Differential Equations

Introduction

- A differential equation (D.E.) is an equation involving one or more derivatives (called differential coefficients) of a function.

- A solution of a D.E. is a relation between the dependent and independent variables that is free of any differential coefficients and satisfies the D.E.

e.g.

① $\dfrac{dy}{dx} - \cos x = 0$ is a D.E. with a solution of

$$\frac{dy}{dx} = \cos x \Rightarrow y(x) = \sin x + C$$

NOTES: y is called "dependent variable", while x is called "independent variable".

② $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 0$ is a D.E. with a solution of $y(x) = Ae^{3x} + Bxe^{3x}$

General form of a D.E.

The general form of a D.E. can be written as follows:

$$\boxed{a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y = f(x)}$$

where $\frac{d^n y}{dx^n}$, $\frac{d^{n-1}y}{dx^{n-1}}$, ..., $\frac{dy}{dx}$ are the differential coefficients.

$a_0(x)$, $a_1(x)$, ..., $a_{n-1}(x)$ are coefficients of the D.E.

If these coefficients are constants, then the D.E. is called "constant coefficient".

But if these coefficients are functions of x, then the D.E. is called "variable coefficient".

e.g.

① $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$ is a constant coefficient D.E. with $a_0 = 1$, $a_1 = -6$, and $a_2 = 9$

② $4x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$ is a variable coefficient D.E. with $a_0(x) = 4x^2$, $a_1(x) = x$, and $a_2(x) = -1$

$n, n-1, n-2, \ldots$ represents the order of the differential coefficients.

$f(x)$ is an arbitrary function.

If $f(x) = 0$, then the D.E. is said to be "homogeneous". But if $f(x) \neq 0$ (either a constant or a function), then the D.E. is said to be "nonhomogeneous".

e.g.

① $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 0$ is a homogeneous D.E.

② $\dfrac{d^2y}{dx^2} + 4y = \sin 2x$ is a nonhomogeneous D.E.

Order and degree of a D.E.

- The order of a D.E. represents the order of the highest differential coefficient in the equation.

e.g.

① $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C} = E\sin\omega t$ is a second order D.E. $\left(\dfrac{d^2q}{dt^2}\right)$

② $\cos x \dfrac{d^2y}{dx^2} + \sin x \left(\dfrac{dy}{dx}\right)^2 + 8y = \tan x$ is a second order D.E. $\left(\dfrac{d^2y}{dx^2}\right)$

③ $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^3 = \left(\dfrac{d^3y}{dx^3}\right)^2$ is a third order D.E. $\left(\dfrac{d^3y}{dx^3}\right)$

- The degree of a D.E. represents the degree of the highest differential coefficient in the D.E. after removing the radical sign and fractional powers.

e.g. Regarding the previous example:
Eqs. 1 and 2 are both first degree.
Eq. 3 is of a second degree (the power of $\frac{d^3y}{dx^3}$).

Example:
Find the order and degree of the following D.E.s:

1. $$\left[1+\left(\frac{dy}{dx}\right)\right]^{3/2} = \frac{d^2y}{dx^2}$$

2. $$y - x\frac{dy}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Solution:

1. Squaring both sides to get rid of the fractional power, then:

$$\left[1+\left(\frac{dy}{dx}\right)\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$$

$\because$ The highest derivative is of order 2, and the degree (power) of that highest derivative is 2
$\therefore$ It is a second order, second degree D.E.

② Squaring both sides to get rid of the fractional power, then:

$$y - \left(x \frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

which is a first order, second degree D.E.

Classification of D.E.s

D.E.s can be classified according to different aspects, such as

- Ordinary vs. partial D.E. : An ordinary D.E. contains only single independent variable, while a partial D.E. has at least two independent variables along with their differential coefficients.

e.g.

① $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$ is an ordinary D.E.

since the function y has only one independent Variable (x) [written as $f(x)$]

② $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u(x, y)$ is a partial D.E.

since the function u has two independent variables (x and y) and their partial derivative $\left(\frac{\partial^2}{\partial x^2} \text{ and } \frac{\partial^2}{\partial y^2}\right)$ are present in the D.E.

- Linear vs. nonlinear D.E.: Any D.E. satisfies the following two conditions is said to be "linear".

  (1) All the differential coefficients are of power 1.

  (2) All the coefficients are either constants or functions of the independent variable only.

  If any one of these conditions is not satisfied, then the D.E. is said to be "nonlinear".

e.g.

all coefficients are constants

① $2y''' + 5y'' - 3y' + 7y = x$ is a linear D.E.

all differential coefficients are of power 1

all coefficients are functions of the independent variable (x)

② $(x+1)^2 \dfrac{d^2y}{dx^2} - 3(x+1)\dfrac{dy}{dx} + 2y = 3$ is a linear D.E.

all differential coefficients are of power 1

③ $\dfrac{d^2y}{dx^2} - 3\left(\dfrac{dy}{dx}\right)^2 + 2y = 4$ is a nonlinear D.E.

this differential coefficient is not of power 1

④ $3y\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 7xy = 0$ is a nonlinear D.E.

this coefficient depends directly on y not on x

- Constant coefficient vs. variable coefficient
- Homogeneous vs. nonhomogeneous