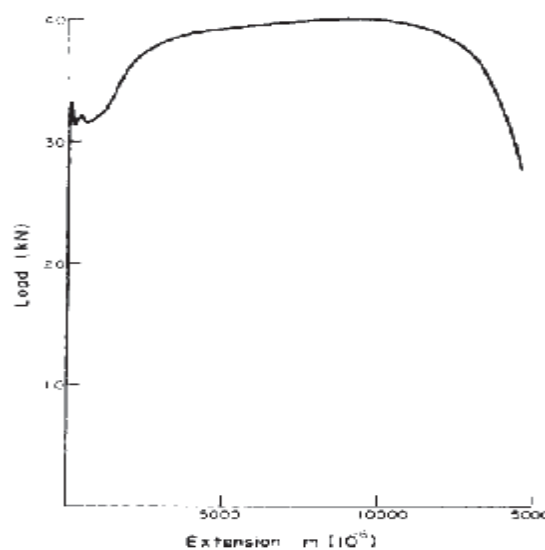$$\text{Lower yield stress} = \frac{31.2 \times 10^3}{100 \times 10^{-6}} = 312 \text{ MN/m}^2$$

Figure (18) Load-extension graph for complete load

(d)   Percentage reduction of area $= \dfrac{\left(\dfrac{\pi}{4}D^2 - \dfrac{\pi}{4}d^2\right)}{\dfrac{\pi}{4}D^2} \times 100$

$$= \dfrac{(D^2 - d^2)}{D^2} \times 100$$

$$= \dfrac{(11.28^2 - 6.45^2)}{11.28^2} = 67.3\%$$

(e)   Percentage elongation $= \dfrac{(70.56 - 56)}{56} \times 100$

$$= 26\%$$

(f)   Nominal stress at fracture $= \dfrac{28 \times 10^3}{100 \times 10^{-6}} = 280 \text{ MN/m}^2$

Actual stress at fracture $= \dfrac{28 \times 10^3}{\dfrac{\pi}{4}(6.45)^2 \times 10^{-6}} = 856.9 \text{ MN/m}^2$

Problems

1. (A). A 25mm square cross-section bar of length 300mm carries an axial compressive load of 50kN. Determine the stress **set** up in the bar and its change of length when the load is applied. For the bar material E = 200 GN/m^2.   [80 MN/m^2; 0.12mm]

2. (A). A steel tube, **25** mm outside diameter and 12mm inside diameter, cames an axial tensile load of 40 kN. What will be the stress in the bar? What further increase in load is possible if the stress in the bar is limited **to** 225 MN/m^2?   [l06 MN/m^2; 45 kN]

3. (A). Define the terms shear stress and shear strain, illustrating your answer by means of a simple sketch. Two circular bars, one of brass and the other of steel, are to be loaded by a shear load of 30 kN. Determine the necessary diameter of the bars (a) in single shear, (b) in double shear, if the shear stress in the two materials must not exceed 50 MN/m^2 and 100 MN/m^2 respectively.   [27.6, 19.5, 19.5, 13.8mm]

4. (A). Two forkend pieces are to be joined together by a single steel pin of 25mm diameter and they are required to transmit 50 kN. Determine the minimum cross-sectional area of material required in one branch of either fork if the stress in the fork material is not to exceed 180 MN/m^2. What will be the maximum shear stress in the pin?

[1.39 **x** 10^{-4}m^2; 50.9MN/m^2.]

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5. (A). A simple turnbuckle arrangement is constructed from a 40 mm outside diameter tube threaded internally at each end to take two rods of 25 mm outside diameter with threaded ends. What will be the nominal stresses set up in the tube and the rods, ignoring thread depth, when the turnbuckle cames an axial load of 30 kN? Assuming a sufficient strength of thread, what maximum load can be transmitted by the turnbuckle if the maximum stress is limited to 180 MN/m^2?

[39.2, 61.1 MN/m2, 88.4 kN]

6. (A). A bar ABCD consists of three sections: AB is 25 mm square and 50 mm long, BC is of 20 mm diameter and40 mm long and CD is of 12 mm diameter and 50 mm long. Determine the stress set up in each section of the bar when it is subjected to an axial tensile load of 20 kN. What will be the total extension of the bar under this load? For the bar material, E = 210GN/m^2.   [32,63.7, 176.8 MN/m^2, 0.062mm]

7.(A). A steel bar ABCD consists of three sections: AB is of 20mm diameter and 200 mm long, BC is 25 mm square and 400 mm long, and CD is of 12 mm diameter and 200mm long. The bar is subjected to an axial compressive load which induces a stress of 30 MN/m^2 on the largest cross-section. Determine the total decrease in the length of the bar when the load is applied. For steel E = 210GN/m^2.

[0.272 mm.]

8. Figure (19) shows a special spanner used to tighten screwed components. A torque is applied at the tommy -bar and is transmitted to the pins which engage into holes located into the end of a screwed component.
 (a) Using the data given in Figure (19) calculate:
 (i) the diameter D of the shank if the shear stress is not to exceed 50N/mm^2,
 (ii) the stress due to bending in the tommy-bar,
 (iii) the shear stress in the pins.
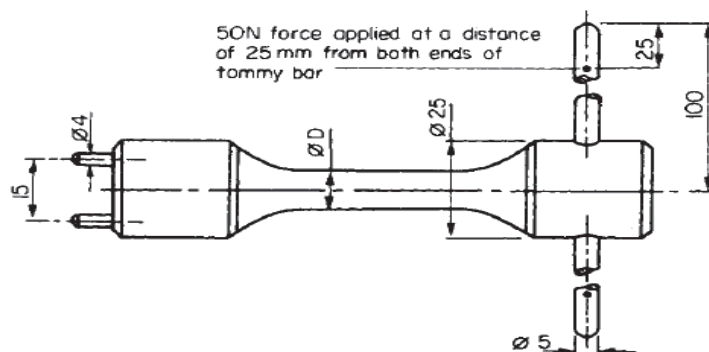
[9.14mm; 254.6 MN/m^2; 39.8 MN/m^2.]

Figure (19)

CH.2/ COMPOUND BARS

1. Compound bars subjected to external load

In certain applications it is necessary to use a combination of elements or bars made from different materials. In overhead electric cables, for example, it is often convenient to carry the current in a set of copper wires surrounding steel wires, the latter being designed to support the weight of the cable over large spans. Such combinations of materials are generally termed **compound** burs. This chapter is concerned with compound bars which are symmetrically proportioned such that no bending results, when an external load is applied to such a compound bar it is shared between the individual component materials in proportions depending on their respective lengths, areas and Young's moduli. A compound bar consisting of n members, each having a different length and cross-sectional area and each being of a different material as shown in figure (1)
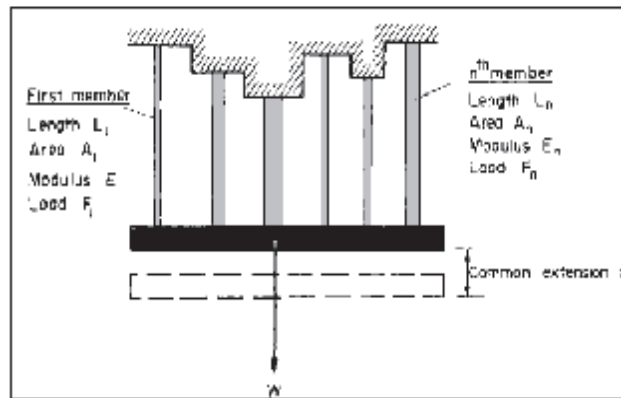
Figure (1) Compound bar formed of different materials

For the nth member,

$$\frac{stress}{strain} = E_n = \frac{F_n.L_n}{A_n.x_n}$$

$$F_n = \frac{E_n.A_n.x}{L_n} \qquad ....(1)$$

where F, is the force in the nth member ,A its cross-sectional area and L_n are its length. The total load carried will be the sum of all such loads for all the members

$$W = \sum \frac{E_n.A_n.x}{L_n} = x.\sum \frac{E_n.A_n}{L_n} \qquad ...(2)$$

Now from equation (1) the force in member 1 is given by

$$F_1 = \frac{E_1.A_1.x}{L_1}$$

But, from equation (2),

$$x = \frac{W}{\sum \dfrac{E_n.A_n}{L_n}}$$

$$F_1 = \frac{\dfrac{E_1.A_1}{L_1}}{\sum \dfrac{E.A}{L}} \; W \quad .....(3)$$

i.e. each member carries a portion of the total load W proportional to its EAIL value. If the wires are all of equal length the above equation reduces to

$$F_1 = \frac{E_1.A_1}{\sum E.A} \; W \quad ....(4)$$

The stress in member 1 is then given by

$$\sigma_1 = \frac{F_1}{A_1} \qquad ....(5)$$

2. Compound bars - "equivalent" or "combined" modulus

In order to determine the common extension of a compound bar it is convenient to consider it as a single bar of an imaginary material with an *equivalent* or *combined* modulus $E_,$. Here it is necessary to assume that both the extension and the original lengths of the individual members of the compound bar are the same; the strains in all members will then be equal.

Now total load on compound bar $= F_1 + F_2 + F_3 + . . . + F,$ where $F_1, F_2,$ etc., are the loads in members 1, 2, etc.

But

$$\text{force} = \text{stress x area}$$

$$\sigma(A_1 + A_2 + ... + A_n) = \sigma_1 A_1 + \sigma_2 A_2 + ... + \sigma_n A_n$$

Where: σ is the stress in the equivalent single bar. Dividing through by the common strain $\varepsilon,$

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$$\frac{\sigma}{\varepsilon}(A_1 + A_2 + ... + A_n) = \frac{\sigma_1}{\varepsilon}A_1 + \frac{\sigma_2}{\varepsilon}A_2 + ... + \frac{\sigma_n}{\varepsilon}A_n$$

$$E_c(A_1 + A_2 + ... + A_n) = E_1 A_1 + E_2 A_2 + ... + E_n A_n$$

where E_c, is the ***equivalent*** or ***combined E*** of the single bar.

$$combined \quad E = \frac{E_1.A_1 + E_2.A_2 + ... + E_n.A_n}{A_1 + A_2 + ... + A_n}$$

$$E_c = \frac{\sum E.A}{\sum A} \qquad ....(6)$$

With an external load W applied,

$$Stress\ in\ the\ equivalent\ bar = \frac{W}{\sum A}$$

$$Strain\ in\ the\ equivalent\ bar = \frac{W}{E_c.\sum A} = \frac{x}{L}$$

$$common\ extension \quad x = \frac{W.L}{E_c.\sum A} \quad ....(7)$$

$$=extension\ of\ single\ bar$$

3. Compound bars subjected to temperature change

When a material is subjected to a change in temperature its length will change by an amount

$$\alpha.L.\Delta T$$

where α is the coefficient of linear expansion for the material, L is the original length and ΔT the temperature change. (An increase in temperature produces an increase in length and a decrease in temperature a decrease in length except in very special cases of materials with zero or negative coefficients of expansion which need not be considered here.) If, however, the free expansion of the material is prevented by some external force, then a stress is set up in the material. This stress is equal in magnitude to that which would be produced in the bar by initially allowing the free change of length and then applying sufficient force to return the bar to its original length.
Now:

$$Change\ in\quad Length = \alpha.L.\Delta T$$

$$Strain = \frac{\alpha.L.\Delta T}{L} = \alpha.\Delta T$$

Therefore, the stress created in the material by the application of sufficient force to remove this strain

$$= strain\ x\ E = E.\alpha.\Delta T$$

Consider now a compound bar constructed from two different materials rigidly joined together as shown in Figure (2) and Figure (3a). For simplicity of description consider that the materials in this case are steel and brass.
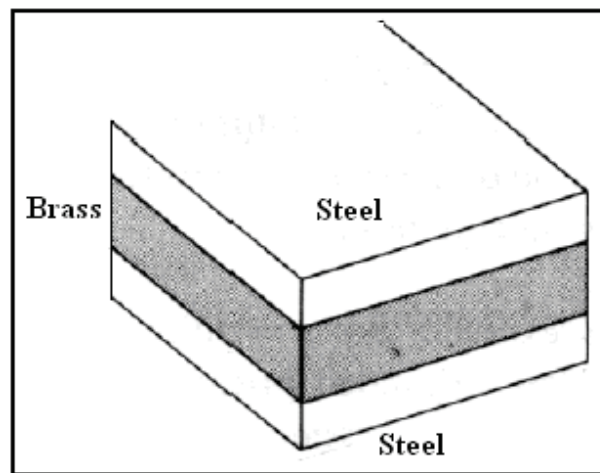
Figure (2)

In general, the coefficients of expansion of the two materials forming the compound bar will be different so that as the temperature rises each material will attempt to expand by different amounts. Figure (3b) shows the positions to which the individual materials will extend if they are completely free to expand (i.e. not joined rigidly together as a compound bar). The extension of any length L is given by $\alpha.L.\Delta T$
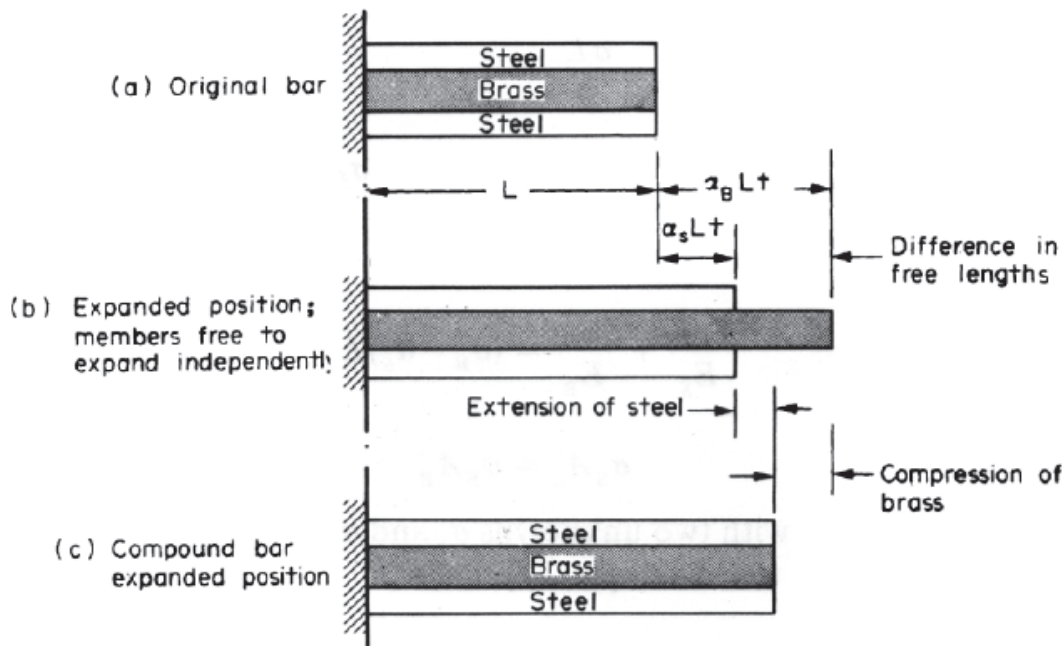
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Figure (3) Thermal expansion of compound bar.

Thus the difference of "free" expansion lengths or so-called free lengths

$$= \alpha_B.L.\Delta T - \alpha_s.L.\Delta T = (\alpha_B - \alpha_s).L.\Delta T$$

since in this case the coefficient of expansion of the brass (α_B) is greater than that for the steel (α_s). The initial lengths L of the two materials are assumed equal. If the two materials are now rigidly joined as a compound bar and subjected to the same temperature rise, each material will attempt to expand to its free length position but each will be affected by the movement of the other, The higher coefficient of expansion material (brass) will therefore seek to pull the steel up to its free length position and conversely the lower position. In practice a compromise is reached, the compound bar extending to the position shown in Figure (3c), resulting in an effective compression of the brass from its free length position and an effective extension of the steel from its free length position. From the diagram it will be seen that the following rule holds.

Rule 1
(Extension of steel + compression of brass = difference in "free" lengths).
Referring to the bars in their free expanded positions the rule may be written as

(Extension of "short" member + compression of"1ong" member = difference in free lengths).

Applying Newton's law of equal action and reaction the following second rule also applies.

Rule 2

The tensile force applied to the short member by the long member is equal in magnitude to the compressive force applied to the long member by the short member.
Thus, in this case,

tensile force in steel **=** compressive force in brass

Now, from the definition of Young's modulus

$$E = \frac{stress}{strain} = \frac{\sigma}{\Delta L / L}$$

where **ΔL** is the change in length.

$$\Delta L = \frac{\sigma.L}{E}$$

Also,

force **=** stress **x** area **=** $\sigma.A$

where: A is the cross-sectional area, Therefore Rule 1 becomes

$$\frac{\sigma_s.L}{E_s} + \frac{\sigma_B.L}{E_B} = (\alpha_B - \alpha_s).L.\Delta T \quad ....(8)$$

and Rule 2 becomes

$$\sigma_s.A_s = \sigma_B.A_B \quad\quad\quad\quad ....(9)$$

4. Compound bar (tube and rod)

Consider now the case of a hollow tube with washers or endplates at each end and a central threaded rod as shown in Figure (4) At first sight there would seem to be no connection with the work of the previous section, yet, in fact, the method of solution to determine the stresses set up in the tube and rod when one nut is tightened.
The compound bar which is formed after assembly of the tube and rod, i.e. with the nuts tightened, is shown in Figure (**4c**), the rod being in a state of tension and the tube in compression. Once again Rule 2 applies, i.e.

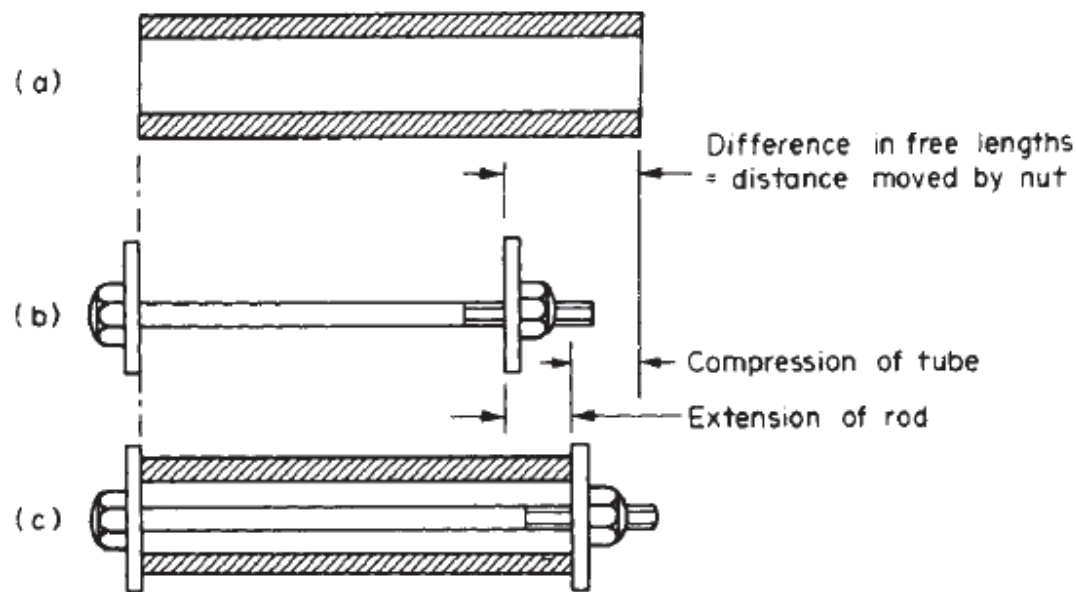**compressive force in tube = tensile force in rod**

Figure (4)

Figure **(4a)** and b show, *diagrammatically,* the effective positions of the tube and rod before the nut is tightened and the two components are combined. As the nut is turned there is a simultaneous compression of the tube and tension of the rod leading to the final state shown in Figure (4c) **.** As before, however, the diagram shows that Rule **1** applies:

compression of tube +extension of rod **=** difference in free lengths
$$= \text{axial advance of nut}$$

i.e. the axial movement of the nut (**=** number of turns n x threads per metre) is taken up by combined compression of the tube and extension of the rod.
Thus, with suffix (t) for tube and (R) for rod,

$$\frac{\sigma_t L}{E_t} + \frac{\sigma_R L}{E_R} = n \times \textbf{threads/metre} \qquad \text{...... (10)}$$

$$\sigma_R A_R = \sigma_t A_t \qquad \text{....... (11)}$$

If the tube and rod are now subjected to a change of temperature they may be treated as a normal compound bar and Rules **1** and **2** again apply
Figure (5),

$$\frac{\sigma'_t L}{E_t} + \frac{\sigma'_R L}{E_R} = (\alpha_t - \alpha_R)Lt \qquad \text{...... (12)}$$
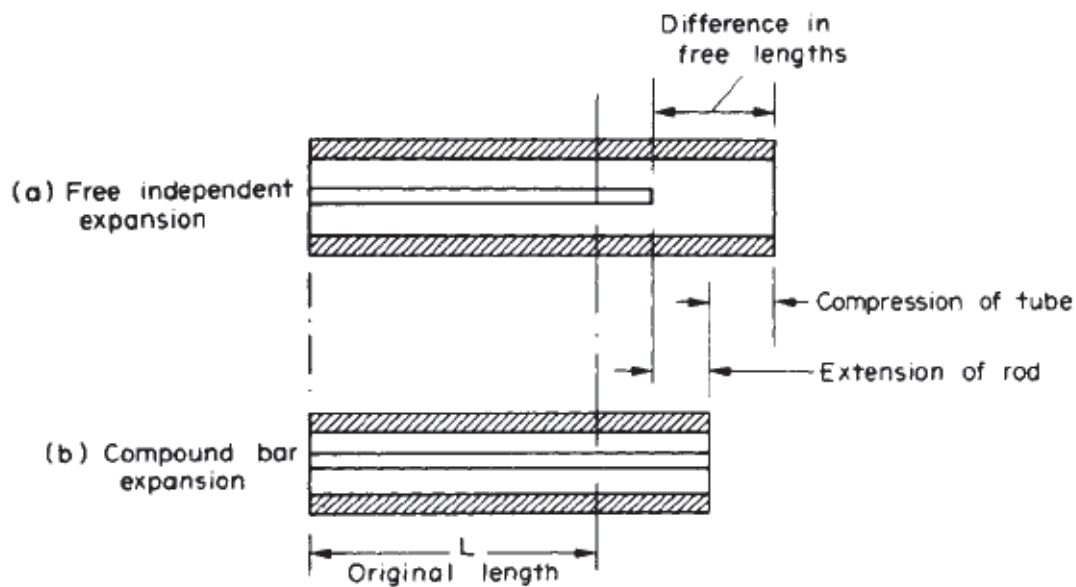
Figure (5)

Where (σ'_R) ; and (σ'_R) ; are the stresses in the tube and rod due to temperature change only and (α_t), is assumed greater than (α_R). If the latter is not the case the two terms inside the final bracket should be interchanged.
Also

$$\sigma'_R A_R = \sigma'_t A_t$$

Examples

Example 1
(a) A compound bar consists of four brass wires of 2.5 mm diameter and one steel wire of 1.5 mm diameter. Determine the stresses in each of the wires when the bar supports a load of 500 N. Assume all of the wires are of equal lengths.
(b) Calculate the "equivalent" or "combined modulus for the compound bar and determine its total extension if it is initially 0.75 m long. Hence check the values of the stresses obtained in part (a).
For brass $E = 100$ GN/m^2 and for steel $E = 200$ GN/m^2.

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Solution

(a) the force in the steel wire is given by

$$F_s = \frac{E_s A_s}{\Sigma EA} W$$

$$= \left[\frac{200 \times 10^9 \times \frac{\pi}{4} \times 1.5^2 \times 10^{-6}}{200 \times 10^9 \times \frac{\pi}{4} \times 1.5^2 \times 10^{-6} + 4(100 \times 10^9 \times \frac{\pi}{4} \times 2.5^2 \times 10^{-6})} \right] 500$$

$$= \left[\frac{2 \times 1.5^2}{(2 \times 1.5^2) + (4 \times 2.5^2)} \right] 500 = 76.27 \text{ N}$$

$$\therefore \qquad \text{total force in brass wires} = 500 - 76.27 = 423.73 \text{ N}$$

$$\therefore \qquad \text{stress in steel} = \frac{\text{load}}{\text{area}} = \frac{76.27}{\frac{\pi}{4} \times 1.5^2 \times 10^{-6}} = \mathbf{43.2 \text{ MN/m}^2}$$

$$\text{and} \qquad \text{stress in brass} = \frac{\text{load}}{\text{area}} = \frac{423.73}{4 \times \frac{\pi}{4} \times 2.5^2 \times 10^{-6}} = \mathbf{21.6 \text{ MN/m}^2}$$

(b) From eqn. (2.6)

$$\text{combined } E = \frac{\Sigma EA}{\Sigma A} = \frac{200 \times 10^9 \times \frac{\pi}{4} \times 1.5^2 \times 10^{-6} + 4(100 \times 10^9 \times \frac{\pi}{4} \times 2.5^2 \times 10^{-6})}{\frac{\pi}{4}(1.5^2 + 4 \times 2.5^2)10^{-6}}$$

$$= \frac{(200 \times 1.5^2 + 400 \times 2.5^2)}{(1.5^2 + 4 \times 2.5^2)} 10^9 = \mathbf{108.26 \text{ GN/m}^2}$$

Now $\qquad E = \dfrac{\text{stress}}{\text{strain}}$

and the stress in the equivalent bar

$$= \frac{500}{\Sigma A} = \frac{500}{\frac{\pi}{4}(1.5^2 + 4 \times 2.5^2)10^{-6}} = 23.36 \text{ MN/m}^2$$

$$\therefore \qquad \text{strain in the equivalent bar} = \frac{\text{stress}}{E} = \frac{23.36 \times 10^6}{108.26 \times 10^9} = 0.216 \times 10^{-3}$$

$$\therefore \qquad \text{common extension} = \text{strain} \times \text{original length}$$

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$$= 0.216 \times 10^{-3} \times 0.75 = 0.162 \times 10^{-3}$$
$$= \mathbf{0.162\ mm}$$

This is also the extension of any single bar, giving a strain in any bar

$$= \frac{0.162 \times 10^{-3}}{0.75} = 0.216 \times 10^{-3} \text{ as above}$$

$\therefore$ $\quad$ stress in steel $=$ strain $\times E_s = 0.216 \times 10^{-3} \times 200 \times 10^9$
$$= \mathbf{43.2\ MN/m^2}$$

and $\quad$ stress in brass $=$ strain $\times E_B = 0.216 \times 10^{-3} \times 100 \times 10^9$
$$= \mathbf{21.6\ MN/m^2}$$

These are the same values as obtained in part (a).

Example 2

(a) A compound bar is constructed from three bars 50 mm wide by $\quad$ 12 mm thick fastened together to form a bar 50 mm wide by 36 mm thick. The middle bar is of aluminium alloy for which E = 70 GN/m^2 and the outside bars are of brass with E = 100 GN/m^2. If the bars are initially fastened at 18°C and the temperature of the whole assembly is then raised to 50°C, determine the stresses set up in the brass and the aluminium.

$$\alpha_B = 18 \text{ x per } °C \text{ and } \alpha_A = 22 \text{ x per } °C$$

(b) What will be the changes in these stresses if an external compressive load of 15 kN is applied to the compound bar at the higher temperature?

Solution

With any problem of this type it is convenient to let the stress in one of the component members or materials, e.g. the brass, be **x.**
Then, since

$$\text{force in brass} = \text{force in aluminium}$$

and

$$\text{force} = \text{stress} \times \text{area}$$

$$x \times 2 \times 50 \times 12 \times 10^{-6} = \sigma_A \times 50 \times 12 \times 10^{-6}$$

i.e.

$$\text{stress in aluminium } \sigma_A = 2x$$

Now, from eqn. (2.8),

$$\text{extension of brass} + \text{compression of aluminium} = \text{difference in free lengths}$$

$$= (\alpha_A - \alpha_B)(T_2 - T_1)L$$

$$\frac{xL}{100 \times 10^9} + \frac{2xL}{70 \times 10^9} = (22 - 18)10^{-6}(50 - 18)L$$

$$\frac{(7x + 20x)}{700 \times 10^9} = 4 \times 10^{-6} \times 32$$

$$27x = 4 \times 10^{-6} \times 32 \times 700 \times 10^9$$

$$x = 3.32 \text{ MN/m}^2$$

The stress in the brass is thus **3.32 MN/m²** (tensile) and the stress in the aluminium is $2 \times 3.32 = $ **6.64 MN/m²** (compressive).

(b) With an external load of 15 kN applied each member will take a proportion of the total load given

$$\text{Force in aluminium} = \frac{E_A A_A}{\Sigma EA} W$$

$$= \left[\frac{70 \times 10^9 \times 50 \times 12 \times 10^{-6}}{(70 \times 50 \times 12 + 2 \times 100 \times 50 \times 12)10^9 \times 10^{-6}} \right] 15 \times 10^3$$

$$= \left[\frac{70}{(70 + 200)} \right] 15 \times 10^3$$

$$= 3.89 \text{ kN}$$

$$\text{force in brass} = 15 - 3.89 = 11.11 \text{ kN}$$

$$\text{stress in brass} = \frac{\text{load}}{\text{area}} = \frac{11.11 \times 10^3}{2 \times 50 \times 12 \times 10^{-6}}$$

$$= \textbf{9.26 MN/m}^2 \textbf{ (compressive)}$$

These stresses represent the changes in the stresses owing to the applied load. The total or resultant stresses owing to combined applied loading plus temperature effects are, therefore,

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$$\text{stress in aluminium} = -6.64 - 6.5 = -13.14 \text{ MN/m}^2$$
$$= 13.14 \text{ MN/m}^2 \text{ (compressive)}$$
$$\text{stress in brass} = +3.32 - 9.26 = -5.94 \text{ MN/m}^2$$
$$= 5.94 \text{ MN/m}^2 \text{ (compressive)}$$

Example 3

A **25** mm diameter steel rod passes concentrically through a bronze tube 400 mm long, 50 mm external diameter and **40** mm internal diameter. The ends of the steel rod are threaded and provided with nuts and washers which are adjusted initially so that there is no end play at **20°C.**

(a) Assuming that there is no change in the thickness of the washers, find the stress produced in the steel and bronze when one of the nuts is tightened by giving it one tenth of a turn, the pitch of the thread being **2.5** mm.

(b) If the temperature of the steel and bronze is then raised to **50°C** find the changes that will occur in the stresses in both materials.

The coefficient of linear expansion per $^{\circ}$C is 11 x 10^{-6} for steel and for bronze and 18 x10^{-6}. E for steel = **200** GN/m^2. E for bronze = **100** GN/m^2.

Solution

(a) Let **x** be the stress in the tube resulting from the tightening of the nut and σ_R the stress in the rod

Then, from eqn. (11),

$$\text{force (stress} \times \text{area) in tube} = \text{force (stress} \times \text{area) in rod}$$

$$x \times \tfrac{\pi}{4}(50^2 - 40^2)10^{-6} = \sigma_R \times \tfrac{\pi}{4} \times 25^2 \times 10^{-6}$$

$$\sigma_R = \frac{(50^2 - 40^2)}{25^2} x = 1.44x$$

And since compression of tube + extension of rod = axial advance of nut, from eqn. (10),

$$\frac{x \times 400 \times 10^{-3}}{100 \times 10^9} + \frac{1.44x \times 400 \times 10^{-3}}{200 \times 10^9} = \frac{1}{10} \times 2.5 \times 10^{-3}$$

$$400 \cdot \frac{(2x + 1.44x)}{200 \times 10^9} 10^{-3} = 2.5 \times 10^{-4}$$

$$\therefore \qquad\qquad 6.88x = 2.5 \times 10^8$$

$$x = 36.3 \text{ MN/m}^2$$

The stress in the tube is thus **36.3 MN/m²** (**compressive**) and the stress in the rod is $1.44 \times 36.3 = $ **52.3 MN/m²** (**tensile**).

(b) Let p be the stress in the tube resulting from temperature change. The relationship between the stresses in the tube and the rod will remain as in part (a) so that the stress in the rod is then $1.44p$. In this case, if free expansion were allowed in the independent members, the bronze tube would expand more than the steel rod and from eqn. (8)

$$\text{compression of tube} + \text{extension of rod} = \text{difference in free length}$$

$\therefore$
$$\frac{pL}{100 \times 10^9} + \frac{1.44pL}{200 \times 10^9} = (\alpha_B - \alpha_S)(T_2 - T_1)L$$

$$\frac{(2p + 1.44p)}{200 \times 10^9} = (18 - 11)10^{-6}(50 - 20)$$

$$3.44p = 7 \times 10^{-6} \times 30 \times 200 \times 10^9$$

$$p = 12.21 \text{ MN/m}^2$$

and
$$1.44p = 17.6 \text{ MN/m}^2$$

The changes in the stresses resulting from the temperature effects are thus 12.2 MN/m² (compressive) in the tube and 17.6 MN/m² (tensile) in the rod.

The final, resultant, stresses are thus:

$$\text{stress in tube} = -36.3 - 12.2 = \textbf{48.5 MN/m}^2 \textbf{ (compressive)}$$

$$\text{stress in rod} = 52.3 + 17.6 = \textbf{69.9 MN/m}^2 \textbf{ (tensile)}$$

Problems

1 (A). A power transmission cable consists of ten copper wires each of 1.6 mm diameter surrounding three steel wires each of 3 mm diameter. Determine the combined E for the compound cable and hence determine the extension of a 30 m length of the cable when it is being laid with a tension of 2 kN.
For steel, $E = 200 \text{ GN/m}^2$; for copper, $E = 100 \text{ GN/m}^2$.   [151.3 GN/m²; 9.6 mm.]

2 (A). If the maximum stress allowed in the copper of the cable of problem 1 is 60 MN/m², determine the maximum tension which the cable can support.   [3.75 kN.]

3 (A). What will be the stress induced in a steel bar when it is heated from 15°C to 60°C, all expansion being prevented?
For mild steel, $E = 210 \text{ GN/m}^2$ and $\alpha = 11 \times 10^{-6}$ per °C.   [104 MN/m²°.]

4 (A). A 75 mm diameter compound bar is constructed by shrinking a circular brass bush onto the outside of a 50 mm diameter solid steel rod. If the compound bar is then subjected to an axial compressive load of 160 kN determine the load carried by the steel rod and the brass bush and the compressive stress set up in each material.
For steel, $E = 210 \text{ GN/m}^2$; for brass, $E = 100 \text{ GN/m}^2$.   [I. Struct. E.] [100.3, 59.7 kN; 51.1, 24.3 MN/m².]

# CH.3/ SHEARING FORCE AND BENDING MOMENT DIAGRAMS

DEFINITION OF A BEAM

A bar subject to forces or couples that lie in a plane containing the longitudinal axis of the bar is called a *beam*. The forces are understood to act perpendicular to the longitudinal axis.

1. Types of Beams

CANTILEVER BEAMS

If a beam is supported at only one end and in such a manner that the axis of the beam cannot rotate at that point, it is called a *cantilever beam*. This type of beam is illustrated in Fig. 1. The left end of the bar is free to deflect but the right end is rigidly clamped. The right end is usually said to be "restrained." The reaction of the supporting wall at the right upon the beam consists of a vertical force together with a couple acting in the plane of the applied loads shown.

Figure (1)
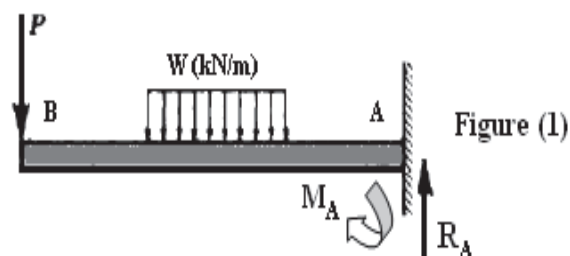
SIMPLE BEAMS

A beam that is freely supported at both ends is called a *simple beam*. The term "freely supported" implies that the end supports are capable of exerting only forces upon the bar and are not capable of exerting any moments. Thus there is no restraint offered to the angular rotation of the ends of the bar at the supports as the bar deflects under the loads. Two simple beams are sketched in Fig. 2.
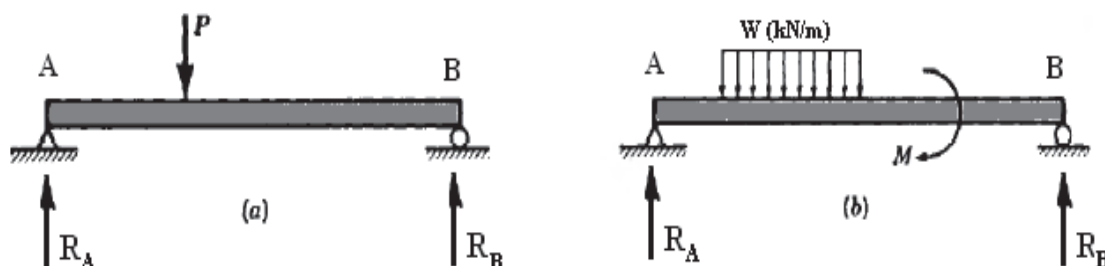

Figure (2)

OVERHANGING BEAMS

A beam freely supported at two points and having one or both ends extending beyond these supports is termed an *overhanging beam*. Two examples are given in Fig. 3.
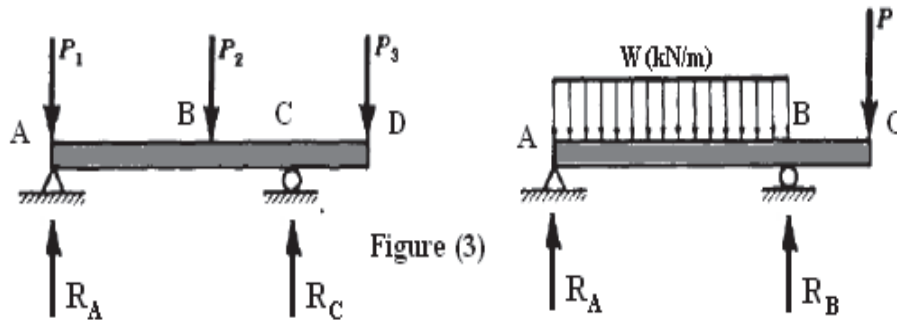

Figure (3)

2. Shearing force and bending moment

At every section in a beam carrying transverse loads there will be resultant forces on either side of the section which, for equilibrium, must be equal and opposite, and whose combined action tends to shear the section in one of the two ways shown in Figure (4 **a** and **b**). The shearing force (S.F.) at the section is defined therefore as the algebraic sum of the forces taken on one side of the section.

2.1. Shearing force (S.F.) sign convention

Forces upwards to the left of a section or downwards to the right of the section are positive. Thus Figure (4a) shows a positive S.F. system at X-X and Figure (4b) shows a negative S.F. system.
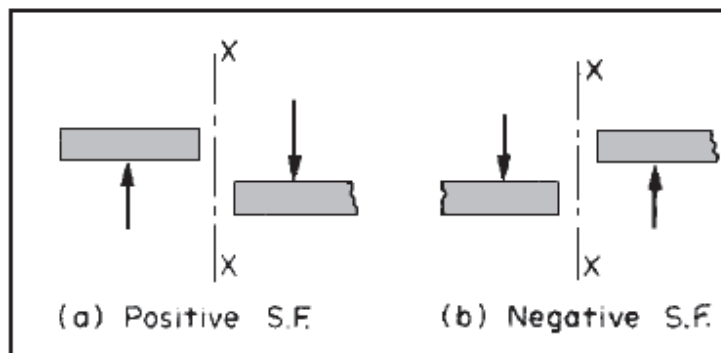

(a) Positive S.F.   (b) Negative S.F.

Figure (4) S.F. sign convention

2.2. Bending moment (B.M.) sign convention

Clockwise moments to the left and counterclockwise to the right are positive. Thus Figure (5a) shows a positive bending moment system resulting in sagging of the beam at X-X and Figure (5b) illustrates a negative B.M. system with its associated hogging beam.
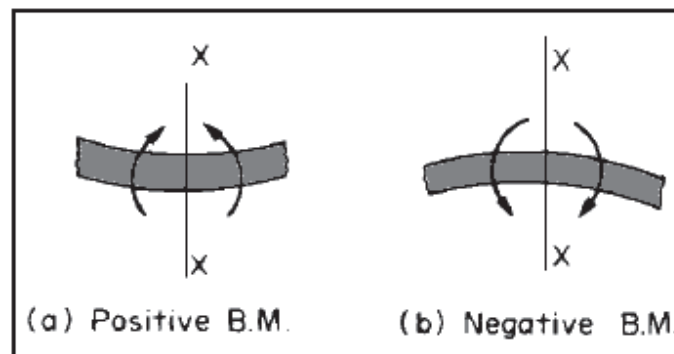
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Figure (5) B.M. sign convention.

It should be noted that whilst the above sign conventions for S.F. and B.M. are somewhat arbitrary and could be completely reversed, the systems chosen here are the only ones which yield the mathematically correct signs for slopes and deflections of beams in subsequent work and therefore are highly recommended.
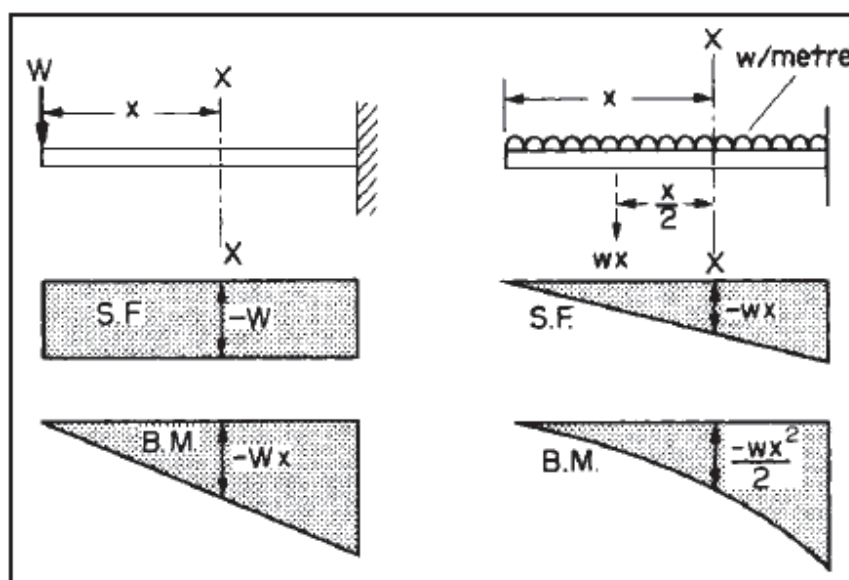

Figure (6) S.F.-B.M. diagrams for standard cases.

Thus in the case of a cantilever carrying a concentrated load (**W**) at the end Figure (6), the S.F. at any section X-X, distance x from the free end, is S.F. **= -** W. This will be true whatever the value of x, and so the S.F. diagram becomes a rectangle. The **B.M.** at the same section X-X is- **W.x** and this will increase linearly with **x.** The **B.M.** diagram is therefore a triangle. If the cantilever now carries a uniformly distributed load, the S.F. at X-X is the net load to one side of X-X, i.e. -wx. In this case, therefore, the S.F. diagram becomes triangular, increasing to a maximum value of - **WL** at the support. The **B.M.** at X-X is obtained by treating the load to the left of X-X as a concentrated load of the same value acting at the centre of gravity,

$$\text{B.M. at } X{-}X = -wx\frac{x}{2} = -\frac{wx^2}{2}$$

Plotted against x this produces the parabolic **B.M.** diagram shown.

3. S.F. and B.M. diagrams for beams carrying concentrated loads only

In order to illustrate the procedure to be adopted for the determination of S.F. and **B.M.** values for more complicated load conditions, consider the simply supported beam shown in Figure (4) carrying concentrated loads only. (The term *simply* supported means that the beam can be assumed to rest on knife-edges or roller supports and is free to bend at the supports without any restraint.)
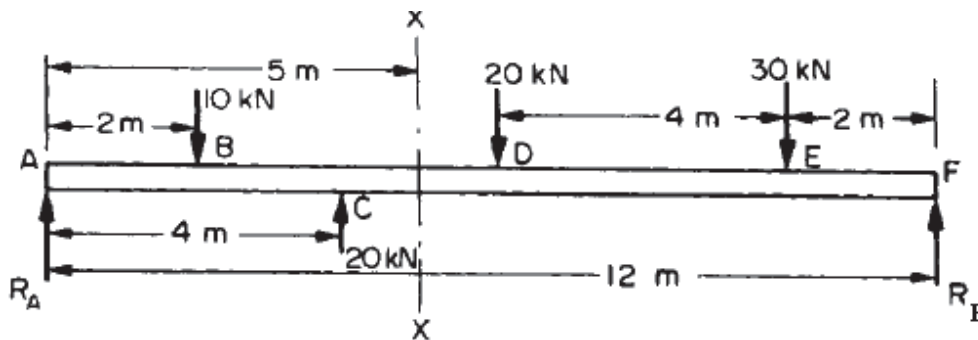

Figure (7)

The values of the reactions at the ends of the beam may be calculated by applying normal equilibrium conditions, i.e. by taking moments about F.
Thus

$$R_A \times 12 = (10 \times 10) + (20 \times 6) + (30 \times 2) - (20 \times 8) = 120$$

R_A = 10 kN
For vertical equilibrium
total force up = total load down
R_A+R_F = 10+20+30-20 = 40
R_F= 30 kN

At this stage it is advisable to check the value of RF by taking moments about A. Summing up the forces on either side of X-X we have the result shown in Figure (8) Using the sign convention listed above, the shear force at X-X is therefore +20kN,i.e the resultant force at X-X tending to shear the beam is 20 kN.
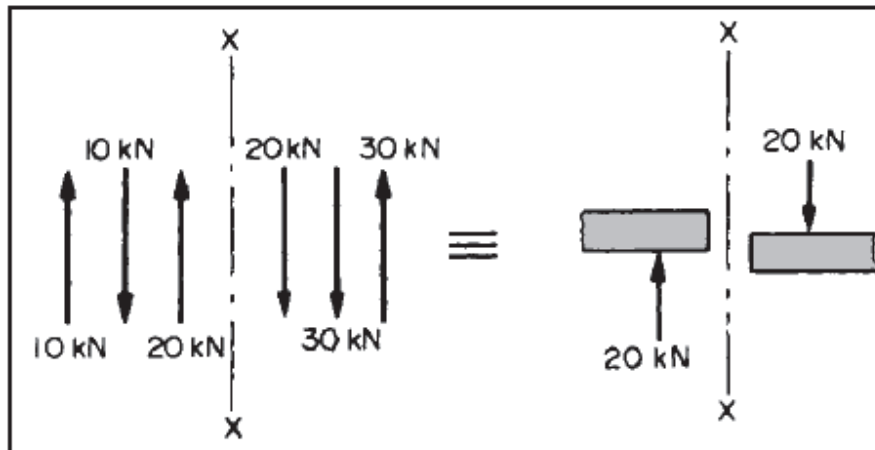
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Strength of Materials

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Figure (8) Total S.F. at X-X.

Similarly, Figure (9) shows the summation of the moments of the forces at *X-X,* the resultant **B.M.** being **40** kNm.

Figure (9)

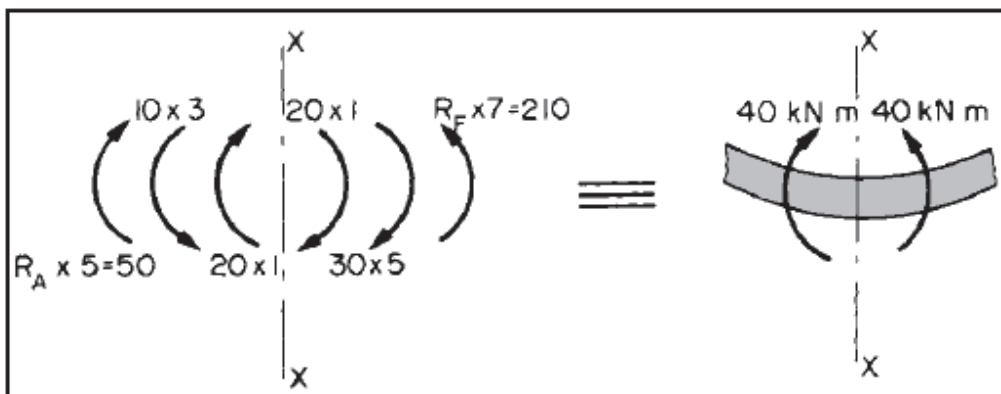In practice only one side of the section is normally considered and the summations involved can often be completed by mental arithmetic. The complete S.F. and **B.M.** diagrams for the beam are shown in Figure (9).

B.M. at A = o

B.M. at B = + (10 x 2) = +20 kN.m

B.M. at C= +(l0 x 4)-(10 x 2) = +20kN.m

B.M. at *D* = +(l0 x 6)+ (20 x 2)- (10 x 4) = +60 kN.m

B.M. at E = + (30 x 2) = +60 kN.m

B.M. at F = 0

All the above values have been calculated from the moments of the forces to the left of each section considered except for E where forces to the right of the section are taken
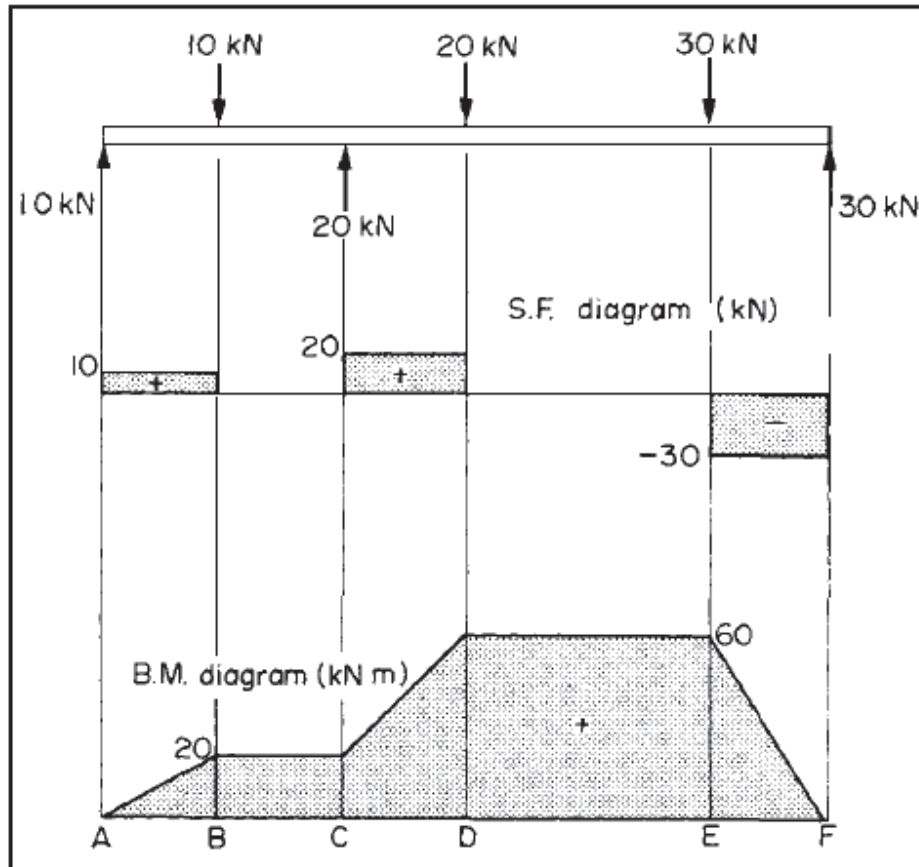

Figure (10)

It may be observed at this stage that the S.F. diagram can be obtained very quickly when working from the left-hand side, since after plotting the S.F. value at the support all subsequent steps are in the direction of and equal in magnitude to the applied loads, e.g. 10 kN up at A, down 10 kN at B, up   20 kN at *C,* etc., with horizontal lines joining the steps to show that the S.F. remains constant between points of application of concentrated loads.

   The S.F. and B.M. values at the left-hand support are determined by considering a section an infinitely small distance to the right of the support. The only load to the left (and hence the S.F.) is then the reaction of 10 kN upwards, Le. positive, and the bending moment

= reaction x zero distance = zero.

The following characteristics of the two diagrams are now evident and will be explained
later in this chapter:
(a) between B and *C* the S.F. is zero and the B.M. remains constant;
(b) between A and *B* the S.F. is positive and the slope of the B.M. diagram is positive; vice
(c) the difference in B.M. between A and *B* = 20 kN m = area of S.F. diagram between A and B.

4. S.F. and B.M. diagrams for uniformly distributed loads
Consider now the simply supported beam shown in Figure (11) carrying a u.d.1. **w** = 25 kN/m across the complete span.
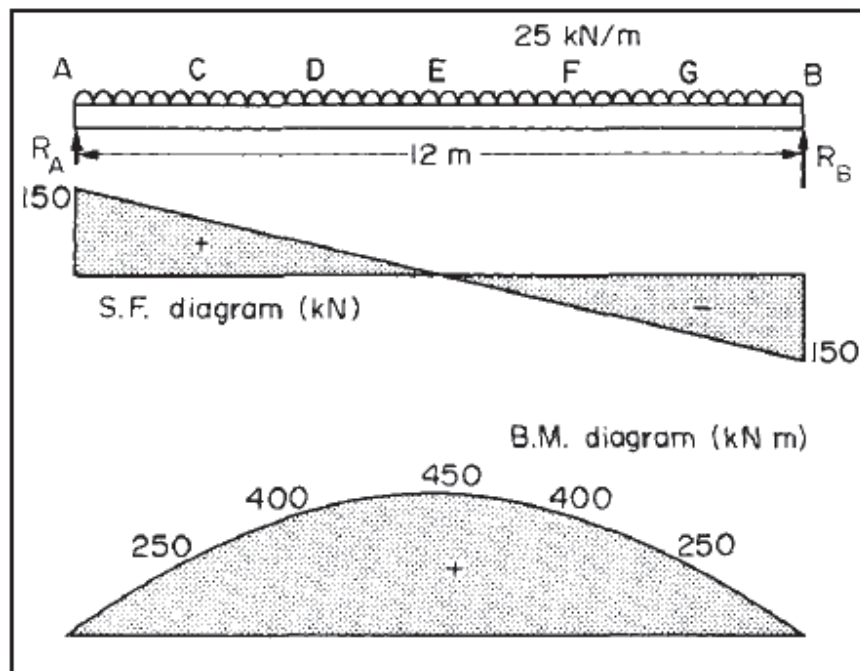

Figure (11)

Here again it is necessary to evaluate the reactions, but in this case the problem is simplified by the symmetry of the beam. Each reaction will therefore take half the applied load,
i.e.

$$R_A = R_B = \frac{25 \times 12}{2} = 150\,kN$$

The S.F. at A, using the usual sign convention, is therefore **+** 150kN.
is, therefore, Consider now the beam divided into six equal parts 2 m long. The S.F. at any other point **C**

$$150 \text{ - load downwards between A and C}$$
$$= 150 - (25 \textbf{ x } 2) = + 100 \text{ kN}$$

The whole diagram may be constructed in this way, or much more quickly by noticing that the S.F. at A is + 150 kN and that between A and **B** the S.F. decreases uniformly, producing the required sloping straight line, shown in Fig. 3.7. Alternatively, the S.F. at A is + 150 kN and between A and B this decreases gradually by the amount of the applied load (By 25 x 12 = 300kN) to - 150kN at B. When evaluating B.M.'s it is assumed that a u.d.1. can be replaced by a concentrated load of equal value acting at the middle of its spread. When taking moments about **C,** therefore, the portion of the u.d.1. between A and **C** has an effect equivalent to that of a concentrated load of
25 x 2 = 50 kN acting the centre of **AC,** i.e. 1 m from **C.**

$$\text{B.M. at } C = (R_A \times 2) - (50 \times 1) = 300 - 50 = 250\,kN\,m$$

Similarly, for moments at **D** the u.d.1. on **AD** can be replaced by a concentrated load of

$$25 \times 4 = 100\,kN \text{ at the centre of } AD, \text{ i.e. at } C.$$

$$\text{B.M. at } D = (R_A \times 4) - (100 \times 2) = 600 - 200 = 400\,kN\,m$$

Similarly,

$$\text{B.M. at } E = (R_A \times 6) - (25 \times 6)3 = 900 - 450 = 450\,kN\,m$$

The B.M. diagram will be symmetrical about the beam centre line; therefore the values of B.M. at F and G will be the same as those at **D** and **C** respectively. The final diagram is therefore as shown in Figure (11) and is parabolic.

Point (a) of the summary is clearly illustrated here, since the B.M. is   a maximum when the S.F. is zero. Again, the reason for this will be shown later.

5. <u>S.F. and B.M. diagrams for combined concentrated and uniformly distributed loads</u>

Consider the beam shown in Figure (12) loaded with a combination of concentrated loads and u.d.1.s.
Taking moments about E

$$(R_A \times 8) + (40 \times 2) = (10 \times 2 \times 7) + (20 \times 6) + (20 \times 3) + (10 \times 1) + (20 \times 3 \times 1.5)$$
$$8R_A + 80 = 420$$
$$R_A = 42.5\,\text{kN}\,(= \text{S.F. at } A)$$

Now
$$R_A + R_E = (10 \times 2) + 20 + 20 + 10 + (20 \times 3) + 40 = 170$$
$$R_E = 127.5\,\text{kN}$$

Working from the left-hand support it is now possible to construct the S.F. diagram, as indicated previously, by following the direction arrows of the loads. In the case of the u.d.l.'s the S.F. diagram will decrease gradually by the amount of the total load until the end of the u.d.1. or the next concentrated load is reached. Where there is no u.d.1. the S.F. diagram remains horizontal between load points. In order to plot the B.M. diagram the following values must be determined:

B.M. at A $=$ 0

B.M. at $B =$ $(42.5 \times 2) - (10 \times 2 \times 1) = 85 - 20$ $=$ $65\,\text{kN m}$

B.M. at $C =$ $(42.5 \times 5) - (10 \times 2 \times 4) - (20 \times 3) = 212.5 - 80 - 60$ $=$ $72.5\,\text{kN m}$

B.M. at $D =$ $(42.5 \times 7) - (10 \times 2 \times 6) - (20 \times 5) - (20 \times 2)$

$\qquad\qquad - (20 \times 2 \times 1) = 297.5 - 120 - 100 - 40 - 40 = 297.5 - 300 = -2.5\,\text{kN m}$

B.M. at $E =$ (-40×2) working from r.h.s. $= -80\,\text{kN m}$

B.M. at F $=$ 0

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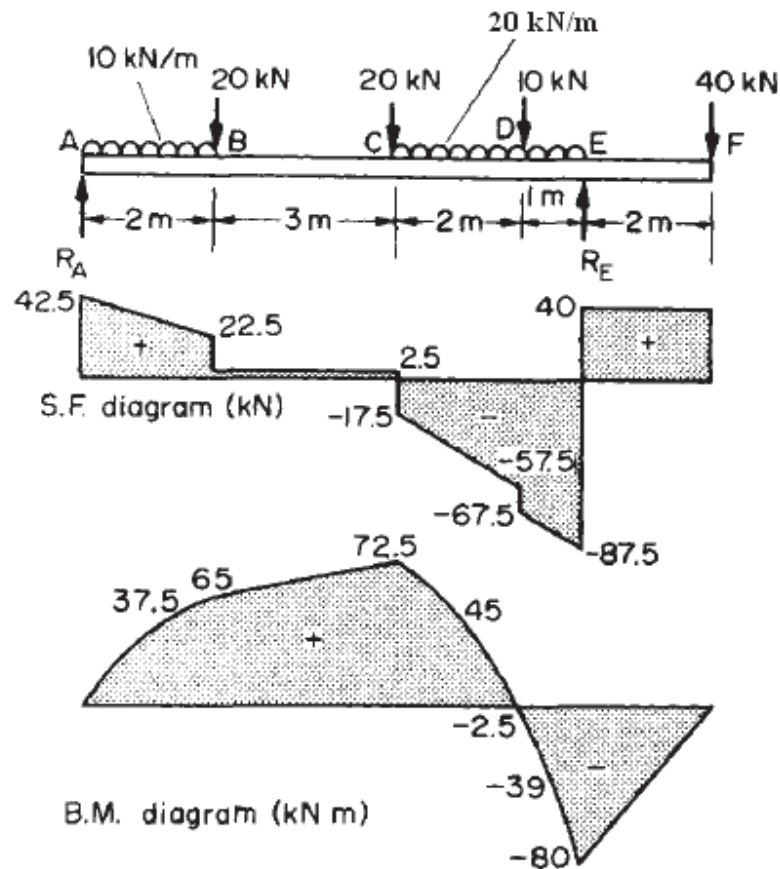
Figure (12)

For complete accuracy one or two intermediate values should be obtained along each u.d.l. portion of the beam,

e.g.   B.M. midway between A and $B = (42.5 \times 1) - (10 \times 1 \times \frac{1}{2})$
$$= 42.5 - 5 = 37.5\,\text{kN m}$$

Similarly, B.M. midway between C and $D = 45\,\text{kN m}$

B.M. midway between D and $E = -39\,\text{kN m}$

The B.M. and S.F. diagrams are then as shown in Figure (12)

5. Points of contraflexure

A point of contraflexure is a point where the curvature of the beam changes sign. It is sometimes referred to as a point of inflexion and will be shown later to occur at the point, or points, on the beam where the B.M. is zero.

For the beam of Figure (**9**) therefore, it is evident from the **B.M.** diagram that this point lies somewhere between C and D (B.M. at C is positive, B.M. at D is negative). If the required point is a distance **x** from C then at that point

$$\text{B.M.} = (42.5)(5+x) - (10 \times 2)(4+x) - 20(3+x) - 20x - \frac{20x^2}{2}$$

$$= 212.5 + 42.5x - 80 - 20x - 60 - 20x - 20x - 10x^2$$

$$= 72.5 - 17.5x - 10x^2$$

Thus the B.M. is zero where

$$0 = 72.5 - 17.5x - 10x^2$$

i.e. where

$$x = 1.96 \text{ or } -3.7$$

Since the last answer can be ignored (being outside the beam), the point of contraflexure must be situated at 1.96 m to the right of *C.*

6. Relationship between shear force Q, bending moment *M* and intensity of loading W (kN/m)

Consider the beam *AB* shown in Figure (10) carrying a uniform loading intensity (uniformly distributed load) of W (kN/m). By symmetry, each reaction takes half the total load, i.e., WL/2.
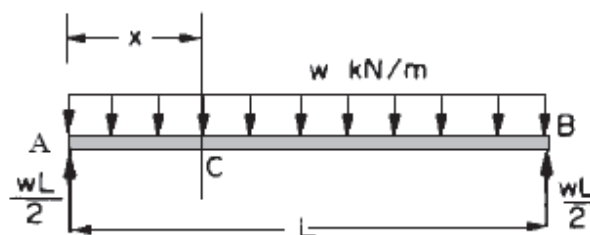

Figure (10)

The B.M. at any point C, distance x from A, is given by

$$M = \frac{wL}{2}x - (wx)\frac{x}{2}$$

i.e.

$$M = \tfrac{1}{2}wLx - \tfrac{1}{2}wx^2$$

Differentiating,

$$\frac{dM}{dx} = \tfrac{1}{2}wL - wx$$

Now

$$\text{S.F. at } C = \tfrac{1}{2}wL - wx = Q \qquad ....(1)$$

∴

$$\frac{dM}{dx} = Q \qquad .....(2)$$

Differentiating equation (1),

$$\frac{dQ}{dx} = -W \qquad .....(3)$$

These relationships are the basis of the rules stated in the summary, the proofs of which are as follows:

(a) The maximum or minimum B.M. occurs where $\dfrac{dM}{dx} = 0$

But $\quad \dfrac{dM}{dx} = Q$

Thus where S.F. is zero B.M. is a maximum or minimum.

(b) The slope of the B.M. diagram $= \dfrac{dM}{dx} = Q$

Thus where $Q = 0$ the slope of the B.M. diagram is zero, and the B.M. is therefore constant.

(c) Also, since Q represents the slope of the B.M. diagram, it follows that where the S.F. is positive the slope of the B.M. diagram is positive, and where the S.F. is negative the slope of the B.M. diagram is also negative.

(d) The area of the S.F. diagram between any two points, from basic calculus, is

$$\int Q\,dx$$

But, $\quad \dfrac{dM}{dx} = Q \qquad$ or $\qquad M = \int Q\,dx$

i.e. the B.M. change between any two points is the area of the S.F. diagram between these points.
This often provides a very quick method of obtaining the B.M. diagram once the S.F. diagram has been drawn.

(e) With the chosen sign convention, when the B.M. is positive the beam is sagging and when it is negative the beam is hogging. Thus when the curvature of the beam changes from sagging to hogging, as at x-x in   Figure (11), or vice versa, the B.M. changes sign, i.e. becomes instantaneously zero. This is termed a point of inflexion or contra flexure. Thus a point of contra flexure occurs where the B.M. is zero.
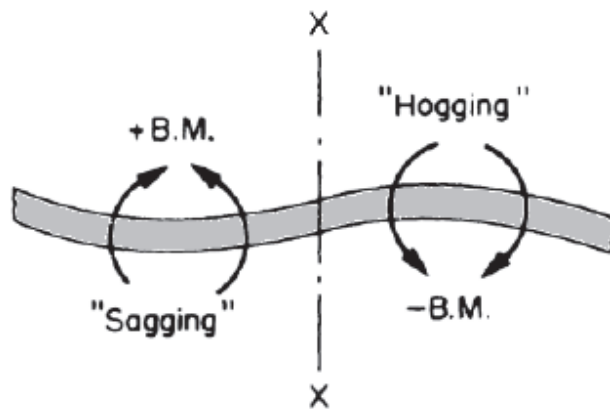
Figure (11) Beam with point contraflexure at X-X

7. S.F. and B.M. diagrams for an applied couple or moment

In general there are two ways in which the couple or moment can be applied: (a) with horizontal loads and (b) with vertical loads, and the method of solution is different for each.

Type (a): couple or moment applied with horizontal loads
        Consider the beam AB shown in Figure (12) to which a moment (F.d) is applied by means of horizontal loads at a point C, distance a from A.

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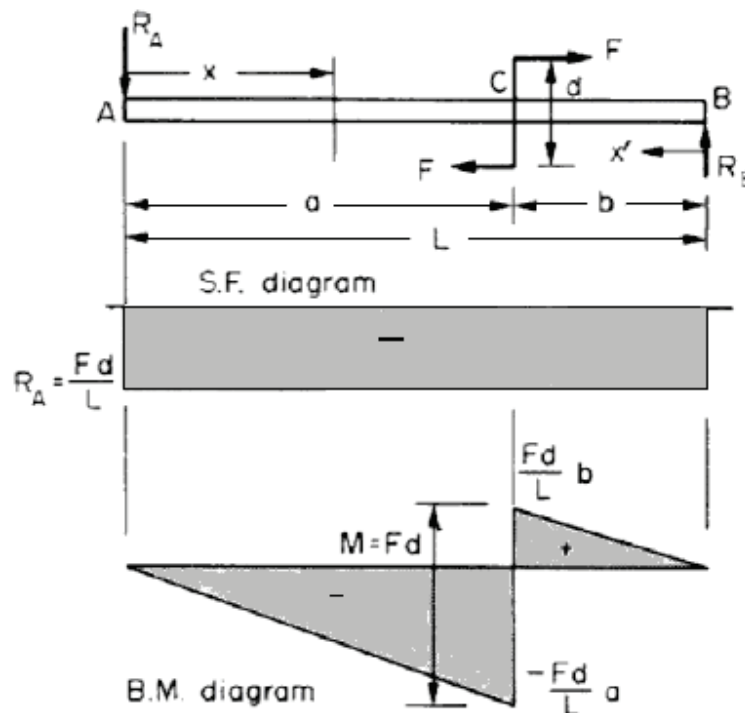
Figure (12)

Since this will tend to lift the beam at A, R_A acts downwards.

Moments about B: $R_A.L = F.d$, $R_A = \dfrac{F.d}{L}$

and for vertical equilibrium $R_B = R_A = \dfrac{F.d}{L}$

The S.F. diagram can now be drawn as the horizontal loads have no effect on the vertical shear.

The B.M. at any section between A and C is

$$M = -R_A.x = \frac{-F.d}{L}.x$$

Thus the value of the B.M. increases linearly from zero at A to $\dfrac{-F.d}{L}.a$ at C
Similarly, the B.M. at any section between C and B is

$$M = -R_A.x + F.d = R_B.x = \frac{-F.d}{L}.x$$

i.e. the value of the B.M. again increases linearly from zero at B to - b at C. The B.M. diagram is therefore as shown in Figure (12).

Type (b): moment applied with vertical loads

Consider the beam AB shown in Figure (13); taking moments about B:

$$R_A L = F(d+b)$$

$$\therefore \qquad R_A = \frac{F(d+b)}{L}$$

Similarly, $\qquad R_B = \frac{F(a-d)}{L}$

The S.F. diagram can therefore be drawn as in Figure **(13)** and it will be observed that in this case (F) does affect the diagram. For the B.M. diagram an equivalent system is used. The offset load **F** is replaced by a moment and a force acting at C, as shown in Figure (13). Thus

B.M. between A and $C = R_A x = \dfrac{F(d+b)}{L} x$

i.e. increasing linearly from zero to $\dfrac{F(d+b)}{L} a$ at C.
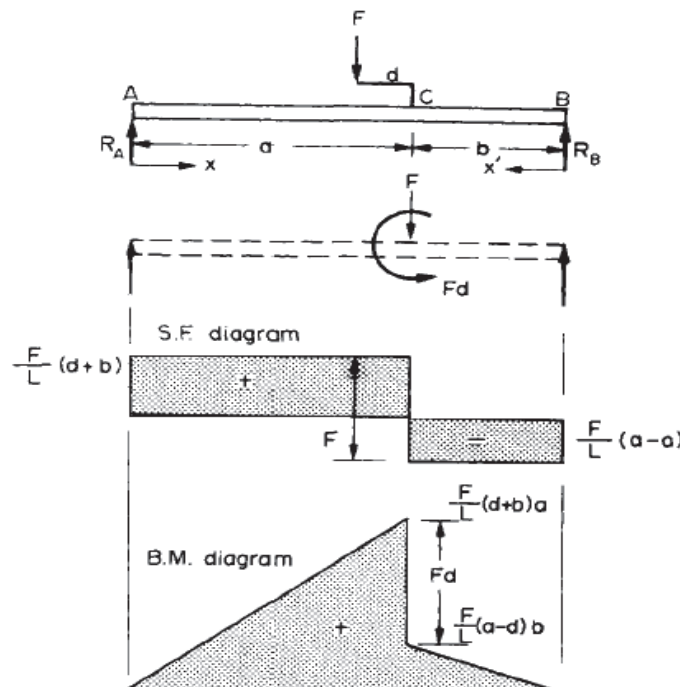
Figure (13)

B.M. between C and $B = R_B x' = \dfrac{F(a-d)}{L} x'$

i.e. increasing linearly from zero to $\dfrac{F(a-d)}{L} b$ at C.

Examples

Example 1

Draw the S.F. and **B.M.** diagrams for the beam loaded as shown in Figure (14), and determine(a) the position and magnitude of the maximum **B.M.,** and (b) the position of any point of contraflexure.

Figure (14)

Solution

Taking the moments about A,

$$5R_B = (5 \times 1) + (7 \times 4) + (2 \times 6) + (4 \times 5) \times 2.5$$

$$\therefore \quad R_B = \frac{5 + 28 + 12 + 50}{5} = 19\,\text{kN}$$

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$$R_A + R_B = 5 + 7 + 2 + (4 \times 5) = 34$$
$$R_A = 34 - 19 = 15\,\text{kN}$$

The S.F. diagram may now be constructed as shown in Figure (14) .

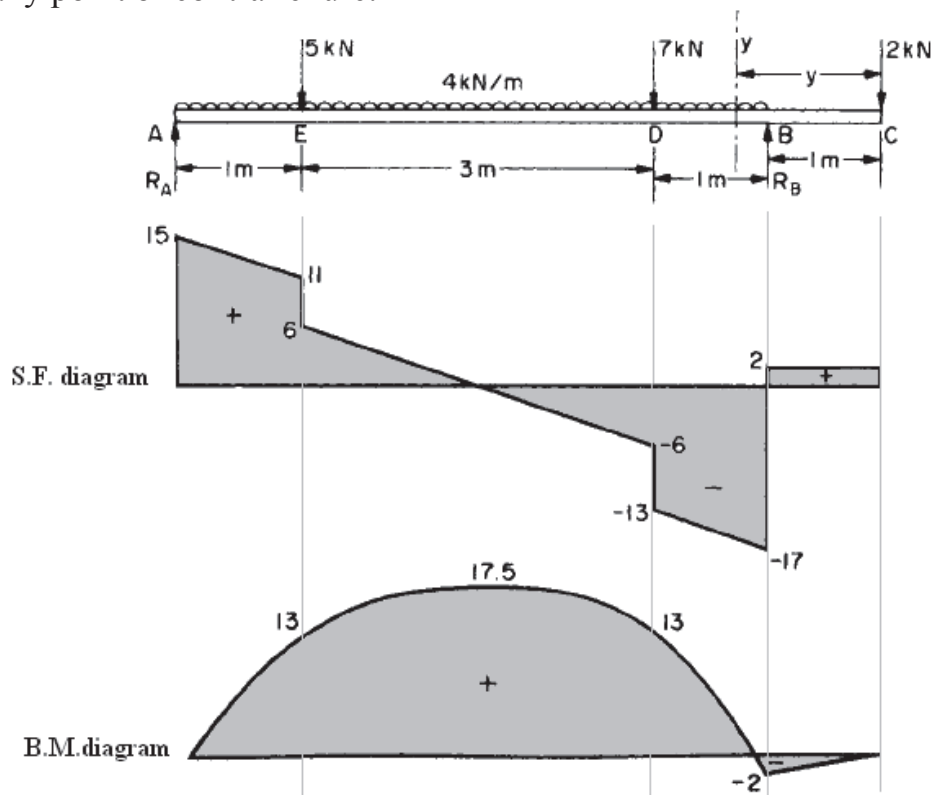Calculation of bending moments

B.M. at A and $C = 0$

B.M. at B $\quad = -2 \times 1 = -2\,\text{kN.m}$

B.M. at D $\quad = -(2 \times 2) + (19 \times 1) - (4 \times 1 \times \frac{1}{2}) = +13\,\text{kN.m}$

B.M. at E $\quad = +(15 \times 1) - (4 \times 1 \times \frac{1}{2}) = +13\,\text{kN.m}$

The maximum B.M. will be given by the point (or points) at which dM/dx (Le. the shear force) is zero. By inspection of the S.F. diagram this occurs midway between D and E, i.e. at 1.5 m from E.

$$\text{B.M. at this point} = (2.5 \times 15) - (5 \times 1.5) - \left(4 \times 2.5 \times \frac{2.5}{2}\right)$$
$$= +17.5\,\text{kN.m}$$

The B.M. diagram is therefore as shown in Figure (14) Alternatively, the B.M. at any point between D and E at a distance of x from A will be given by

$$M_{xx} = 15x - 5(x - 1) - \frac{4x^2}{2} = 10x + 5 - 2x^2$$

The maximum B.M. position is then given where $\dfrac{dM}{dx} = 0$.

$$\frac{dM}{dx} = 10 - 4x = 0 \qquad \therefore \qquad x = 2.5\,\text{m}$$

1.5 m from E, as found previously.

(b) Since the B.M. diagram only crosses the zero axis once there is only one point of contraflexure, i.e. between B and D. Then, B.M. at distance y from C will be given by

$$M_{yy} = -2y + 19(y - 1) - 4(y - 1)\tfrac{1}{2}(y - 1)$$
$$= -2y + 19y - 19 - 2y^2 + 4y - 2 = 0$$

The point of contraflexure occurs where B.M. = 0, i.e. where $M_{yy} = 0$,

$$0 = -2y^2 + 21y - 21$$
$$2y^2 - 21y + 21 = 0$$
$$y = \frac{21 \pm \sqrt{(21^2 - 4 \times 2 \times 21)}}{4} = 1.12\,\text{m}$$

i.e. point of contraflexure occurs 0.12 m to the left of B.

Example 2

A beam ABC is 9 m long and supported at B and C, **6** m apart as shown in Figure **(15).** The beam carries a triangular distribution of load over the portion **BC** together with an applied counterclockwise couple of moment 80 kN m at B and a uniform distributed load (u.d.1.) of 10 kN/m over AB, as shown. Draw the S.F. and B.M. diagrams for the beam.
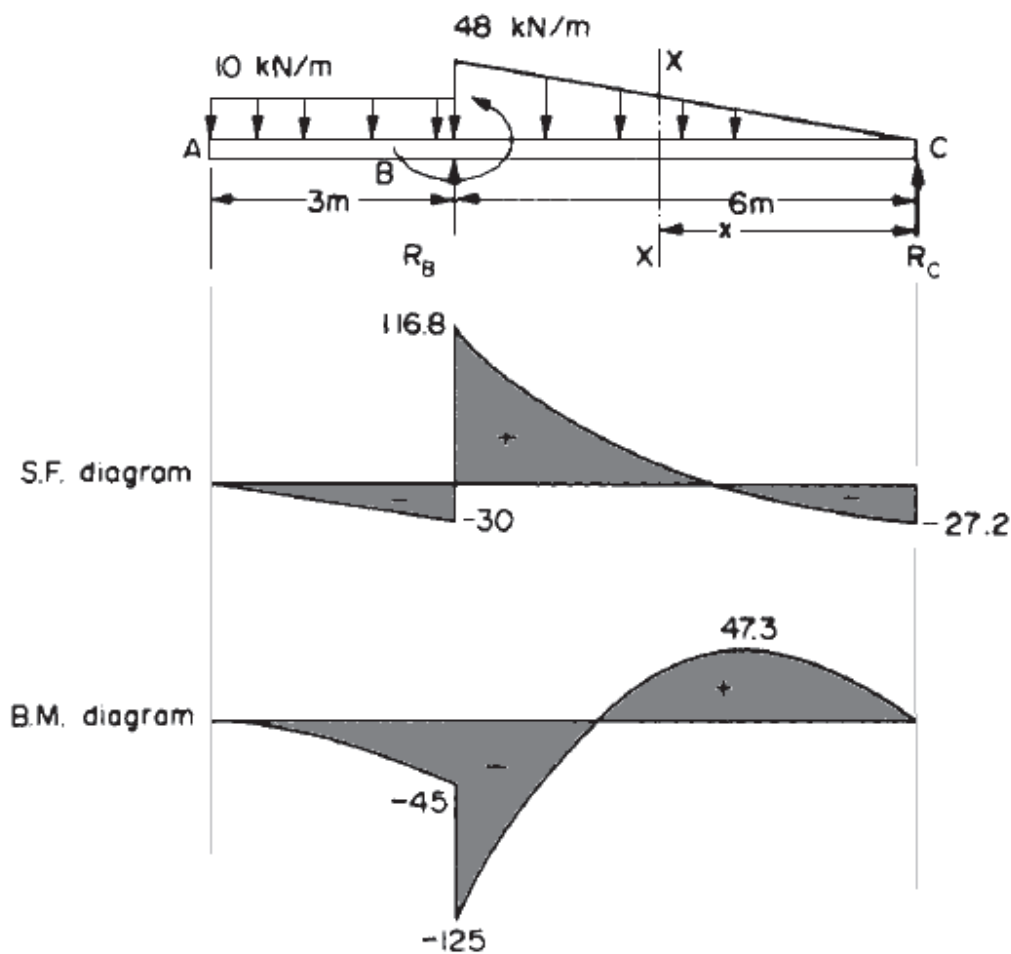

Figure **(15)**

Solution

Taking moments about B,

$$(R_C \times 6) + (10 \times 3 \times 1.5) + 80 = (\tfrac{1}{2} \times 6 \times 48) \times \tfrac{1}{3} \times 6$$
$$6R_C + 45 + 80 = 288$$
$$R_C = 27.2 \text{ kN}$$

and
$$R_C + R_B = (10 \times 3) + (\tfrac{1}{2} \times 6 \times 48)$$
$$= 30 + 144 = 174$$
$$\therefore \qquad R_B = 146.8 \text{ kN}$$

At any distance x from C between C and B the shear force is given by

$$\text{S.F.}_{XX} = -\tfrac{1}{2}wx + R_C$$

and by proportion
$$\frac{w}{x} = \frac{48}{6} = 8$$

i.e.
$$w = 8x \text{ kN/m}$$
$$\therefore \qquad \text{S.F.}_{XX} = -(R_C - \tfrac{1}{2} \times 8x \times x)$$
$$= -R_C + 4x^2$$
$$= -27.2 + 4x^2$$

The S.F. diagram is then as shown in Figure (15)

Also
$$\text{B.M.}_{XX} = -(\tfrac{1}{2}wx)\frac{x}{3} + R_C x$$

$$= 27.2x - \frac{4x^3}{3}$$

For a maximum value, $\dfrac{d\,(\text{B.M.})}{dx} = \text{S.F.} = 0$

i.e.,where
$$4x^2 = 27.2$$
or
$$x = 2.61 \text{ m from } C$$
$$\text{B.M.}_{max} = 27.2(2.61) - \frac{4}{3}(2.61)^3$$
$$= 47.3 \text{ kN m}$$

B.M. at A and $C = 0$

B.M. immediately to left of $B = -(10 \times 3 \times 1.5) = -45 \text{ kN m}$

At the point of application of the applied moment there will be a sudden change in B.M. of 80 kN. m. (There will be no such discontinuity in the S.F. diagram; the effect of the moment will merely be reflected in the values calculated for the reactions.) The B.M. diagram is therefore as shown in Figure (15).

Problems

1. **A** beam AB, 1.2 m long, is simply-supported at its ends A and B and carries two concentrated loads, one of 10 kN at C, the other 15 kN at D. Point C is 0.4 m from A, point D is 1 m from A. Draw the S.F. and B.M. diagrams for the beam inserting principal values.

    [9.17, - 0.83, -15.83 kN;   3.67, 3.17 kN.m]

2 . The beam of question (1) carries an additional load of **5** kN upwards at point E, 0.6 m from A. Draw the S.F. and B.M. diagrams for the modified loading. What is the maximum B.M.?

    [6.67, -3.33, 1.67, -13.33 kN;   2.67, 2, 2.67 kN.m.]

3. **A** cantilever beam AB, 2.5 m long is rigidly built in at A and carries vertical concentrated loads of 8 kN at B and 12 kN at C, 1 m from A. Draw **S.F.** and **B.M.** diagrams for the beam inserting principal values.

    [-8, -20 kN;   -11.2, -31.2kN.m]

4. A beam AB, 5 m long, is simply-supported at the end B and at a point C, 1 m from A. It carries vertical loads of 5 kN at A and 20kN at D, the centre of the span BC. Draw S.F. and B.M. diagrams for the beam inserting principal values.   [- 5 , 11.25, - 8.75kN; - 5 , 17.5 kN.m]

5. A beam AB, 3 m long, is simply-supported at A and E. It carries a 16 kN concentrated load at C, 1.2 m from A, and a u.d.1. of 5 kN/m over the remainder of the beam. Draw the S.F. and B.M. diagrams and determine the value of the maximum B.M.

    [12.3, -3.7, -12.7kN;   14.8 kN.m.]

6. A simply supported beam has a span of 4m and carries a uniformly distributed load of 60 kN/m together with a central concentrated load of 40 kN. Draw the S.F. and B.M. diagrams for the beam and hence determine the maximum B.M. acting on the beam.

    [S.F. 140, k20, -140 kN; B.M. 0, 160,0 kN.m]

7. A 2 m long cantilever is built-in at the right-hand end and carries a load of 40 kN at the free end. In order to restrict the deflection of the cantilever within reasonable limits an upward load of 10 kN is applied at mid-span. Construct the S.F. and B.M.

diagrams for the cantilever and hence determine the values of the reaction force and moment at the support. [30 kN, 70 kN. m.]

8. A beam 4.2 m long overhangs each of two simple supports by 0.6 m. The beam carries a uniformly distributed load of 30 kN/m between supports together with concentrated loads of 20 kN and 30 kN at the two ends. Sketch the S.F. and B.M. diagrams for the beam and hence determine the position of any points of contraflexure.
[S.F. -20, +43, -47, +30 kN; B.M. - 12, 18.75, - 18kN.m; 0.313 and 2.553 from left hand support.]

9. A beam ABCDE, with A on the left, is 7 m long and is simply supported at Band E. The lengths of the various portions are AB = 1.5 m, BC = 1.5 m, CD = 1 m and DE = 3 m. There is a uniformly distributed load of 15 kN/m between B and a point 2 m to the right of B and concentrated loads of 20 kN act at A and D with one of 50 kN at C.
(a) Draw the S.F. diagrams and hence determine the position from A at which the S.F. is zero.
(b) Determine the value of the B.M. at this point.
(c) Sketch the B.M. diagram approximately to scale, quoting the principal values.
[3.32 m; 69.8 kN,m; 0, -30, 69.1, 68.1, 0 kN.m]

10. A beam ABCDE is simply supported at A and D. It carries the following loading: a distributed load of 30 kN/m between A and B a concentrated load of 20 kN at B; a concentrated load of 20 kN at C; a concentrated load of 10 kN at E; a distributed load of 60 kN/m between D and E. Span AB = 1.5 m, BC = CD = DE = 1 m. Calculate the value of the reactions at A and D and hence draw the S.F. and B.M. diagrams. What are the magnitude and position of the maximum B.M. on the beam?
[41.1, 113.9kN;   28.15kN.m; 1.37 m from A.]

CH.8/ TORSION

Simple torsion theory

When a uniform circular shaft is subjected to a torque it can be shown that every section of the shaft is subjected to a state of pure shear Figure (1), the moment of resistance developed by the shear stresses being everywhere equal to the magnitude, and opposite in sense, to the applied torque. For the purposes of deriving a simple theory, to make the following basic assumptions:

(1) The material is homogeneous, i.e. of uniform elastic properties throughout.
(2) The material is elastic, following Hooke's law with shear stress proportional to shear strain.
(3) The stress does not exceed the elastic limit or limit of proportionality.
(4) Circular Sections remain circular.
(5) Cross-sections remain plane. (This is certainly not the case with the torsion of non circular Sections.)
(6) Cross-sections rotate as if rigid, i.e. every diameter rotates through the same angle.
Practical tests carried out on circular shafts have shown that the theory developed below on the basis of these assumptions shows excellent correlation with experimental results.
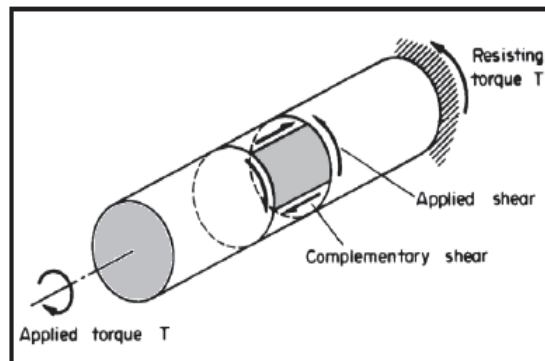
Figure (1) Shear system set up on an element in the surface
of a shaft subjected to torsion.

(a) Angle of twist

Consider now the solid circular shaft of radius (R) subjected to a torque (T) at one end, the other end being fixed Figure (2). Under the action of this torque   a radial line at the free end of the shaft twists through an angle (θ), point A moves to B, and AB subtends an angle (γ) at the fixed end. This is then the angle of distortion of the shaft, i.e. the shear strain.

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angle in radians = arc /radius

$$\text{arc } AB = R\theta = L\gamma$$

$$\therefore \quad \gamma = R\theta/L \quad \quad \text{... (1)}$$

From the definition of rigidity modulus

$$G = \frac{\text{shear stress } \tau}{\text{shear strain } \gamma}$$

Figure (2)

$$\therefore \quad \gamma = \frac{\tau}{G} \quad \quad \text{....(2)}$$
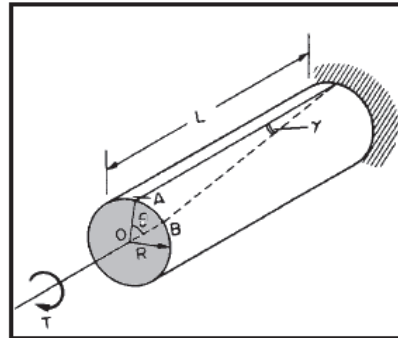
where τ is the shear stress set up at radius R.

Therefore equating eqns. (1) and (2),

$$\frac{R\theta}{L} = \frac{\tau}{G}$$

$$\frac{\tau}{R} = \frac{G\theta}{L}\left(=\frac{\tau'}{r}\right) \quad \quad \text{....(3)}$$

where τ' is the shear stress at any other radius r.

(b) Stresses

Let the cross-section of the shaft be considered as divided into elements of radius r and thickness (dr) as shown in Figure (3) each subjected to a shear stress (τ').

Figure (3) Shaft cross-section

The force set up on each element,
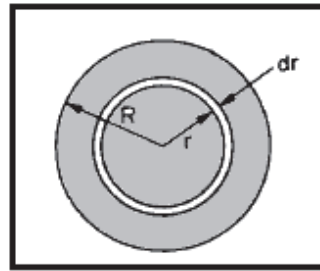
= stress x area= τ' x $2.\pi.r$ dr (approximately)

This force will produce a moment about the centre axis of the shaft, providing a contribution to the torque

$= (\tau'$ x $2.\pi.r$ dr$).r = \tau'$ x $2.\pi.r^2$ dr

The total torque on the section (**T**) will then be the sum of all such contributions across the section,

$$T = \int_0^R 2\pi\tau'r^2 \, dr$$

Now the shear stress (τ') will vary with the radius rand must therefore be replaced in terms of r before the integral is evaluated. From eqnuation (3)

$$\tau' = \frac{G\theta}{L}r$$

$$T = \int_0^R 2\pi \frac{G\theta}{L} r^3 \, dr$$

$$= \frac{G\theta}{L} \int_0^R 2\pi r^3 \, dr$$

The integral $\int_0^R 2.\pi.r^3.dr$ is called the polar second moment of area (J), and may be

evaluated as a

standard form for solid and hollow shafts .

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$$\therefore \quad T = \frac{G\theta}{L} J$$

or $$\quad \frac{T}{J} = \frac{G\theta}{L} \quad ....(4)$$

Combining eqns. *(3)* and *(4)* produces the so-called simple theory of torsion:

$$\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L} \quad .....(5)$$

Polar second moment of area

As stated above the polar second moment of area *J* is defined as

$$J = \int_{0}^{R} 2\pi r^3 \, dr$$

For a *solid shaft,*

$$J = 2\pi \left[\frac{r^4}{4} \right]_0^R$$

$$= \frac{2\pi R^4}{4} \quad \text{or} \quad \frac{\pi D^4}{32} \quad .... (6)$$

For a hollow *shaft* of internal radius *r,*

$$J = 2\pi \int_{r}^{R} r^3 \, dr = 2\pi \left[\frac{r^4}{4} \right]_r^R$$

$$= \frac{\pi}{2}(R^4 - r^4) \quad \text{or} \quad \frac{\pi}{32}(D^4 - d^4) \quad .... (7)$$

For thin-walled hollow shafts the values of (D) and (d) may be nearly equal, and in such cases there can be considerable errors in using the above equation involving the difference of two large quantities of similar value. It is therefore convenient to obtain an alternative form of expression for the polar moment of area.
Now

$$J = \int_0^R 2\pi r^3 \, dr = \Sigma (2\pi r \, dr) r^2$$

$$= \Sigma \, A r^2$$

Where; A $= 2\pi r \, dr$ is the area of each small element of Figure (3).
If a thin hollow cylinder is therefore considered as just one of these small elements with its wall thickness t = dr, then

$$J = A r^2 = (2\pi r \, t) r^2$$

$$= 2\pi r^3 t \text{ (approximately)} \qquad \ldots (8)$$

Shear stress and shear strain in shafts

The shear stresses which are developed in a shaft subjected to pure torsion are indicated in Figure (1),their values being given by the simple torsion theory as

$$\tau = \frac{G\theta}{L} R$$

Now from the definition of the shear or rigidity modulus (**G**),

$$\tau = G\gamma$$

It therefore follows that the two equations may be combined to relate the shear stress and strain in the shaft to the angle of twist per unit length, thus

$$\tau = \frac{G\theta}{L} R = G\gamma \qquad \ldots (9)$$

or, in terms of some internal radius *r,*

$$\tau' = \frac{G\theta}{L} r = G\gamma \qquad \ldots (10)$$

These equations indicate that the shear stress and shear strain vary linearly with radius and have their maximum value at the outside radius Figure (4) .
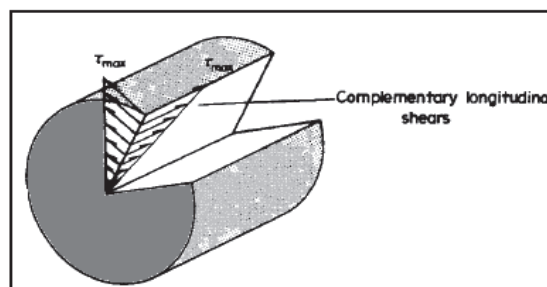

Figure (4) Complementary longitudinal shear stress in
a shaft subjected to torsion.

Section modulus

It is sometimes convenient to re-write part of the torsion theory formula to obtain the maximum shear stress in shafts as follows:

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\tau = \frac{TR}{J}$$

With (R) the outside radius of the shaft the above equation yields the greatest value possible for **T,** Figure (4).

$$\tau_{max} = \frac{TR}{J}$$

$$\tau_{max} = \frac{T}{Z} \qquad \text{.... (11)}$$

Where; $z = J/R$ is termed the polar section modulus. It will be seen from the preceding section that:

for solid shafts, $\qquad Z = \frac{\pi D^3}{16} \qquad$ ..... (12)

and for hollow shafts, $\qquad Z = \frac{\pi(D^4 - d^4)}{16D} \qquad$ .... (13)

Torsional rigidity

The angle of twist per unit length of shafts is given by the torsion theory as

$$\frac{\theta}{L} = \frac{T}{GJ}$$

The quantity (GJ) is termed the *torsional rigidity* of the shaft and is thus given by

$$GJ = \frac{T}{\theta/L} \qquad \text{.... (14)}$$

i.e. the torsional rigidity is the torque divided by the angle of twist (in radians) per unit length.

Torsion of hollow shafts

It has been shown above that the maximum shear stress in a solid shaft is developed in the outer surface, values at other radii decreasing linearly to zero at the centre. In applications where weight reduction is of prime importance as in the aerospace industry, for instance, it is often found advisable to use hollow shafts.

Composite shafts - series connection

If two or more shafts of different material, diameter or basic form are connected together in such a way that each carries the same torque, then the shafts are said to be connected in series and the composite shaft so produced is therefore termed *series-connected* Figure (5) .

$$T = \frac{G_1 J_1 \theta_1}{L_1} = \frac{G_2 J_2 \theta_2}{L_2} \qquad \text{.... (15)}$$
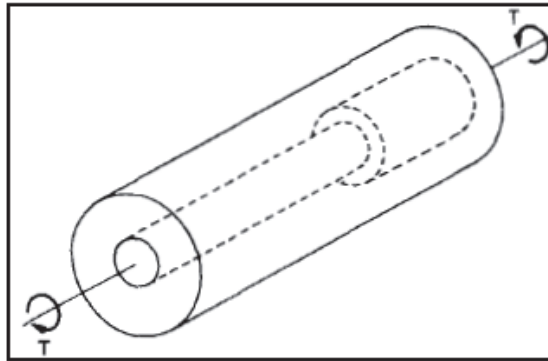
Figure (5)"Series-connected" shaft – common torque.

$$\frac{J_1}{L_1} = \frac{J_2}{L_2}$$

$$\frac{L_1}{L_2} = \frac{J_1}{J_2} \qquad \text{.... (16)}$$

Composite shafts - parallel connection

If two or more materials are rigidly fixed together such that the applied torque is shared between them then the composite shaft so formed is said to be connected in parallel Figure (6).
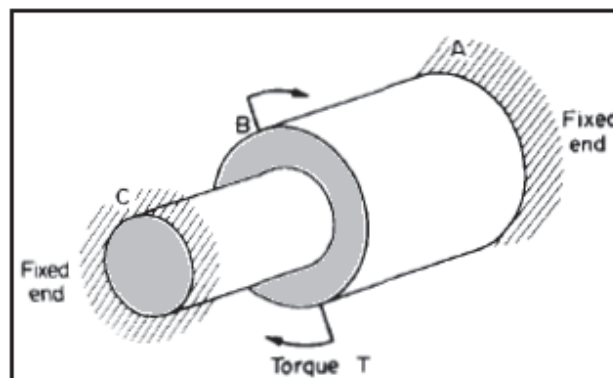

Figure (6)"Parallel-connected" shaft – shared torque.

For parallel connection,

$$\text{total torque } T = T_1 + T_2 \quad ....(17)$$

In this case the angles of twist of each portion are equal and

$$\frac{T_1 L_1}{G_1 J_1} = \frac{T_2 L_2}{G_2 J_2} \quad .....(18)$$

i.e. for equal lengths (as is normally the case for parallel shafts)

$$\frac{T_1}{T_2} = \frac{G_1 J_1}{G_2 J_2} \quad ....(19)$$

The maximum stresses in each part can then be found from

$$\tau_1 = \frac{T_1 R_1}{J_1} \quad \text{and} \quad \tau_2 = \frac{T_2 R_2}{J_2}$$

Strain energy in torsion
The strain energy stored in a solid circular bar or shaft subjected to a torque (T) is given by the alternative expressions.

$$U = \tfrac{1}{2}T\theta = \frac{T^2 L}{2GJ} = \frac{GJ\theta^2}{2L} = \frac{\tau^2}{4G} \times \text{volume} \quad .....(20)$$

Power transmitted by shafts
If a shaft carries a torque T Newton metres and rotates at ω rad/s it will do work at the rate of;

$$T.\omega \text{ Nm/s (or joule/s)}.$$

Now the rate at which a system works is defined as its power, the basic unit of power being the

$$\text{Watt (1 Watt = 1 N.m/s)}.$$

Thus, the power transmitted by the shaft:

$$= T.\omega \text{ Watts}.$$

Since the Watt is a very small unit of power in engineering terms use is normally made of SI. multiples, i.e. kilowatts (kW) or megawatts (MW).

Combined bending and torsion - equivalent bending moment
For shafts subjected to the simultaneous application of a bending moment (**M**) and torque (T) the principal stresses set up in the shaft can be shown to be equal to those produced by an *equivalent bending moment,* of a certain value (**M_e**) acting alone.
From the simple bending theory the maximum direct stresses set up at the outside surface of the shaft owing to the bending moment (M) are given by

$$\sigma = \frac{M y_{max}}{I} = \frac{MD}{2I}$$

Similarly, from the torsion theory, the maximum shear stress in the surface of the shaft is given by

$$\tau = \frac{TR}{J} = \frac{TD}{2J}$$

But for a circular shaft J = 2I,

$$\tau = \frac{TD}{4I}$$

The principal stresses for this system can now be obtained by applying the formula derived in

$$\sigma_1 \text{ or } \sigma_2 = \tfrac{1}{2}(\sigma_x + \sigma_y) \pm \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau^2]}$$

and, with $\sigma_y = 0$, the maximum principal stress (σ_1) is given by

$$\sigma_1 = \frac{1}{2}\left(\frac{MD}{2I}\right) + \frac{1}{2}\sqrt{\left[\left(\frac{MD}{2I}\right)^2 + 4\left(\frac{TD}{4I}\right)^2\right]}$$

$$= \frac{1}{2}\left(\frac{D}{2I}\right)[M + \sqrt{(M^2 + T^2)}]$$

Now if (M_e) is the bending moment which, acting alone, will produce the same maximum stress, then

$$\sigma_1 = \frac{M_e y_{max}}{I} = \frac{M_e D}{2I}$$

$$\frac{M_e D}{2I} = \frac{1}{2}\left(\frac{D}{2I}\right)[M + \sqrt{(M^2 + T^2)}]$$

i.e. the equivalent bending moment is given by

$$M_e = \tfrac{1}{2}[M + \sqrt{(M^2 + T^2)}] \qquad \text{.....(21)}$$

and it will produce the same maximum direct stress as the combined bending and torsion effects.

Combined bending and torsion - equivalent torque

Again considering shafts subjected to the simultaneous application of a bending moment (M) and a torque (T) the *maximum shear stress* set up in the shaft may be determined by the application of an *equivalent torque* of value (T_e) acting alone. From the preceding section the principal stresses in the shaft are given by

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$$\sigma_1 = \frac{1}{2}\left(\frac{D}{2I}\right)[M + \sqrt{(M^2 + T^2)}] = \frac{1}{2}\left(\frac{D}{J}\right)[M + \sqrt{(M^2 + T^2)}]$$

$$\sigma_2 = \frac{1}{2}\left(\frac{D}{2I}\right)[M - \sqrt{(M^2 + T^2)}] = \frac{1}{2}\left(\frac{D}{J}\right)[M - \sqrt{(M^2 + T^2)}]$$

Now the maximum shear stress is given by equation (12)

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}\left(\frac{D}{J}\right)\sqrt{(M^2 + T^2)}$$

But, from the torsion theory, the equivalent torque T_e , will set up a maximum shear stress of

$$\tau_{max} = \frac{T_e D}{2J}$$

Thus if these maximum shear stresses are to be equal,

$$T_e = \sqrt{(M^2 + T^2)} \qquad \text{.....(22)}$$

Examples

Example 1

(a) A solid shaft, 100 mm diameter, transmits 75 kW at 150 rev/min. Determine the value of the maximum shear stress set up in the shaft and the angle of twist per metre of the shaft length if $G = 80$ GN/m^2.

(b) If the shaft were now bored in order to reduce weight to produce a tube of   100 mm outside diameter and 60mm inside diameter, what torque could be carried if the same maximum shear stress is not to be exceeded? What is the percentage increase in power/weight ratio effected by this modification?

Solution

(a) $\text{Power} = T\omega$ $\therefore$ torque $T = \dfrac{\text{power}}{\omega}$

$\therefore \; T = \dfrac{75 \times 10^3}{150 \times 2\pi/60} = 4.77 \text{ kN m}$

From the torsion theory ·

$\dfrac{T}{J} = \dfrac{\tau}{R}$ and $J = \dfrac{\pi}{32} \times 100^4 \times 10^{-12} = 9.82 \times 10^{-6} \text{ m}^4$

$\therefore \tau_{max} = \dfrac{TR_{max}}{J} = \dfrac{4.77 \times 10^3 \times 50 \times 10^{-3}}{9.82 \times 10^{-6}} = 24.3 \text{ MN/m}^2$

Also from the torsion theory

$\theta = \dfrac{TL}{GJ} = \dfrac{4.77 \times 10^3 \times 1}{80 \times 10^9 \times 9.82 \times 10^{-6}} = 6.07 \times 10^{-3} \text{ rad/m}$

$= 6.07 \times 10^{-3} \times \dfrac{360}{2\pi} = 0.348 \text{ degrees/m}$

(b) When the shaft is bored, the polar moment of area J is modified thus:

$J = \dfrac{\pi}{32}(D^4 - d^4) = \dfrac{\pi}{32}(100^4 - 60^4)10^{-12} = 8.545 \times 10^{-6} \text{ m}^4$

The torque carried by the modified shaft is then given by

$$T = \frac{\tau J}{R} = \frac{24.3 \times 10^6 \times 8.545 \times 10^{-6}}{50 \times 10^{-3}} = \mathbf{4.15 \times 10^3 \ Nm}$$

Now, weight/metre of original shaft

$$= \frac{\pi}{4}(100)^2 \times 10^{-6} \times 1 \times \rho g = 7.854 \times 10^{-3} \ \rho g$$

where ρ is the density of the shaft material.

Also, weight/metre of modified shaft $= \frac{\pi}{4}(100^2 - 60^2)10^{-6} \times 1 \times \rho g$

$$= 5.027 \times 10^{-3} \ \rho g$$

Power/weight ratio for original shaft $= \dfrac{T\omega}{\text{weight/metre}}$

$$= \frac{4.77 \times 10^3 \ \omega}{7.854 \times 10^{-3} \ \rho g} = 6.073 \times 10^5 \frac{\omega}{\rho g}$$

Power/weight ratio for modified shaft

$$= \frac{4.15 \times 10^3 \ \omega}{5.027 \times 10^{-3} \ \rho g} = 8.255 \times 10^5 \frac{\omega}{\rho g}$$

Therefore percentage increase in power/weight ratio

$$= \frac{(8.255 - 6.073)}{6.073} \times 100 = \mathbf{36\%}$$

Example 2

Determine the dimensions of a hollow shaft with a diameter ratio of 3:4 which is to transmit 60 kW at 200 rev/min. The maximum shear stress in the shaft is limited to 70 MN/m^2 and the angle of twist to 3.8$^{\text{o}}$ in a length of 4 m. For the shaft material   G = 80 GN/m^2.

Solution

Maximum shear stress condition

Since power $= T\omega$ and $\omega = 200 \times \dfrac{2\pi}{60} = 20.94$ rad/s

$$T = \frac{60 \times 10^3}{20.94} = 2.86 \times 10^3 \ Nm$$

From the torsion theory $J = \dfrac{TR}{\tau}$

$$\therefore \quad \frac{\pi}{32}(D^4 - d^4) = \frac{2.86 \times 10^3 \times D}{70 \times 10^6 \times 2}$$

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But $d/D = 0.75$

$$\therefore \frac{\pi}{32} D^4 (1 - 0.75^4) = 20.43 \times 10^{-6} D$$

$$D^3 = \frac{20.43 \times 10^{-6}}{0.0671} = 304.4 \times 10^{-6}$$

$$D = 0.0673 \text{ m} = 67.3 \text{ mm}$$

$$d = 50.5 \text{ mm}$$

Angle of twist condition

Again from the torsion theory $\quad J = \dfrac{TL}{G\theta}$

$$\frac{\pi}{32} (D^4 - d^4) = \frac{2.86 \times 10^3 \times 4 \times 360}{80 \times 10^9 \times 3.8 \times 2\pi}$$

$$\frac{\pi}{32} D^4 (1 - 0.75^4) = \cdot 2.156 \times 10^{-6}$$

$$\frac{\pi}{32} (D^4 - d^4) = \frac{2.86 \times 10^3 \times 4 \times 360}{80 \times 10^9 \times 3.8 \times 2\pi}$$

$$\frac{\pi}{32} D^4 (1 - 0.75^4) = \cdot 2.156 \times 10^{-6}$$

$$D^4 = \frac{2.156 \times 10^{-6}}{0.0671} = 32.12 \times 10^{-6}$$

$$D = 0.0753 \text{ m} = 75.3 \text{ mm}$$

$$d = 56.5 \text{ mm}$$

Thus the dimensions required for the shaft to satisfy both conditions are outer diameter 75.3mm; inner diameter 56.5 mm.

Example 3

(a) A steel transmission shaft is 510 mm long and 50 mm external diameter. For part of its length it is bored to a diameter of 25 mm and for the rest to 38 mm diameter. Find the maximum power that may be transmitted at a speed of 210 rev/min if the shear stress is not to exceed 70 MN/m^2.

(b) If the angle of twist in the length of 25 mm bore is equal to that in the length of 38 mm bore, find the length bored to the latter diameter.

Solution

(a) This is, in effect, a question on shafts in series since each part is subjected to the same torque. From the torsion theory ;

$$T = \frac{\tau J}{R}$$

and as the maximum stress and the radius at which it occurs (the outside radius) are the same for both shafts the torque allowable for a known value of shear stress is dependent only on the value of (J). This will be least where the internal diameter is greatest since

$$J = \frac{\pi}{32}(D^4 - d^4)$$

least value of $J = \frac{\pi}{32}(50^4 - 38^4)10^{-12} = 0.41 \times 10^{-6}\, m^4$

Therefore maximum allowable torque if the shear stress is not to exceed 70 MN/m² (a 25 mm radius) is given by

$$T = \frac{70 \times 10^6 \times 0.41 \times 10^{-6}}{25 \times 10^{-3}} = 1.15 \times 10^3\, Nm$$

Maximum power $= T\omega = 1.15 \times 10^3 \times 210 \times \frac{2\pi}{60}$

$= 25.2 \times 10^3 = \mathbf{25.2\, kW}$

(b)

Let suffix 1 refer to the 38 mm diameter bore portion and suffix 2 to the other part. Now for shafts in series, equation (16) applies,

$$\frac{J_1}{L_1} = \frac{J_2}{L_2}$$

$$\frac{L_2}{L_1} = \frac{J_2}{J_1} = \frac{\frac{\pi}{32}(50^4 - 25^4)10^{-12}}{\frac{\pi}{32}(50^4 - 38^4)10^{-12}} = 1.43$$

$L_2 = 1.43\, L_1$

$L_1 + L_2 = 510\, mm$

$L_1(1 + 1.43) = 510$

$L_1 = \frac{510}{2.43} = \mathbf{210\, mm}$

PROBLEMS

1 - A solid steel shaft (A) of 50 mm diameter rotates at 250 rev/min. Find the greatest power that can be transmitted for a limiting shearing stress of 60 MN/m^2 in the steel.
It is proposed to replace (A) by a hollow shaft (B), of the Same external diameter but with a limiting shearing stress of 75 MN/m^2. Determine the internal diameter of (B) to transmit the same power at the same speed.   **Ans.[38.6kW, 33.4 mm]**

2 - Calculate the dimensions of a hollow steel shaft which is required to transmit 750 kW at a speed of 400 rev/min if the maximum torque exceeds the mean by 20 % and the greatest intensity of shear stress is limited to75 MN/m^2. The internal diameter of the shaft is to be   80 % of the external diameter. (The mean torque is that derived from the horsepower equation.)
                                                        Ans. [135.2mm, 108.2 mm.]

3 - A steel shaft 3 m long is transmitting 1 MW at 240 rev/min. The working conditions to be satisfied by the
shaft are:
(a) that the shaft must not twist more than 0.02 radian on a length of 10 diameters;
(b) that the working stress must not exceed 60 MN/m^2.
If the modulus of rigidity of steel is 80 GN/m^2 what is
   (i) the diameter of the shaft required
   (ii) the actual working stress;
   (iii) the angle of twist of the 3 m length?   **Ans. [l50 mm; 60MN/m^2; 0.030 rad.]**

4 - A hollow shaft has to transmit 6MW at 150 rev/min. The maximum allowable stress is not to exceed 60 MN/m^2 and the angle of twist 0.3^o per metre length of shafting. If the outside diameter of the shaft is 300 mm find the minimum thickness of the hollow shaft to satisfy the above conditions. G = 80 GN/m^2.   **Ans. [61.5mm.]**

5 - A flanged coupling having six bolts placed at a pitch circle diameter of 180 mm connects two lengths of solid steel shafting of the same diameter. The shaft is required to transmit 80 kW at 240 rev/min. Assuming the allowable intensities of shearing stresses in the shaft and bolts are 75 MN/m^2 and 55 MN/m^2 respectively, and the maximum torque is 1.4 times the mean torque, calculate:
   (a) the diameter of the shaft;
   (b) the diameter of the bolts.   **Ans. [67.2mm, 13.8 mm.]**

6 - A hollow low carbon steel shaft is subjected to a torque of 0.25 MN. m. If the ratio of internal to external diameter is 1 to 3 and the shear stress due to torque has to be limited to 70 MN/m^2 determine the required diameters and the angle of twist in degrees per metre length of shaft.
   G = 80GN/m^2.   **Ans. [264mm, 88 mm; 0.38^o]**