7.1 Introduction

For the bracketing methods in Chap. 5, the root is located within an interval prescribed by a lower and an upper bound. Repeated application of these methods always results in closer estimates of the true value of the root. Such methods are said to be convergent because they move closer to the truth as the computation progresses. **In contrast**, the open methods described in this chapter are based on formulas that require only a single starting value of x or two starting values that do not necessarily bracket the root. As such, they sometimes diverge or move away from the true root as the computation progresses. However, when the open methods converge, they usually do so much more quickly than the bracketing methods.

7.2 Simple Fixed Iteration Method

As mentioned above, open methods employ a formula to predict the root. Such a formula can be developed for **simple fixed-point iteration** (or, as it is also called, one-point iteration or successive substitution) by rearranging the function $f(x) = 0$ so that x is on the left-hand side of the equation:

$$x = g(x) \tag{1}$$

This transformation can be accomplished either by algebraic manipulation or by simply adding x to both sides of the original equation. For example,

$$x^2 - 2x + 3 = 0$$

can be simply manipulated to yield

$$x = \frac{x^2+3}{2}$$

whereas sin x = 0 could be manipulated by adding x to both sides to yield

$$x = \sin x + x$$

The utility of Eq. (1) is that it provides a formula to predict a new value of x as a function of an old value of x. Thus, given an initial guess at the root x_i, Eq. (1) can be used to compute a new estimate x_{i+1} as expressed by the iterative formula

$x_{i+1} = g(x_i)$

The approximate error for this equation can be determined using the error estimator

$\varepsilon_a = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| 100\%$

Important Note: the condition for choosing the function in the right hand side of equation (1) is $|g'(x)| \leq 1$

Example (1): Use simple fixed-point iteration to locate the root of $f(x) = e^{-x} - x$. Starting with an initial guess of $x_0 = 0$.

Solution:

The function can be separated directly and expressed in the form of

$x_{i+1} = e^{-x_i}$

i	x_i	ε_a
0	0	
1	1.0000	100.0
2	0.367879	171.8
3	0.962291	46.9
4	0.500473	38.3
5	0.606244	17.4
6	0.545396	11.2
7	0.579612	5.90
8	0.560115	3.48
9	0.571143	1.93
10	0. 564879	1.11

H.W. (1): Use simple fixed-point iteration to locate the root of $x - \sin(x) = 0.25$ Starting with an initial guess of $x_0 = 1.2$.

7.3 Newton-Raphson Method

Perhaps the most widely used of all root-locating formulas is the Newton-Raphson equation, which can be written as:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad (2)$$

which is called the Newton-Raphson formula.

Example (2): Use the Newton-Raphson method to estimate the root of $f(x) = e^{-x} - x$. Employing an initial guess of $x_0 = 0$.

Solution:

The first derivative of the function can be evaluated as

$$f'(x) = -e^{-x} - 1$$

Which can be substituted along with the original function into Eq. (2) to give

$$x_{i+1} = x_i - \frac{e^{-x_i} - x_i}{-e^{-x_i} - 1}$$

Starting with an initial guess of $x_0 = 0$, this iterative equation can be applied to compute

i	x_i	ε_a %
1	0	
2	0.50000	100
3	0.56631	11.9
4	0.56714	$1.46e^{-3}$
5	0.56714	0

H.W.2: Use the Newton-Raphson method to estimate the root of $f(x) = e^{x} - 1.5 - tan^{-1}x$. Employing an initial guess of $x_0 = -10$.