6.1 Introduction

The range of differential equations that can be solved by straightforward analytical methods is relatively restricted. Even solution in series may not always be satisfactory, either because of the slow convergence of the resulting series or because of the involved manipulation in repeated stages of differentiation. In such cases, where a differential equation and known boundary conditions are given, an approximate solution is often obtainable by the application of numerical methods, where a numerical solution is obtained at discrete values of the independent variable.

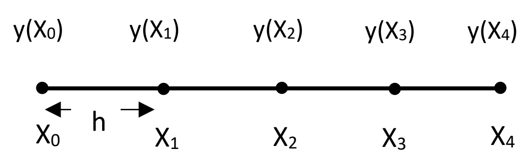
6.2 Taylor Series

The Taylor series, is of great value in the study of numerical methods. In essence, the Taylor series provides a means to predict a function value at one point in terms of the function value and its derivatives at another point. In particular, the theorem states that any smooth function can be approximated as a polynomial.

6.2.1 Taylor's Theorem

$$\boldsymbol{y(x_1) = y(x_0) + \frac{(x_1 - x_0)}{1!} y'(x_0) + \frac{(x_1 - x_0)^2}{2!} y''(x_0) + \frac{(x_1 - x_0)^3}{3!} y'''(x_0) + \cdots}$$

$(x_1 - x_0)$ represents the difference between two consecutive x value which represents (h)



$$\therefore \boldsymbol{y(x_1) = y(x_0) + \frac{h}{1!} y'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \cdots}$$

In general

$$\therefore \boldsymbol{y(x_{i+1}) = y(x_i) + \frac{h}{1!}y'(x_i) + \frac{h^2}{2!}y''(x_i) + \frac{h^3}{3!}y'''(x_i) + \cdots}$$

A special case of Taylor series can be obtained by taking the value of $x_0 = 0$.

In this case the series is called **Maclaurin Series**

$$\boldsymbol{y(x_1) = y(0) + \frac{(x_1)}{1!}y'(0) + \frac{(x_1)^2}{2!}y''(0) + \frac{(x_1)^3}{3!}y'''(0) + \cdots}$$

Example (1):

Given $y' = x + y$, $y(0) = 0, h = 0.2$. Use Taylor Series to determine the value of y for three steps.

Solution:

$y(x_{i+1}) = y(x_i) + \frac{h}{1!}y'(x_i) + \frac{h^2}{2!}y''(x_i) + \frac{h^3}{3!}y'''(x_i)$

$y'' = 1 + y'$

$y''' = y''$

$y(0.2) = 0 + \frac{0.2}{1!}0 + \frac{0.2^2}{2!}1 + \frac{0.2^3}{3!}1 = 0.0213$

$y(0.4) = 0.0213 + \frac{0.2}{1!}0.2213 + \frac{0.2^2}{2!}1.2213 + \frac{0.2^3}{3!}1.2213 = 0.0916144$

$y(0.6) = 0.0916144 + \frac{0.2}{1!}0.4916144 + \frac{0.2^2}{2!}1.4916144 + \frac{0.2^3}{3!}1.4916144$

H.W.(1): given $y'' = 13\cos(x) - 45y, y(0) = y'(0) = 0, h = 0.01$. Use the first three terms of Taylor Series to determine $y(0.01), y(0.02)$