Ch1 Infinite Series (Ch9)

9.1 Sequences

A sequence of numbers is a function whose domain is the set of integer numbers greater than or equal to some integer n_0

$a_n = a(n)$ for $n \geq n_0$

Ex: The sequence defined by $a(n) = \frac{n-1}{n}$

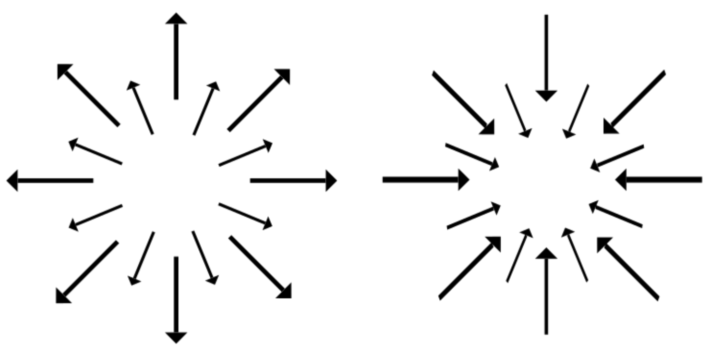
$a(1)=0$, $a(2)=1/2$, $a(3)=2/3$,…… $a(n)=(n-1)/n$ (the n[th] term)

$a_1=0$, $a_2=1/2$, $a_3=2/3$,…… $a_n=(n-1)/n$

Ex:

0, 1, 2, 3, , n-1,	$a_n=n-1$
1, ½, 1/3, 1/4, ……..1/n,	$a_n=1/n$
1, -1/2, 1/3, -1/4,……… $(-1)^{n+1}\frac{1}{n}$,	$a_n = (-1)^{n+1}\frac{1}{n}$
0, -1/2, 2/3, -3/4, ……. $(-1)^{n+1}\left(\frac{n-1}{n}\right)$	$a_n = (-1)^{n+1}\left(\frac{n-1}{n}\right)$
3, 3, 3, 3,……………..3	$a_n=3$
3, -3, 3, -3,…………….. $(-1)^{n+1}3$	$a_n = (-1)^{n+1}3$

If the sequence approaches a limit L as $n \to \infty$ (n approaches infinity), then it is convergent, if not it is divergent.



If a sequence a_n converges to A as n approaches infinity we write

$$\lim_{n\to\infty} a_n = A$$

Theorems

If $\lim\limits_{n\to\infty} a_n = A$ and $\lim\limits_{n\to\infty} b_n = B$

1- $\lim\limits_{n\to\infty}[a_n + b_n] = A + B$

2- $\lim\limits_{n\to\infty}[a_n - b_n] = A - B$

3- $\lim\limits_{n\to\infty}[a_n \cdot b_n] = AB$

4- $\lim\limits_{n\to\infty}[kb_n] = kB$ (k is any number)

5- $\lim\limits_{n\to\infty}\left[\dfrac{a_n}{b_n}\right] = \dfrac{A}{B}$ (B≠0)

Ex:

- $\lim\limits_{n\to\infty}\left(\dfrac{-1}{n}\right) = (-1)\lim\limits_{n\to\infty}\dfrac{1}{n}$=(-1)(0)=0
- $\lim\limits_{n\to\infty}\left(\dfrac{n-1}{n}\right) = \lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right) = \lim(1) - \lim\left(\dfrac{1}{n}\right) = 1 - 0 = 1$
- $\lim\limits_{n\to\infty}\dfrac{5}{n^2} = \lim(5)\lim\left(\dfrac{1}{n}\right)\lim\left(\dfrac{1}{n}\right) = (5)(0)(0) = 0$
- $\lim\limits_{n\to\infty}\dfrac{4-7n^6}{n^6+3} = \lim\limits_{n\to\infty}\dfrac{4/n^6-7}{1+3/n^6} = \dfrac{0-7}{1+0} = -7$

Theorem:

If $\lim\limits_{n\to\infty} a_n = L$ and f is a function that is continuous at L then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

Ex: $a_n = \sqrt{\dfrac{n+1}{n}}$

We know that $\dfrac{n+1}{n} \to 1$

Taking $f(x) = \sqrt{x}$

Therefore $\sqrt{\dfrac{n+1}{n}} \to f(1) = \sqrt{1} = 1$

Theorem:

If $f(x)$ is a function defined for all $x \geq n_0$ and $\{a_n\}$ is a sequence such that $a_n = f(n)$ when $n \geq n_0$

If $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L$

Ex: Show that $\dfrac{\ln(n)}{n} \to 0$

Let $f(x) = \dfrac{\ln x}{x}$, $a_n = f(n)$

$\lim_{x\to\infty} \dfrac{\ln x}{x} = \lim_{x\to\infty} \dfrac{1/x}{1} = 0$ (by applying L'Hopital's rule)

Therefore $\lim_{n\to\infty} \dfrac{\ln(n)}{n} = 0$

Ex: find $\lim_{n\to\infty} \dfrac{2^n}{5n}$

$$\lim_{x\to\infty} \frac{2^x}{5x} = \lim_{x\to\infty} \frac{2^x \ln 2}{5} = \infty$$

Limits Arise Frequently

- $\lim_{n\to\infty} \dfrac{1}{n} = 0$
- $\lim_{n\to\infty} x^{1/n} = 1$
- $\lim_{n\to\infty} \left(1 + \dfrac{x}{n}\right)^n = e^x$
- $\lim_{n\to\infty} \sqrt[n]{n} = 1$
- $\lim_{n\to\infty} x^n = 0$ $(|x|<1)$
- $\lim_{n\to\infty} \dfrac{x^n}{n!} = 0$

<u>Ex:</u> Specify if the sequence is convergent or divergent

1- $\dfrac{n+(-1)^n}{n} = 1 + \dfrac{(-1)^n}{n} \to 1$ convergent

2- $\dfrac{2n+1}{1-3n} = \dfrac{2+1/n}{1/n-3} \to -\dfrac{2}{3}$ convergent

3- $1+(-1)^n$ divergent

4- $\dfrac{10^{n+1}}{10^n} = 10\dfrac{10^n}{10^n} = 10$ convergent

<u>Ex:</u> Find the limit of

1- $\left(-\frac{1}{2}\right)^n \to 0$

4- $\sqrt{\frac{2n}{n+1}}$

Let $f(x)=\sqrt{x}$

$$\frac{2n}{n+1}=\frac{2}{1+1/n}\to 2$$

$$\lim f\left(\frac{2n}{n+1}\right)= f(2)=\sqrt{2}$$

5- sin(π/2+1/*n*)

Let *f*(*x*)=sin(*x*)

$$\lim_{n\to\infty}\left(\frac{\pi}{2}+\frac{1}{n}\right)=\frac{\pi}{2}$$

$$\lim_{n\to\infty} f\left(\frac{\pi}{2}+\frac{1}{n}\right)= f(\pi/2)=\sin(\pi/2)=1$$

6- $\ln(n)-\ln(n+1)=\ln\left(\frac{n}{n+1}\right)$

Let *f*(*x*)=ln*x*

$$\lim_{n\to\infty}\frac{n}{n+1}=1$$

$$\lim f\left(\frac{n}{n+1}\right)\to f(1)=\ln(1)=0$$

7- $\left(1+\frac{7}{n}\right)^n$

Compare to $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x$

$$\left(1+\frac{7}{n}\right)^n \to e^7$$

8- $\frac{(n+1)!}{n!}=\frac{(n+1)n!}{n!}=n+1\to\infty$ (6!=6·5·4·3·2·1=6·5!)

9- $\sqrt[n]{n^2}=\sqrt[n]{n\cdot n}=\sqrt[n]{n}\sqrt[n]{n}\to(1)(1)=1$

9.2 Infinite Series

Definition:

For the sequence $\{a_n\}$

$a_1+a_2+a_3+\ldots..+a_n+\ldots.$ is infinite series

a_n is the n^{th} term

$S_1=a_1$

$S_2= a_1 + a_2$

$S_3= a_1 + a_2 + a_3$

$\vdots$

$S_n= a_1+a_2+a_3+\ldots..+a_n$ Partial Sum

$S_1, S_2, S_3,\ldots\ldots.S_n$ is a sequence of partial sums

The Series converges if

$\sum_{n=1}^{\infty} a_n = L$ or $\lim_{n\to\infty} S_n = L$

The Geometric Series

$$a+ar+ar^2+ar^3+\ldots\ldots.+ar^{n-1}+\ldots\ldots=\sum_{n=1}^{\infty} ar^{n-1}$$

or

$$a+ar+ar^2+ar^3+\ldots\ldots.+ar^{n-1}+\ldots\ldots=\sum_{n=0}^{\infty} ar^{n}$$

$S_n= a+ar+ar^2+ar^3+\ldots\ldots.+ar^{n-1}$ (1)

$rS_n= ar+ar^2+ar^3+ar^4+\ldots\ldots.+ar^n$ (2)

Subtract (2) from (1)

$S_n-rS_n=a-ar^n$

$S_n = \frac{a(1-r^n)}{1-r}$ $r\neq 1$

If $|r|<1$ then $r^n\to 0$, then the geometric series converges to

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

Ex: Determine whether $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges, if so, find the sum.

From partial fractions, we know

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

We can write the partial sum

$$S_k = \sum_{n=1}^{k} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + ... + \frac{1}{k\cdot(k+1)}$$

$$S_k = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + ... + \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$S_k = 1 - \frac{1}{k+1}$$ (Telescopic Sum)

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(1 - \frac{1}{1+k}\right) = 1$$

Then the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ (convergent)

Divergent Series

$\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + ... + \frac{n+1}{n} + ...$ Divergent ($\lim_{n\to\infty} a_n = 1$)

$\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + 1 - ..... + 1 - .....$ Divergent ($\lim_{n\to\infty} a_n$ does not exist)

The *n*th Term Test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ fails to exist, then

$\sum_{n=1}^{\infty} a_n$ diverges

Necessary Condition for Convergence

If $\sum_{n=1}^{\infty} a_n$ converges $\underset{then}{\Rightarrow}$, $a_n \to 0$

Theorems

If $\sum a_n = A$ and $\sum b_n = B$, then

$\sum (a_n + b_n) = A + B$ (Sum Rule)

$\sum (a_n - b_n) = A - B$ (Difference Rule)

$\sum (ka_n) = kA$ (Constant Multiple Rule)

Ex:

1- $\sum_{n=1}^{\infty}\frac{4}{2^{n-1}}=4\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=4\frac{1}{1-1/2}=8$

2- $\sum_{n=1}^{\infty}\frac{3^{n-1}-1}{6^{n-1}}=\sum_{n=1}^{\infty}\left(\frac{3^{n-1}}{6^{n-1}}-\frac{1}{6^{n-1}}\right)$

$=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}-\sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^{n-1}=\frac{1}{1-1/2}-\frac{1}{1-1/6}=2-\frac{6}{5}=\frac{4}{5}$

3- $\sum_{n=0}^{\infty}\frac{1}{4^n}=\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^{n}=\frac{1}{1-1/4}=\frac{4}{3}$

4- $\sum_{n=2}^{\infty}\frac{1}{4^n}$, let $k=n-2 \Rightarrow n=k+2$

$\sum_{n=2}^{\infty}\frac{1}{4^n}=\sum_{k=0}^{\infty}\frac{1}{4^{k+2}}=\sum_{k=0}^{\infty}\frac{1}{4^2\cdot 4^k}=\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^{2}\left(\frac{1}{4}\right)^{k}$

$=\frac{1}{4^2}\cdot\frac{1}{1-1/4}=\frac{1}{4^2}.\frac{4}{3}=\frac{1}{12}$

Or

$\sum_{n=2}^{\infty}\frac{1}{4^n}=1+\frac{1}{4}+\sum_{n=2}^{\infty}\frac{1}{4^n}-1-\frac{1}{4}=\sum_{n=0}^{\infty}\frac{1}{4^n}-1-\frac{1}{4}=\frac{1}{1-1/4}-1-\frac{1}{4}$

$=\frac{4}{3}-1-\frac{1}{4}=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$

5- $\sum_{n=0}^{\infty}\left(\frac{5}{2^n}+\frac{1}{3^n}\right)=5\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n}+\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{n}=5\frac{1}{1-1/2}+\frac{1}{1-1/3}$

$=10+\frac{3}{2}=\frac{23}{2}$

6- $\sum_{n=0}^{\infty}\left(\frac{1}{2^n}+\frac{(-1)^n}{5^n}\right)=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n}+\sum_{n=0}^{\infty}\left(\frac{-1}{5}\right)^{n}=\frac{1}{1-1/2}+\frac{1}{1+1/5}$

$=2+\frac{5}{6}=\frac{17}{6}$

$\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^{2}=\frac{1}{1-\frac{1}{\sqrt{2}}}=\frac{\sqrt{2}}{\sqrt{2}-1}$

$$\sum_{n=1}^{\infty} \ln\frac{1}{n}, \; a_n = \ln\frac{1}{n} = -\ln n$$

$\lim_{n\to\infty} a_n \to -\infty$ Divergent

$\sum_{n=0}^{\infty} \cos n\pi = 1-1+1-1+....$ Divergent

$$\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} = \sum_{n=0}^{\infty} \left(\frac{-1}{5}\right)^n = \frac{1}{1+\frac{1}{5}} = \frac{5}{6}$$

$$\sum_{n=0}^{\infty} e^{-2n} = \sum_{n=0}^{\infty} \left(e^{-2}\right)^n = \frac{1}{1-e^{-2}} = \frac{e^2}{e^2-1}$$

$$\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^n, \; a_n = \left(1-\frac{1}{n}\right)^n$$

$\lim_{n\to\infty} a_n = e^{-1} \neq 0$ Divergent

9.3 Series without Negative Terms: Comparison and Integration Tests

<u>Theorem</u>

A nondecreasing sequence converges if and only if its terms are bounded from above. If all terms are less than or equal to *M* then the limit (*L*) of the sequence is less than or equal to *M* ($L \leq M$)

Ex: The series

$$\sum_{n=0}^{\infty}\frac{1}{n!}=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+.... \qquad (0!=1)$$

Converges because all of its terms are positive and less than or equal to the corresponding term of

$$1+\sum_{n=0}^{\infty}\frac{1}{2^n}=1+1+\frac{1}{2}+\frac{1}{2^2}+.....$$

$$\sum_{n=0}^{\infty}\frac{1}{n!}<1+\sum_{n=0}^{\infty}\frac{1}{2^n}=1+\frac{1}{1-1/2}=3$$

Then the upper limit of our series is 3. This does not mean that our series converges to 3. Actually it converges to e=2.718281828459045

Ex: The Harmonic Series

$$\sum_{1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n}+...$$

Can be written as

$$=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+....+\frac{1}{16}\right)+....$$

$$>1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\frac{1}{16}+....+\frac{1}{16}\right)+....$$

$$= \qquad >\frac{2}{4}=\frac{1}{2} \qquad >\frac{4}{8}=\frac{1}{2} \qquad >\frac{8}{16}=\frac{1}{2}$$

In other words

$$\sum_{n=1}^{\infty}\frac{1}{n}>1+\frac{1}{2}+\frac{2}{4}+\frac{4}{8}+....+\frac{2^n}{2^{n+1}}+....$$

$$\sum_{n=1}^{\infty}\frac{1}{n}>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+....+\frac{1}{2}+....\rightarrow\infty$$

$$\sum_{n=1}^{\infty}\frac{1}{n} \quad \text{diverges}$$

Comparison Test (Term-by-Term Comparison) for series of Nonnegative Terms

1- If $\sum c_n$ is a convergent series and $a_n<c_n$ for some $n>n_0$ then $\sum a_n$ converges
2- If $\sum d_n$ is a divergent series and $a_n>d_n$ for some $n>n_0$ then $\sum a_n$ diverges

Our standard series are

1- Geometric series with $|r|<1$ convergent
2- Harmonic series divergent
3- Any series with $\lim_{n\to\infty} a_n \neq 0$ divergent

The integral Test

Let $a_n=f(n)$ where $f(x)$ is a continuous, positive, decreasing function of x for all $x\geq 1$ then the series $\sum a_n$ and the integral $\int_1^\infty f(x)dx$ both converge or diverge both

Ex: The p-Series (p is a real constant)

$$\sum_1^\infty \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + ... + \frac{1}{n^p} + ...$$

Let $f(x) = \frac{1}{x^p}$, $p>1$ => $p-1>0$ =>$1-p<0$

$$\int_1^\infty x^{-p}dx = \lim_{b\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^b = \frac{1}{1-p}\lim_{b\to\infty}(b^{1-p}-1) = \frac{1}{1-p}\lim_{b\to\infty}(\frac{1}{b^{p-1}}-1) = \frac{1}{p-1}$$

Which is finite, hence the p-series converges for p>1

If p=1 we have

$$\sum_1^\infty \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n} + ...$$

The harmonic series which we know it diverges or we can use the integral test

$$\int_1^\infty \frac{1}{x}dx = \lim_{b\to\infty} \ln x\Big|_1^b = \infty$$

Since the integral diverges, the series diverges

If p<1, by comparison test we find that each term is greater that the harmonic series terms which is divergent

$$\frac{1}{2^p} > \frac{1}{2},\ \frac{1}{3^p} > \frac{1}{3},\ \frac{1}{4^p} > \frac{1}{4} \ ..... \frac{1}{n^p} > \frac{1}{n}$$

$\sum_{n=1}^\infty \frac{1}{n^p}$ diverges for p<1

The Limit Comparison Test

If $a_n \geq 0$ for $n > n_o$ and there is a convergent series Σc_n such that $c_n > 0$ and

$\lim_{n\to\infty} \frac{a_n}{c_n} < \infty$ (finite positive number)

Then Σa_n is convergent

If $a_n \geq 0$ for $n > n_o$ and there is a divergent series Σd_n such that $d_n > 0$ and

$\lim_{n\to\infty} \frac{a_n}{d_n} < \infty$ (finite positive number)

Then Σa_n is divergent

Ex: a- $\sum_{n=2}^{\infty} \frac{2n}{n^2 - n + 1}$ b- $\sum_{n=2}^{\infty} \frac{2n^3 + 100n^2 + 1000}{(1/8)n^6 - n + 2}$

a- For large values of n the series behaves like $\sum \frac{2n}{n^2} = \sum \frac{2}{n}$, then we can compare our series with the divergent harmonic series

$\lim_{n\to\infty} \left(\frac{2n}{n^2 - n + 1} \right) / (1/n)) = \lim_{n\to\infty} \frac{2n^2}{n^2 - n + 1} = 2$

The series is divergent

b- For large values of n the series behaves like $\sum \frac{2n^3}{(1/8)n^6} = 16 \sum \frac{1}{n^3}$, then we can compare is with the series $1/n^3$ which we know it is convergent.

$\lim_{n\to\infty} \left(\frac{2n^3 + 100n^2 + 1000}{(1/8)n^6 - n + 2} \right) / \left(\frac{1}{n^3} \right) = \lim_{n\to\infty} \left(\frac{2n^6 + 100n^5 + 1000n^3}{(1/8)n^6 - n + 2} \right) = 2/(1/8) = 16$

The series is convergent because $\sum \frac{1}{n^3}$ .is a p-series with $p = 3 > 1$

Exercises 9.3 Which series converges and which diverges

1- $\sum_{n=1}^{\infty} \frac{1}{10^n}$ Converges, geometric series with $r = 1/10 < 1$

2- $\sum_{n=1}^{\infty} \frac{n}{n+1}$ Diverges by the n^{th} term test for divergence $a_n \to 1 \neq 0$

3- $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$ Converges because $\frac{\sin^2 n}{2^n} < \frac{1}{2^n}$, (term-by-term comparison)

10- $\sum_{n=1}^{\infty}\frac{-2}{n+1}$ By limit comparison test with $1/n$

$\lim_{n\to\infty}\left(\frac{1}{n+1}\right)/(1/n)=\lim_{n\to\infty}\frac{n}{n+1}$ =1 diverges

15- $\sum_{1}^{\infty}\frac{1}{\sqrt{n^3+2}}<\sum_{1}^{\infty}\frac{1}{n^{1.5}}$ convergent

16- $\sum_{2}^{\infty}\frac{\sqrt{n}}{\ln n}$

$f(x)=\frac{\sqrt{x}}{\ln x}$, $\lim_{x\to\infty}\frac{\sqrt{x}}{\ln x}=\lim_{x\to\infty}\frac{1/(2\sqrt{x})}{(1/x)}=\lim_{x\to\infty}\frac{x}{2\sqrt{x}}=\lim_{x\to\infty}\frac{\sqrt{x}}{2}=\infty$

$\lim_{n\to\infty}a_n=\infty\neq 0$ (n^{th} term test) divergent

18- $\sum_{n=1}^{\infty}\frac{1}{n\sqrt[n]{n}}$

Use limit comparison test with the divergent series $\sum\frac{1}{n}$

$\lim_{n\to\infty}\left(\frac{1}{n\sqrt[n]{n}}\right)/\left(\frac{1}{n}\right)=\lim_{n\to\infty}\frac{1}{\sqrt[n]{n}}$ =1 divergent series

Series with Nonnegative terms: Ratio and Root Test
The Ratio Test

Let Σa_n be a series with positive terms and suppose

$\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$ Then

1- The series converges if ρ<1
2- The series diverges if ρ>1
3- The test fails if ρ=1

Ex:

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} = \sum_{n=1}^{\infty} \frac{n!n!}{(2n)!},$$

$$a_{n+1} = \frac{(n+1)!(n+1)!}{(2n+2)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!(n+1)!(2n)!}{n!n!(2n+2)!} = \frac{(n+1)n!(n+1)n!(2n)!}{n!n!(2n+2)(2n+1)(2n)!} = \frac{(n+1)}{2(2n+1)}$$

$\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{(n+1)}{2(2n+1)} = 1/4$ the series is convergent

The nth Root Test

Let Σa_n be a series with an≥0 for n≥no and suppose that $\sqrt[n]{n} \to \rho$. Then

1- The series converges if ρ<1
2- The series diverges if ρ>1
3- The test fails if ρ=1

Ex: $a_n = \frac{n^2}{2^n}$

$\sqrt[n]{a_n} = \sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n}\sqrt[n]{n}}{2} \to 1/2$ the series is convergent

Ex: $a_n = \frac{e^n}{n^{10}}$

$\sqrt[n]{a_n} = \sqrt[n]{\frac{e^n}{n^{10}}} = \frac{\sqrt[n]{e^n}}{\left(\sqrt[n]{n}\right)^{10}} = \frac{e}{(1)^{10}} = e > 1$ the series is divergent

Exercises 9.4

2- $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

By ratio test

$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} = \frac{(n+1)n!}{10 \cdot 10^n} \cdot \frac{10^n}{n!} = \frac{n+1}{10}$ divergent

$\lim_{n\to\infty} \frac{n+1}{10} = \infty$ divergent

4- $\sum_{n=1}^{\infty} n^2 e^{-n}$

By n[th] root test

$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{n^2 e^{-n}} = e^{-1} \lim_{n\to\infty} \sqrt[n]{n}\sqrt[n]{n} = e^{-1}(1)(1) = e^{-1} < 1 \Rightarrow$ convergent

9- $\sum_{n=1}^{\infty} \left(1 - \frac{3}{n}\right)^n$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 - \frac{3}{n}\right)^n = e^{-3} \neq 0 \Rightarrow$ divergent

10- $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n^2}\right)^n$

$a_n = \left(1 - \frac{1}{n^2}\right)^n = \left(1 - \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right)^n \to e^{-1}e^{1} = 1 \Rightarrow$ divergent

11- $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

$\ln(n) > 1 \Rightarrow \frac{lnn}{n} > \frac{1}{n} \Rightarrow$ divergent

12- $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right) = \sum_{n=1}^{\infty}\frac{1}{n}-\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow$ divergent

14- $\sum_{n=1}^{\infty}\frac{n\ln n}{2^n}$

$(\ln n < n)\ \times \frac{n}{2^n} = \frac{nln}{2^n} < \frac{n^2}{2^n}$

By n[th] root test of the right-hand-side

$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\frac{n^2}{2^n}}$ =1/2<1 $\Rightarrow$ convergent

By the comparison test the series is convergent

20 $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ using ratio test

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{(n+1)}}\cdot\frac{n^n}{n!} = \frac{(n+1)n!}{(n+1)(n+1)^n}\cdot\frac{n^n}{n!} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$\to \frac{1}{e} = e^{-1}$ convergent

22- $\sum_{n=2}^{\infty}\frac{1}{(\ln n)^2}$

$$\frac{1}{(lnn)^2} > \frac{1}{nlnn}$$

Let $f(x) = \frac{1}{xlnx}$

$$\int_2^{\infty}\frac{dx}{xlnx} = \ln(lnx)|_2^{\infty} = \infty$$

Then $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ diverges

Then $\sum_{n=2}^{\infty}\frac{1}{(\ln n)^2}$ divergent by comparison test

9.5 Alternating Series

The alternating series Theorem

The Series

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots \dots .$$

converges if all three of the following conditions are satisfied

1- $a_n > 0$ for all n.

2- $a_{n+1} \leq a_n$ for $n > n_o$.

3- $\lim_{n\to\infty} a_n = 0$

Ex: The alternating harmonic series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

1- $a_n = \frac{1}{n} > 0$

2- $a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$

3- $\lim_{n\to\infty}\frac{1}{n} = 0$

This is a convergent series.

Definition

A series Σa_n **converges absolutely (absolutely convergent)** if the corresponding series of absolute values $\Sigma|a_n|$ is convergent.

A series that converges but does not converge absolutely **converges conditionally**.

Absolute Convergence Theorem

If $\Sigma|a_n|$ converges then Σa_n converges

Ex:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}=1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\cdots$$

The corresponding series of absolute values

$$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n^2}\right|=\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots$$

is a p-series with p=2>1, therefore, it converges absolutely, therefore

$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}$ is convergent

Alternating p-series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^p}=1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots$$

When p is a positive constant then $a_n=\frac{1}{n^p}$

1- $a_n=\frac{1}{n^p}>0$

2- $\frac{1}{(n+1)^p}<\frac{1}{n^p}$

3- $\lim_{n\to\infty}\frac{1}{n^p}=0$

Therefore

$\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^p}$, is a convergent series $p>0$

If $p>1$ the series converges absolutely

If $p\leq 1$ the series converges conditionally

Exercises 9.5

Show if the series is absolutely convergent, conditionally convergent or divergent.

2- $\sum_{n=2}^{\infty}(-1)^n \frac{1}{\ln n}$

This is an alternating series with $a_n = \frac{1}{\ln n}$

$$a_n = \frac{1}{\ln n} > 0$$

$$\frac{1}{\ln(n+1)} < \frac{1}{\ln n}$$

$$\lim_{n\to\infty} \frac{1}{\ln n} = 0$$

Therefore, it is a convergent harmonic series. But the series

$\sum_{n=2}^{\infty}\left|(-1)^n \frac{1}{\ln n}\right| = \sum_{n=2}^{\infty}\frac{1}{\ln n}$ is divergent

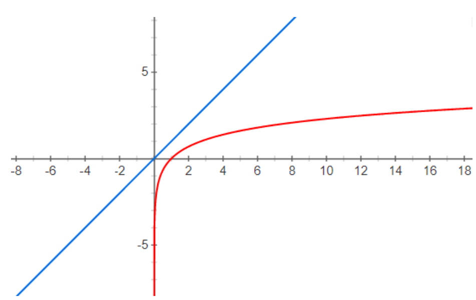
Because $\frac{1}{\ln n} > \frac{1}{n}$

Therefore the series $\sum_{n=2}^{\infty}(-1)^n \frac{1}{\ln n}$ is conditionally convergent.

6- $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{\ln n}{n}$

This is a harmonic series with $a_n = \frac{\ln n}{n}$

$$a_n = \frac{\ln n}{n} > 0$$



$$\frac{\ln(n+1)}{n+1} < \frac{\ln n}{n}$$

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

$$\lim_{n\to\infty} \frac{\ln x}{x} = \lim_{n\to\infty} \frac{1/x}{1} = 0$$

Therefore, it is a convergent harmonic series. But the series

$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{\ln n}{n} \right| = \sum_{n=1}^{\infty} \frac{\ln n}{n}$ is divergent

Because $\frac{\ln n}{n} > \frac{1}{n}$

Therefore, the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$ is conditionally convergent.

12- $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$ converges conditionally (show the details)

16- $\sum_{n=1}^{\infty} (-1)^{n} \frac{\sin n}{n^2}$

This is not a harmonic series because the term $\frac{\sin n}{n^2}$ is not always positive. By applying the absolute convergence theorem

$\left| (-1)^n \frac{\sin n}{n^2} \right| = \left| (-1)^n \right| \frac{|\sin n|}{|n^2|} = \frac{|\sin n|}{n^2} < \frac{1}{n^2}$ (*p*-series with *p*=2>1)

Therefore

$\sum_{n=1}^{\infty} \left| (-1) \frac{\sin n}{n^2} \right|$ is convergent therefore

$\sum_{n=1}^{\infty} (-1)^n \frac{\sin n}{n^2}$ is absolutely convergent

33- $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n\sqrt{n}}$

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$$

This is a harmonic series with $a_n = \frac{1}{n^{3/2}}$

$$a_n = \frac{1}{n^{3/2}} > 0$$

$$\frac{1}{(n+1)^{3/2}} < \frac{1}{n^{3/2}}$$

$$\lim_{n\to\infty} \frac{1}{n^{3/2}} = 0$$

The harmonic series is convergent, but the series

$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^{3/2}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a p-series with $p = \frac{3}{2} > 1$ which is a convergent series

Therefore the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$ is absolutely convergent.

9.6 Power Series

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots$$

Or

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n + \cdots$$

Where a is the center of the series and c_0, c_1, c_2,… are the coefficients.

Ex: the geometric series is a special case of the power series with all coefficients equal to 1.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

It converges to $\frac{1}{1-x}$ for $|x|<1$

The Radius and interval of convergence

The series of the form

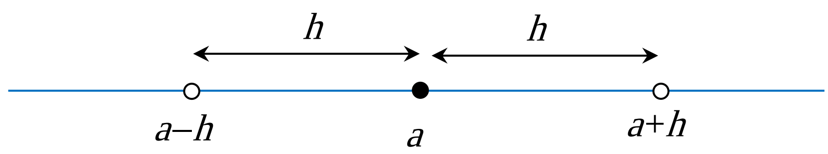
$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

Can have either of the following behaviors

1- The series converges at a and diverges elsewhere.

2- There is a positive number h such that the series diverges for $|x-a|>h$ but converges absolutely for $|x-a|<h$. The series may or may not converge at the endpoints $x=a+h$ and $x=a-h$.

3- The series converges for all values of x.



In case #2, the set of points at which the series converges is called the **Interval of Convergence** and the value of ***h*** is called the **radius of convergence**.

Ex:

For what values of *x* do the following series converge or diverge. (Find the interval of convergence of the following series.)

a) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$

By applying the absolute convergence theorem, we test the convergence of the corresponding series with absolute values $\Sigma|a_n|$.

$$|a_n| = \left|(-1)^{n+1}\frac{x^n}{n}\right| = \frac{|x|^n}{n}$$

$$|a_{n+1}| = \frac{|x|^{n+1}}{n+1}$$

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{|x|^{n+1}/(n+1)}{|x|^n/n} = |x|\lim_{n\to\infty}\frac{n}{n+1} = |x| < 1$$

The series converges absolutely for $|x|<1 \Rightarrow -1<x<1$.

Now we have to repeat the test at the endpoints of the interval

At $x=1$:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$$

This is a convergent alternating harmonic series.

At x=–1:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-1)^n}{n}=\sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n}=\sum_{n=1}^{\infty}\frac{(-1)((-1)^2)^n}{n}=(-1)\sum_{n=1}^{\infty}\frac{1}{n}$$

This is a divergent harmonic series.

Therefore the interval of convergence is $-1<x\leq 1$.

b) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{2n-1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots$

$$|a_n|=\left|(-1)^{n+1}\frac{x^{2n-1}}{2n-1}\right|=\frac{|x|^{2n-1}}{2n-1}$$

$$|a_{n+1}|=\frac{|x|^{2(n+1)-1}}{2(n+1)-1}=\frac{|x|^{2n+1}}{2n+1}$$

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{|x|^{2n+1}/(2n+1)}{|x|^{2n-1}/(2n-1)}=|x|^2\lim_{n\to\infty}\frac{2n-1}{2n+1}=|x|^2<1$$

The series converges absolutely for $|x|^2<1 \Rightarrow -1<x<1$.

At x=1:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n-1}$$

This is an alternating series with

$$\frac{1}{2n-1}>0$$

$$\frac{1}{2(n+1)-1}=\frac{1}{2n+1}<\frac{1}{2n-1}$$

$$\lim_{n\to\infty}\frac{1}{2n-1}=0$$

Therefore the series is convergent at x=1

At x=–1

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-1)^{2n-1}}{2n-1}=\sum_{n=1}^{\infty}(-1)^{n}\frac{(-1)^{2n}}{2n-1}=\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{2n-1}$$

This is also an alternating series that converge as above.

Therefore the series is convergent for $-1 \leq x \leq 1$

c) $\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$

$$|a_n|=\left|\frac{x^n}{n!}\right|=\frac{|x|^n}{n!}$$

$$|a_{n+1}|=\frac{|x|^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{|x|^{n+1}/(n+1)!}{|x|^n/n!}=|x|\lim_{n\to\infty}\frac{1}{n+1}=0<1$$

The series converges for all values of x.

d) $\sum_{n=0}^{\infty}n!x^n=1+x+2!x^2+3!x^3+\dots$

$$|a_n|=|n!\,x^n|=n!\,|x|^n$$

$$|a_{n+1}|=|(n+1)!\,x^{n+1}|=(n+1)!\,|x|^{n+1}$$

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{(n+1)!\,|x|^{n+1}}{n!\,|x|^n}=|x|\lim_{n\to\infty}(n+1)=\infty>1$$

The series diverges for all values of $x \neq 0$.

Exercises 9.6

3- $\sum_{n=0}^{\infty}(-1)^n(x+1)^n$

This is a power series and a geometric series Σr^n

$a_n = (-1)^n(x+1)^n = (-(x+1))^n = r^n$

It converges for $|r|<1 \Rightarrow$

$|-(x+1)| < 1 \;\Rightarrow\; |x+1| \Rightarrow\; -1 < x+1 < 1 \;\Rightarrow\; -2 < x+< 0$

The series is divergent at –2 and 0.

5- $\sum_{n=0}^{\infty}\frac{(x-2)^n}{10^n}$

This is a geometric series and it can be written in the form

$$\sum_{n=0}^{\infty}\frac{(x-2)^n}{10^n} = \sum_{n=0}^{\infty}\left(\frac{x-2}{10}\right)^n = \sum_{n=0}^{\infty} r^n$$

The series is convergent for $|r| = \left|\frac{x-2}{10}\right| < 1 \Rightarrow$

$$-1 < \frac{x-2}{10} < 1 \;\Rightarrow\; -10 < x-2 < 10 \;\Rightarrow\; -8 < x < 12$$

17- $\sum_{n=1}^{\infty}\frac{x^n}{n\sqrt{n}}$

$$|a_n| = \left|\frac{x^n}{n\sqrt{n}}\right| = \frac{|x|^n}{n^{3/2}}$$

$$|a_{n+1}| = \frac{|x|^{n+1}}{(n+1)^{3/2}}$$

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{n^{3/2}|x|^{n+1}}{(n+1)^{3/2}|x|^n} = |x|\lim_{n\to\infty}\left(\frac{n}{n+1}\right) = |x| < 1$$

The series is convergent for $-1<x<1$

At x=1:

$$\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$$

This is a p-series with p=3/2>1 ⇒ convergent

At $x=-1$:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n\sqrt{n}}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{3/2}}$$

This is an alternating p-series ⇒ convergent

Therefore, the series is convergent for $-1\le x\le 1$

Taylor Series and McLaurin Series

Let f be a function with derivatives of all order throughout some interval containing a as interior, then the **Taylor Series** generated by f at a is

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots$$

And the Mclaurin series generated by f is

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots$$