1.1 Introduction

The basic idea for this method is **finding the value of (x)** by arranging several simulation equations in a manner that simplify the solution.

A simple example can explain the above text:

Suppose we have several equations that contain the unknown value of (x), we cannot find the value of (x_s) from one equation, like

$$\boldsymbol{a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1}$$

Where

The values of $\boldsymbol{x_1} \rightarrow \boldsymbol{x_n}$ is unknown

$\boldsymbol{a_{11}} \rightarrow \boldsymbol{a_{1n}}$ Constant

$\boldsymbol{b_1}$ Constant

So:

If we want to know the values of (x), we should have several equations having a number equal to the number of unknown,

$$\begin{gathered}
\boldsymbol{a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1} \\
\boldsymbol{a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2} \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
\boldsymbol{a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m}
\end{gathered}$$

Where the $\boldsymbol{a_s}$ are constant coefficients and the $\boldsymbol{b_s}$ are constants.

There are two types for solving the system of equations.

- **A- Indirect Method (Gauss-Seidel, Jacobi).**
- **B- Direct Method (Gauss Elimination, Gauss Jordan).**

1.2 Indirect Method (Iterative Technique)

1.2.1 Gauss-Seidel Iteration Method

Iterative technique method is one in which we start from an approximate and end up with the true solution as the number of iterations becomes large.

Assume the equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

The solution for the new values is:

$$x_1^{(k+1)} = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(k)} - \frac{a_{13}}{a_{11}} x_3^{(k)}$$

$$x_2^{(k+1)} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(k+1)} - \frac{a_{23}}{a_{22}} x_3^{(k)}$$

$$x_3^{(k+1)} = \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_2^{(k+1)} - \frac{a_{32}}{a_{33}} x_3^{(k+1)}$$

In General

$$x_1^{(k+1)} = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(k)} - \frac{a_{13}}{a_{11}} x_3^{(k)} \ldots\ldots - \frac{a_{1n}}{a_{11}} x_n^{(k)}$$

$$x_2^{(k+1)} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(k+1)} - \frac{a_{23}}{a_{22}} x_3^{(k)} \ldots\ldots - \frac{a_{2n}}{a_{22}} x_n^{(k)}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$x_n^{(k+1)} = \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(k+1)} - \frac{a_{n2}}{a_{nn}} x_2^{(k+1)} \ldots\ldots - \frac{a_{n\,n-1}}{a_{nn}} x_{n-1}^{(k+1)}$$

Important Note

The **first step** for solving any **gauss-Seidel** problem is to apply the condition below

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

Example (1-1)

Solve the following set of linear algebraic equations using **Gauss-Seidel** method.

$$5x_1 - 2x_2 + x_3 = 4$$

$$x_1 + 4x_2 - 2x_3 = 3$$

$$x_1 + 2x_2 + 4x_3 = 17$$

Solution

Obtain the condition

$|5| > |-2| + |1| \rightarrow 5 > 3$ .. Right

$|4| > |1| + |-2| \rightarrow 4 > 3$ .. Right

$|4| > |1| + |2| \rightarrow 4 > 3$ .. Right

∴ We can solve the problem by Gauss-Seidel Method.

$$x_1^{(k+1)} = 0.8 + 0.4x_2^{(k)} - 0.2x_3^{(k)}$$

$$x_2^{(k+1)} = 0.75 - 0.25x_1^{(k+1)} + 0.5x_3^{(k)}$$

$$x_3^{(k+1)} = 4.25 - 0.25x_1^{(k+1)} - 0.5x_2^{(k+1)}$$

For (k=0)/first iterative, initial guess $x_1 = x_2 = x_3 = 1$

$$x_1^1 = 0.8 + 0.4 - 0.2 = 1$$

$x_2^1 = 0.75 - (0.25 * 1) - 0.5 = 0$

$x_3^1 = 4.25 - 0.25 - 0 = 4$

The process is continued by substituting the new $\boldsymbol{x_2^2}$ & $\boldsymbol{x_3^2}$

Iteration	X_1	X_2	X_3
1	1	0	4
2	0	2.75	2.875
3	1.324	1.8565	2.990
4	0.9446	2.008	3.007
5	1.002	2.003	2.998
6	1.001	1.999	3.000
7	0.999	2.000	3.000

$$\therefore\ x_1 = 1, \quad x_2 = 2, \quad x_3 = 3$$

H.W (1)

Solve the following sets of linear equations using Gauss-Seidel and Jacobi iteration methods and compare the results of the two methods.

$$8x_1 + 2x_2 + 3x_3 = 30$$

$$x_1 - 9x_2 + 2x_3 = 1$$

$$2x_1 + 3x_2 + 6x_3 = 31$$

H.W (2)

Solve the following sets of linear equations using Gauss-Seidel and Jacobi iteration methods.

$$\begin{aligned} 10x_1 - x_2 + 2x_3 \qquad &= 6 \\ -x_1 + 11x_2 - x_3 + 3x_4 &= 25 \\ 2x_1 - x_2 + 10x_3 - x_4 &= -11 \\ 3x_2 - x_3 + 8x_4 &= 15 \end{aligned}$$

1.3 Direct Method (Exact Method)

1.3.1 Gauss Elimination Method

Engineers often need to solve large systems of linear equations; for example in determining the forces in a large framework or finding currents in a complicated electrical circuit. The method of Gauss elimination provides a systematic approach to their solution.

The technique involves combining equations in order to eliminate unknown, although it is one of the earliest methods for solving simultaneous equations, it remains among the most important algorithms in use today.

Simple Example

Consider we have three simultaneous equations

$$\begin{aligned} 3x_1 - x_2 + 2x_3 &= 12 \quad \rightarrow r_1 \\ x_1 + 2x_2 + 3x_3 &= 11 \quad \rightarrow r_2 \\ 2x_1 - 2x_2 - x_3 &= 2 \quad \rightarrow r_3 \end{aligned}$$

First step eliminating $\boldsymbol{x_1}$ from equations (1&2)

$(3r_2 - r_1)$ & multiply r_1 by 2 & r_3 by 3 and subtract r_3 from r_1

$$3x_1 - x_2 + 2x_3 = 12$$

$$0 + 7x_2 + 7x_3 = 21$$

$$0 - 4x_2 - 7x_3 = -18$$

Eliminate x_2 from equation (3) by multiplying equation (2 by 4) and equation (3 by 7).

$$3x_1 - x_2 + 2x_3 = 12$$
$$0 + 7x_2 + 7x_3 = 21$$
$$0 + 0 - 21x_3 = -42$$

$x_3 = 2$ From third equation

By back substitution

$x_2 = 1\ , x_1 = 3$

H.W (1)

Solve the following sets of linear equations using Gauss-Elimination method.

$$\begin{bmatrix} 0.729 & 0.81 & 0.9 \\ 1 & 1 & 1 \\ 1.331 & 1.2 & 1 \end{bmatrix} \begin{bmatrix} 0.6867 \\ 0.8338 \\ 1 \end{bmatrix}$$

H.W (2)

Solve the following sets of linear equations using Gauss-Elimination method.

$$x_1 + x_2 - x_3 = 0$$

$$3x_1 - x_2 + x_3 = 4$$

$$x_1 - 2x_2 + 2x_3 = 3$$

1.3.2 Gauss-Jordan Method

The Gauss-Jordan method is a variation of Gauss Elimination, the major difference is that the unknowns eliminated from all other equations rather than just the subsequent ones.

All rows are normalized by dividing them by their pivot equations, this illustrated by an example.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} \vdots & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{bmatrix}$$

$$\downarrow\downarrow\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & c_1^{(n)} \\ 0 & 1 & 0 \vdots & c_2^{(n)} \\ 0 & 0 & 1 & c_3^{(n)} \end{bmatrix}$$

Hint

The superscript (n) means that the elements of the right-hand side vector have been modified n- times (for this case n=3).

Example

Solve the following set of algebraic equations by using the Gauss-Jordan method

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.3$$

Solution

The first step normalized the first row by dividing by 3

$$\begin{bmatrix} 1 & -0.0333 & -0.06667 & 2.61667 \\ 0.1 & 7 & -0.31 \quad \vdots & -19.3 \\ 0.3 & -0.2 & 10 & 71.3 \end{bmatrix} \begin{matrix} R_1 \\ \rightarrow R_2 \\ R_3 \end{matrix}$$

Now: (R_2 - 0.1R_1), (R_3 – 0.3R_1)

$$\begin{bmatrix} 1 & -0.0333 & -0.06667 & 2.61667 \\ 0 & 7.0033 & -0.2933 \vdots & -19.56166 \\ 0 & -0.1900 & 10.02 & 70.51499 \end{bmatrix}$$

Now: (R_2/7.0033), (R_3 +0.1900R_2)

$$\begin{bmatrix} 1 & -0.0333 & -0.06667 & \vdots & 2.61667 \\ 0 & 1 & -0.0417 & \vdots & -2.7814 \\ 0 & 0 & 10.192 & & 70.086 \end{bmatrix}$$

Now: ($R_1 + 0.0333R_2$)

$$\begin{bmatrix} 1 & 0 & -0.068 & \vdots & 2.524 \\ 0 & 1 & -0.0417 & \vdots & -2.7814 \\ 0 & 0 & 10.192 & & 70.081 \end{bmatrix}$$

Now: ($R_3/10.192$),($R_1 + 0.068R_3$), ($R_1+0.0417R_2$)

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & 2.991 \\ 0 & 1 & 0 & \vdots & -2.4946 \\ 0 & 0 & 1 & & 6.8765 \end{bmatrix}$$

$$\therefore\ x_1 = 2.991\ , x_2 = -2.4946\ , x_3 = 6.8765$$

H.W (1)

Solve the following set of algebraic equations by using Gauss-Jordan method

$$\begin{bmatrix} 3 & -1 & 2 & \vdots & 12 \\ 1 & 2 & 3 & \vdots & 11 \\ 2 & -2 & -1 & & 2 \end{bmatrix}$$