4.1 Introduction

In many applications, the engineer often encounters integrals that are very difficult or even impossible to solve analytically. For example, the error function, we then need methods from numerical analysis to evaluate such integrals. Methods that address these kinds of problems are called methods of numeric integration.

Numeric integration means the numeric evaluation of integrals

$$R = \int_a^b f(x)dx$$

Where:

a & b : beginning and ending the interval

f(x): a function given analytically by a formula or empirically by a table of values.

R: is the area under the curve of ***f*** between ***a*** and ***b***

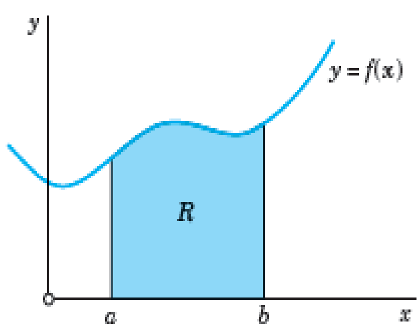

Figure (4-1) Geometric interpretation of a definite integral

4.2 Rectangular Rule. Trapezoidal Rule

Numeric integration methods are obtained by approximating the integrand f by functions that can easily be integrated. The first order represent a straight line cube can be represented as follows:

$$f_1(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

The area under the straight line is an estimate of the integral of $f(x)$ between the limits ***a & b***

$$I \cong \int_a^b \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a)\right] dx$$

The result of integration is:

$$I \cong (b - a)\frac{f(a) + f(b)}{2}$$

This is called trapezoidal rule.

Example (4-1)

Integrate the following function numerically

$$f(x) = 0.2 + 2.5x - 2x^2 + 6.7x^3 - 9x^4 + 4x^5$$

From $\boldsymbol{a = 0}$ to $\boldsymbol{b = 0.8}$, (I_{exact} = 1.6405)

Solution:

The function values rule $f(0) = 0.2, f(0.8) = 1.9747$

Due to Trapezoidal rule

$$I \cong (b - a)\frac{f(a) + f(b)}{2}$$

$$I \cong (0.8)\frac{0.2 + 1.9747}{2} \cong 0.87$$

$$Error\ = 1.6405 - 0.87 = 0.7706$$

4.2.1 Multiple – application Trapezoidal rule

To improve the accuracy of trapezoidal rule is to divide the integration interval from ***a*** to ***b*** into a number of segments and apply the method to each segment then added to yield the integral.

There are ***n*** segments of equal width:

$$h = \frac{b-a}{n}$$

If ***a*** & ***b*** are designated as x_0 and x_n the total integral can be represented as:

$$I = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \ldots + \int_{x_{n-1}}^{x_n} f(x)dx$$

Substituting the Trapezoidal rule for each integral yields

$$I \cong \frac{h}{2}\big(f(x_0) + f(x_1)\big) + \frac{h}{2}(f\big(x_1 + f(x_2)\big) + \cdots + \frac{h}{2}(f(x_{n-1} + f(x_n))$$

Re arranging

$$I \cong \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

Example (4-2)

Use two segments Trapezoidal rule to integrate numerically the function

$$f(x) = 0.2 + 2.5x - 2x^2 + 6.7x^3 - 9x^4 + 4x^5$$

From $\boldsymbol{a = 0}$ to $\boldsymbol{b = 0.8}$

The function values rule $f(0) = 0.2, f(0.8) = 1.9747$

$$h = \frac{b-a}{n} = \frac{0.8-0}{2} = 0.4$$

$$f(0.4) = 1.11936$$

$$I \cong \frac{0.4}{2}[0.2 + 2(1.11936) + 1.9747] \cong 0.8826$$

4.2.2 Error of the Trapezoidal Rule

When we employ the integral under a straight-line segment to approximate the integral under a curve, we obviously can incur an error that may be substantial (figure 4.2). An estimate for the local truncation error of a single application of the trapezoidal rule is

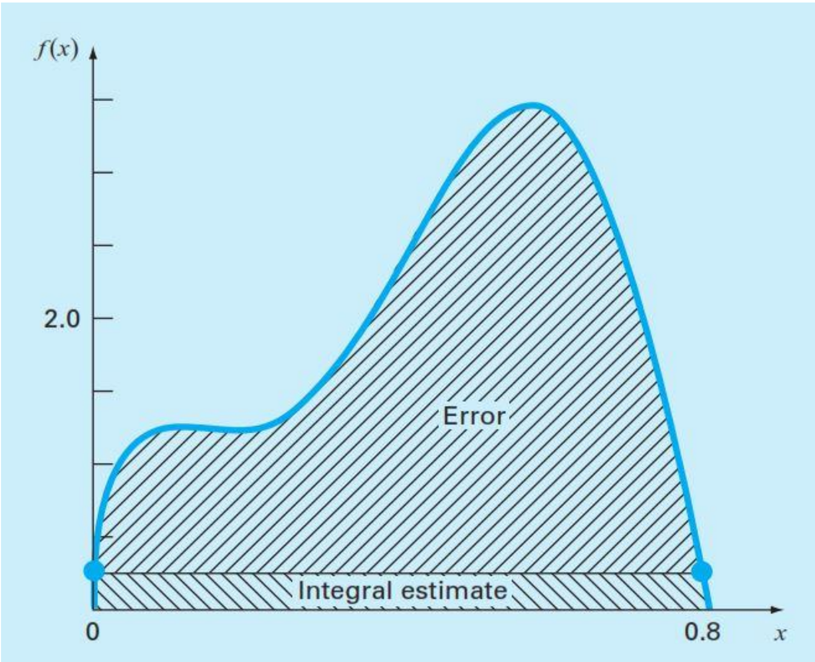

Figure (4.2) Graphical depiction of the use of a single application of the trapezoidal rule to approximate the integral of $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ from $x = 0$ to 0.8

$$E_a = -\frac{1}{12} f''(\xi)(b-a)^3$$

Where ξ lies somewhere in the interval from a to b. Equation above indicates that if the function being integrated is linear, the trapezoidal rule will be exact. Otherwise, for functions with second and higher-order derivatives (that is, with curvature), some error can occur. This is the truncation error for a trapezoidal rule type (rectangular rule). In the case of a trapezoidal of multi segment, the error can be computed from

$$E_t = -\frac{(b-a)^3}{12n^3}\sum_{i=1}^{n} f''(\xi_i)$$

Where $f''(\xi_i)$ is the second derivative at a point ξ_i located in segment i. This result can be simplified by estimating the mean or average value of the second derivative for the entire interval as

$$\bar{f}'' \cong \frac{\sum_{i=1}^{n} f''(\xi_i)}{n}$$

Therefore, $\sum f''(\xi_i) \cong n\,\bar{f}''$

$$E_a = -\frac{(b-a)^3}{12n^2}\bar{f}''$$

Thus, if the number of segments is doubled, the truncation error will be quartered

Example (4.3)

Use the trapezoidal rule to integrate the function

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 500x^5$$

From $a = 0$ to $b = 0.8$. You must know that the exact value of the integral can be determined analytically to be 1.640533. After that use a multiple trapezoidal rule for two segments and compute the error for each method

Solution

$$f(0) = 0.2$$

$$f(0.4) = 2.456$$

$$f(0.8) = 0.232$$

$$I \cong (b-a)\frac{f(a)+f(b)}{2}$$

$$I \cong (0.8-0)\frac{0.2+0.232}{2} = 0.1728$$

Which represents an error of

$$E_t = 1.640533 - 0.1728 = 1.467733$$

Which corresponds to a percent relative error of ε_t = 89.5%. The reason for this large error is evident from the graphical depiction in figure (4.2). Notice that the area under the straight line neglects a significant portion of the integral lying above the line.

In actual situations, we would have no foreknowledge of the true value. Therefore, an approximate error estimate is required. To obtain this estimate, the function's second derivative over the interval can be computed by differentiating the original function twice to give

$$f''(x) = -400 + 4050x - 10800x^2 + 8000x^3$$

The average value of the second derivative can be computed as followed

$$\bar{f}''(x) = \frac{\int_0^{0.8}(-400 + 4050x - 10800x^2 + 8000x^3)dx}{0.8-0} = -60$$

Which can be substituted into the truncation error equation

$$E_a = -\frac{(b-a)^3}{12n^2}\bar{f}'' = -\frac{(0.8-0)^3}{12(2)^2}(-60) = 0.64$$

H.W (1)

Use Trapezoidal rule to evaluate $\int e^{-x^2}$ from $a = 0$ to $b = 1$with $n = 10$, compare the result with rectangular rule.

H.W (2)

To get a feel for increase in accuracy, integrate x^2 from 0 to 1 with $h = 1, 0.5, 0.25, 0.1$.

4.3 Simpson's rule

The trapezoidal rule approximates the area under the curve by assuming the area of uniform width trapezoidal formed by connecting successive points on the curve by straight line.

Simpson's rule gives a more accurate approximation since it consists of connecting successive groups of three points on the curve by a second degree parabola.

Simpson's rule results from integrating over [a,b] the second Lagrange polynomial with equally-spaced nodes $x_0 = a$, $x_2 = b$, and $x_1 = a + h$, where $h = (b - a)/2$. As in figure (4-3).

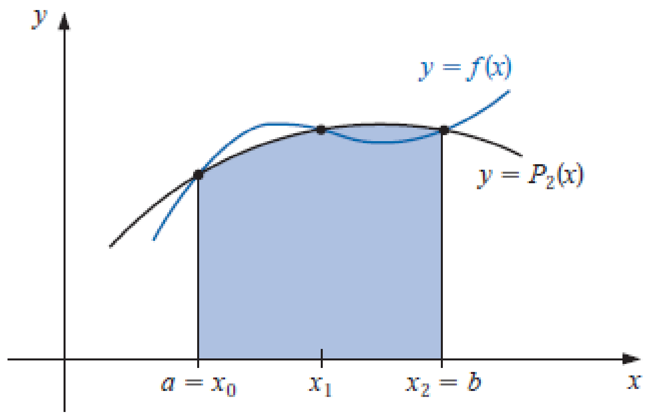

Figure (4-3) Simpson's rule

4.3.1 Simpson's 1/3 rule

Simpson's 1/3 rule results when a second order interpolating polynomial is substituting into the equation below:

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_2(x)dx$$

If a & b designated as x_0&x_2 and $f_2(x)$ is represented by a second order Lagrange polynomial, then the integral becomes:

$$I \cong \int_{x_0}^{x_2} \left[\frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \right] dx$$

After integration and algebraic manipulation the following formulation results:

$$I \cong \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

$$h = \frac{b-a}{2}$$

Example (4-3)

Use 1/3 Simpson's rule to integrate

$$f(x) = 0.2 + 2.5x - 2x^2 + 6.7x^3 - 9x^4 + 4x^5$$

From $a = 0$ to $b = 0.8$

Solution:

$$f(0) = 0.2, f(0.4) = 1.11936, f(0.8) = 1.9747$$

$$I \cong \frac{(0.8-0)}{3*2}[0.2 + 4(1.11936 + 1.97472] = 0.8869$$

H.W (3)

Evaluate $\int_0^2 \sqrt{1+x^2}\,dx$ using Simpson's rule and find the percentage error, $I_{exact} = 2.958$

4.3.2 Derivation and Error Estimate of Simpson's 1/3 Rule

For a single 1/3 Simpson's rule, the truncation error can be calculated from

$$E_a = -\frac{1}{90}h^5 f^4(\xi)$$

Or, because that $h = (b-a)/2$

$$E_a = -\frac{(b-a)^5}{2880} f^4(\xi)$$

Where ξ lies somewhere in the interval from a to b. Thus, Simpson's 1/3 rule is more accurate than the trapezoidal rule.

Example (4.4)

Estimate the truncation and exact error for example (4.3)

Solution

The exact error is

$$E_t = 1.640533 - 1.367467 = 0.2730667$$

The truncation estimated error is

$$E_a = -\frac{(b-a)^5}{2880} f^4(\xi)$$

$$E_a = -\frac{(0.8)^5}{2880}(-2400) = 0.2730667$$

Where (-2400) is the average fourth derivative for the interval as obtained using the equation below

$$Mean = \frac{\int_a^b f(x)dx}{b-a}$$

4.3.3 Simpson's 3/8 rule

In a similar manner to derivation of the trapezoidal rule, and Simpson's 1/3 rule, a third order Lagrange polynomial can be fit to four points, and integrated

$$I = \int_a^b f(x)dx$$

To yield

$$I = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$$

$$h = \frac{b-a}{3}$$

This equation is called Simpson's 3/8 rule, because h multiplied by 3/8. It is the third Newton-Cotes integration formula.

4.3.4 Estimated Error of Simpson's 3/8 Rule

The estimated trunction error of a Simpson's 3/8 rule can be computed from the following equation

$$E_t = -\frac{3}{80}h^5 f^4(\xi)$$

Or, because $h = (b-a)/3$

$$E_t = -\frac{(b-a)^5}{6480}f^4(\xi)$$

4.4 Composite integration

Another method for evaluate the integration is to connect to ways of integration, like Trapezoidal and Simpson's rule, as illustrate in example below.

Example (4-4)

Use 3/8 conjunction with Simpson's 1/3 rule to integrate the function, for five segments from $a = 0$ to $b = 0.8$

$$f(x) = 0.2 + 2.5x - 2x^2 + 6.7x^3 - 9x^4 + 4x^5$$

Solution:

The data for five segments ($h = 0.16$)

$$f(0) = 0.2, \quad f(0.16) = 0.57053, \quad f(0.32) = 1.03379$$

$$f(0.48) = 1.3044, \quad f(0.64) = 1.6534, \quad f(0.8) = 1.97472$$

The integral for first two segments is obtained by Simpson's 1/3 rule and last three segments 3/8 rule can be used, finally evaluate the result from

$$I_{total} = I_{1/3} + I_{3/8}$$